Time Interval Errors of a Flicker-Noise Generator

C. A. Greenhall
Communications Systems Research Section

Time interval error (TIE) is the error of a clock at time t after it has been synchronized and syntonized at time zero. Previous simulations of Flicker FM noise have yielded a mean-square TIE proportional to t^2. This study shows that the order of growth is actually t^2 log t, explains the earlier t^2 result, and gives a modified version of the Barnes-Jarvis simulation algorithm.

I. Introduction

Let the time base of a clock be an oscillator whose output is $\sin(2\pi\nu_0 t + \phi(t))$, where ν_0 is a nominal frequency and $\phi(t)$ the phase. Then we call $x_m(t) = \phi(t)/(2\pi\nu_0)$ the measured raw time deviation of the clock. Its derivative $y(t) = dx_m/dt$ is the instantaneous fractional frequency deviation. The τ-average fractional frequency deviation is defined by

$$\bar{y}(t,\tau) = \frac{1}{\tau}\int_{t-\tau}^{t} y(s)ds = \frac{x_m(t)-x_m(t-\tau)}{\tau} \qquad (1)$$

The standard measure of frequency stability is the two-sample (Allan) variance $\sigma_y^2(\tau)$, defined by

$$\sigma_y^2(\tau) = \lim_{m\to\infty} \frac{1}{2(m-1)} \sum_{j=2}^{m} [\bar{y}(j\tau,\tau) - \bar{y}((j-1)\tau,\tau)]^2 \qquad (2)$$

if the limit exists. For most models of clock noise, the limit does exist and can be computed by an ensemble average, so that

$$\sigma_y^2(\tau) = \frac{1}{2}E\,[\bar{y}(t,\tau) - \bar{y}(t-\tau,\tau)]^2$$

in which E is mathematical expectation and t is arbitrary. Thus, the two-sample variance gives the mean-square change in τ-average frequency during a time interval of length τ.

On the other hand, the time interval error (TIE), discussed by Kartaschoff (Refs. 1,2), directly measures the error of the clock after it has been synchronized and syntonized. Let the clock be calibrated at time zero, that is, we assume that $x_m(0)$ and Y_0 are available, where Y_0 is some estimate of initial frequency. We could take $Y_0 = y(0)$ if the latter were observable, or $Y_0 = \bar{y}(0,\tau)$ for some τ. The TIE at a later time t is defined by

$$x(t) = x_m(t) - x_m(0) - Y_0 t \qquad (3)$$

Since $y = dx_m/dt$, we have

$$x(t) = \int_0^t (y(s) - Y_0)ds \qquad (4)$$

The ensemble RMS average of $x(t)$ is called $\sigma_x(t)$ (Ref. 1); thus, by definition,

$$\sigma_x^2(t) = E\,x^2(t) \qquad (5)$$

One must keep in mind that $\sigma_x^2(t)$ depends on the choice of Y_0. Also, it is stipulated that Y_0 must not vary with t.

Kartaschoff (Ref. 1) investigated the behavior of TIE for the three random frequency-noise models whose names and properties are given below:

Name	$S_y(f)$	$\sigma_y^2(\tau)$
White FM	h_0	$\frac{h_0}{2\tau}$
Flicker FM	h_{-1}/f	$h_{-1} \ln 4$
Random Walk FM	h_{-2}/f^2	$h_{-2} \frac{2\pi^2}{3} \tau$

where $S_y(f)$ is the one-sided spectral density of the process y, and h_α is a constant. Having conjectured the approximation

$$\sigma_x^2(t) = C t^2 \sigma_y^2(t) \tag{6}$$

for some constant C, Kartaschoff verified Eq. (6) with $C = 1$ by a discrete-time computer simulation of the three FM processes.

On the other hand, a recently-developed structure theory for clock noise processes (Ref. 3) yields the following exact formulas, whose derivation is given in the Appendix:

White FM, $Y_0 = 0$.

$$\sigma_x^2(t) = \frac{h_0}{2} t \tag{7}$$

Flicker FM, $Y_0 = \bar{y}(0,\tau_1)$.

$$\sigma_x^2(t) = h_{-1} t^2 \left(1 + \frac{\tau_1}{t}\right) \left[\ln \frac{t}{\tau_1} + \left(1 + \frac{t}{\tau_1}\right) \ln \left(1 + \frac{\tau_1}{t}\right)\right] \tag{8}$$

$$= h_{-1} t^2 \ln \left(\frac{et}{\tau_1}\right) \left[1 + O\left(\frac{\tau_1}{t}\right)\right] \quad \text{as } t/\tau_1 \to \infty \tag{9}$$

Random Walk FM, $Y_0 = y(0)$.

$$\sigma_x^2(t) = h_{-2} \frac{2\pi^2}{3} t^3 \tag{10}$$

Thus, provided that the calibration frequency Y_0 is properly defined, Eq. (6) holds exactly for White and Random Walk FM with $C = 1$. For Flicker FM, however, the theoretical result Eq. (8) differs from Kartaschoff's simulation result

$$\sigma_x^2(t) = (h_{-1} \ln 4) t^2 \tag{11}$$

by a factor that grows logarithmically with time.

Our purpose here is to explain the discrepancy between Eqs. (8) and (11). Kartaschoff used a flicker-noise generation algorithm of Barnes and Jarvis (Ref. 4) for his simulations. A study of this generator, which is a certain linear filter acting on white noise, shows that the discrepancy is caused, not by any design defect of the filter, nor by any defect of theory, but by the procedure for initializing the generator. In our results, the initialization used by Barnes-Jarvis and Kartaschoff (setting all variables to zero) yields Eq. (11) (although with a different constant factor), while a more complex procedure given in Section IV (making the output process stationary) yields an asymptotic result like Eq. (9), with e replaced by a different constant.

In contrast, the initialization has practically no effect on the observed two-sample variance. No matter which of the two initializations is used, one observes

$$\sigma_y^2(\tau) = h_{-1} \ln 4$$

for τ greater than three times the sample time.

II. The Barnes-Jarvis Generator

The discrete-time n-stage Barnes-Jarvis flicker-noise generator is a cascade of n first-order filters with transfer functions

$$G_j(z) = \frac{z - (1-3\gamma_j)}{z - (1-\gamma_j)}, \quad j = 1,2,\ldots,n \tag{12}$$

where

$$\gamma_j = \frac{1}{6 \cdot 9^{j-1}}$$

For $n = 5$, the frequency response $|H(e^{i\omega})|^2$ of the overall transfer function

$$H(z) = G_1(z) \ldots G_n(z) \tag{13}$$

differs from the ideal two-sided spectrum $h_{-1}\pi/\omega$ by at most 0.25 dB over four decades of frequency, where $h_{-1} = 0.2757$.

The difference equations that implement the generator are

$$y_j(t+1) = (1-\gamma_j)y_j(t) + y_{j-1}(t+1) - (1-3\gamma_j)y_{j-1}(t), \quad j = 1,2,\ldots,n \tag{14}$$

where $y_0(t)$ is the input, consisting of "standard" (mean zero and variance one) white noise, and $y_n(t)$ is the output. This implementation differs from that of Barnes-Jarvis and Kartaschoff only in the overall gain and in the order of application of the filter modules.

The system (14) implies that y_n and y_0 are related by an nth-order difference equation whose coefficients can be obtained by expanding the numerator and denominator of $H(z)$ [Eq. (13)] in powers of z. Let the sequence $\psi(0)$, $\psi(1)$, $\psi(2), \ldots$ be the impulse response of the overall filter H; thus

$$H(z) = \sum_{t=0}^{\infty} \psi(t)z^{-t}, \qquad |z| \geqslant 1 \tag{15}$$

Assume that the standard white noise $y_0(t)$ is available for all integer t, positive and negative. The unique *stationary* solution $y(t) = y_n(t)$ of the nth order equation is

$$y(t) = \sum_{s=-\infty}^{t} \psi(t-s)y_0(s), \quad -\infty < t < \infty \tag{16}$$

in which $y(t)$ depends on $y_0(s)$ for $-\infty < s \leqslant t$. In the terminology of Box and Jenkins (Ref. 5), y is an ARMA (n,n) process. For a practical simulation, it can be assumed that $y_0(s)$ is available only for $s \geqslant 1$. Thus, it is natural to split $y(t)$ into two independent parts,

$$y(t) = y_+(t) + y_-(t) \tag{17}$$

where

$$y_+(t) = \sum_{s=1}^{t} \psi(t-s)y_0(s), \qquad t \geqslant 1$$
$$= 0, \qquad t \leqslant 0 \tag{18}$$

is called the *present* part of $y(t)$ because it depends only on the random shocks y_0 after time zero, while

$$y_-(t) = \sum_{s=-\infty}^{0} \psi(t-s)y_0(s), \qquad t \geqslant 1$$
$$= y(t), \qquad t \leqslant 0 \tag{19}$$

is called the *past* part of $y(t)$ because it depends only on the random shocks before time one. In another terminology, $y_-(t)$ is just the mean-square best linear prediction of $y(t)$ from the past of y_0, while $y_+(t)$ is the prediction error. The stationary process $y(t)$ will also be called the *complete* Barnes-Jarvis (BJ) process.

The time interval error $x(t)$ of the discrete-time frequency process $y(t)$, defined by

$$x(t) = \sum_{s=1}^{t} [y(s) - y(0)], \qquad t \geqslant 1$$
$$x(0) = 0 \tag{20}$$

also breaks up into present and past parts $x_+(t)$ and $x_-(t)$, obtained from Eq. (20) with y replaced by y_+ and y_-.

The BJ generator, Eq. (14), is initialized by assigning values to $y_1(0), \ldots, y_n(0)$. The usual way of doing this, called the *zero* initialization here, is to set all these values, plus $y_0(0)$, to zero (Ref. 4). By an induction on n, one can prove the following assertion: *If the zero initialization is used, then the output $y_n(t)$ of the BJ generator is exactly $y_+(t)$ for $t \geqslant 1$.* Moreover, since $y_+(0) = 0$, $x_+(t)$ is exactly the TIE simulated by Kartaschoff.

Two questions arise. First, are there any practical differences between $y(t)$ and $y_+(t)$? Second, inasmuch as the infinite past of $y_0(t)$ is unavailable, can the complete process $y(t)$ be simulated accurately? Since, for $t \geqslant 1$, both $y(t)$ and $y_+(t)$ (alias $y_n(t)$) satisfy Eq. (14) with identical inputs $y_0(t)$, the difference $y(t) - y_+(t)$ is merely a transient of the filter H; consequently,

$$y(t) - y_+(t) \to 0 \qquad \text{as } t \to \infty \tag{21}$$

Nevertheless, the stationary output $y(t)$ behaves like true nonstationary $1/f$ noise only over a time span of the order of magnitude $6 \cdot 9^{n-1}$, the longest time constant found in H. Over the useful time span of the generator, one cannot expect the transient to be small. As we shall soon see, its effect causes the variances of $x(t)$ and $x_+(t)$ to differ greatly.

The second question also has a yes answer. To simulate the complete BJ process $y(t)$, one has only to set up the initial vector $y_0(0), y_1(0), \ldots, y_n(0)$ with *random* values having the correct joint distribution. The difference equations [Eq. (14)] do the rest. The *stationary* initialization algorithm is given in Section IV.

III. Statistical Results

The mean-square time interval error $\sigma_x^2(t)$ and the two-sample variance $\sigma_y^2(\tau)$ were calculated theoretically for the complete five-stage Barnes-Jarvis process $y(t)$ and for its present part $y_+(t)$. Then, the same quantities were estimated by computer simulation, the zero initialization being used to generate $y_+(t)$ and the stationary initialization (Section IV) being used to generate $y(t)$. Gaussian pseudorandom numbers were used for the white input noise $y_0(t)$. The results are given in Figs. 1 and 2. Figure 1 plots $\sigma_x^2(t)/t^2$ (linear scale) against t/τ_1 (logarithmic scale), where τ_1 is the sample time of the simulation. The mean-square TIE of the complete process dominates that of the present part. To put it another way, *as time goes on, more and more of the TIE (on the average) comes from the remote past.* The results can be summarized by empirical formulas. For the complete process,

$$\sigma_x^2(t) = h_{-1} t^2 \ln (5.5\, t/\tau_1) \tag{22}$$

and for the present part,

$$\sigma_x^2(t) = 2 h_{-1}\, t^2 \tag{23}$$

valid for $16 \leqslant t/\tau_1 \leqslant 16000$. Equations (22) and (9) have different constants inside the logarithm. This discrepancy is caused by high-frequency spectral deviation of the Barnes-Jarvis process from pure sampled flicker FM. Equation (23) agrees with Kartaschoff's result Eq. (11), except for the constant factor. This discrepancy is perhaps caused by statistical variations, since Kartaschoff generated only 100 sample functions.

The mean-square TIE $\sigma_x^2(t)$ of the present part can also be interpreted as the minimal mean-square prediction error of the *complete* $x(t)$ from the history of the process before time one. Our result, Eq. (23), is in approximate agreement with a result of Percival for this prediction error variance (Fig. 3.1 in Ref. 6) although our factor $2h_{-1}$ appears to differ slightly from his.

Figure 2 plots $\sigma_y^2(\tau)$ (*not* $\sigma_y(\tau)$) on a linear scale against τ/τ_1 on a logarithmic scale. Here, the situation is quite different. It is apparent that *the two-sample variances of the complete BJ process and of its present part are practically indistinguishable*. In fact, the *past* part of the BJ process, which accounts for most of the TIE for large t, contributes only 9 percent of the complete $\sigma_y^2(4096\tau_1)$ and 0.00015 percent of the complete $\sigma_y^2(\tau_1)$. This explains why earlier BJ users, who relied on a constant $\sigma_y^2(\tau)$ as an indicator of the success of the simulations, were not aware that a large part y_- of the process y was missing.

The error bars in Fig. 2 concern a side issue addressed also by Kartaschoff (Ref. 1), namely, the calculation of the variance of the classical $\sigma_y^2(\tau)$ estimator $S_y^2(\tau,m)$, which is just the right side of Eq. (2) without the limit. The solid bars are the ± one-standard-deviation error bars of $S_y^2(\tau,m)$ as computed for pure Gaussian Flicker FM by Yoshimura (Ref. 7), and later by the author (Ref. 3); the dashed bars come from the sample variances observed during the 2048 simulation runs. The agreement is satisfactory.

IV. Simulation of the Complete BJ Process

It is desired to simulate the stationary solution $y(t) = y_n(t)$ of the Barnes-Jarvis system, Eq. (14). To this end, consider the stationary n-vector process $Z(t) = (Z_1(t), \ldots, Z_n(t))$, $-\infty < t < \infty$, where $Z_j(t) = y_j(t) - y_{j-1}(t)$. (The Z_j are used instead of the y_j because the Z_j are less highly correlated.) The jth component Z_j is obtained by acting on y_0 with the filter whose transfer function is

$$K_j(z) = G_1(z) \ldots G_{j-1}(z)[G_j(z) - 1] \tag{24}$$

[Recall Eqs. (12), (13).] Each $Z_j(t)$ is orthogonal to $y_0(t)$, and the covariance matrix R_Z is given by

$$R_Z(i,j) = E[Z_i(t)Z_j(t)] = \oint_{|z|=1} K_i(z)K_j(1/z)\,\frac{dz}{2\pi i z} \tag{25}$$

which can easily be evaluated by residues.

Suppose now that we create new random variables $y_0(0)$, $Z_1(0), \ldots, Z_n(0)$ with the above covariances, set $y_j(0) = y_{j-1}(0) + Z_j(0)$, $j=1, \ldots, n$, and generate $y_n(t)$, $t \geqslant 1$, by Eq. (14) from the white noise inputs $y_0(t)$, $t \geqslant 1$. One can then prove that these new $y_n(t)$ have the same covariance structure as the original stationary process.

To set up the $Z_j(0)$, one can use the Choleski factorization of R_Z, namely, $R_Z = LL^T$, where L is an $n \times n$ lower-triangular matrix. The coefficients L_{ij} are given in Table 1 for $n = 6$; for smaller n, one simply truncates the matrix. These coefficients give the $Z_j(0)$ as linear combinations of a standard white noise vector $u_1, \ldots, u_n$. If the simulated process is to be Gaussian, then it is important to make the u_j Gaussian because there are so few of them and their effect is large. Although one could then save time by using uniformly distributed noise inputs $y_0(t)$ (with mean zero, variance one), our simulations used Gaussian noise inputs throughout.

Here, then, is the simulation algorithm for the complete Barnes-Jarvis process. Let GRAN be the routine that generates independent Gaussian random numbers with mean zero, variance one.

1. Generate $y_0(0), u_1, \ldots, u_n$ with GRAN
2. For $i = 1$ to n

   Let $Z_i = \sum_{j=1}^{i} L_{ij} u_j$ (L_{ij} from Table 1)

   Let $y_i(0) = y_{i-1}(0) + Z_i$

   Next i
3. For $t = 0, 1, 2, \ldots$

   Generate $y_0(t+1)$ with GRAN

   For $j = 1$ to n

   Compute $y_j(t+1)$ by Eq. (14)

   Next j

   Next t

Remark: To generate the present part of the BJ process, set $y_0(0), y_1(0), \ldots, y_n(0) = 0$, then go to Step 3.

V. Concluding Remarks

As we have seen, the Barnes-Jarvis generator, if given the stationary initialization, produces an output whose mean-square TIE $\sigma_x^2(t)$ has a $t^2 \log t$ behavior, just as theory predicts. The previously-observed t^2 behavior, which appears when the BJ generator is given the zero initialization, can be interpreted as the minimal mean-square error of a linear clock-time predictor based on knowledge of the clock's behavior over the entire past, $-\infty$ to 0. Naturally, this error is smaller than $\sigma_x^2(t)$, the mean-square error of a predictor $x_m(0) + Y_0 t$ for the time deviation $x_m(t)$. We conclude that if the diagnosis of the noise of a clock includes a Flicker FM component, then its $\sigma_x^2(t)$ must include a $t^2 \log t$ component. If $\bar{y}(0,\tau_1)$ is used for calibrating the initial frequency, then this component is given by Eq. (8).

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Table 1. Coefficients L_{ij} for the stationary initialization of the Barnes-Jarvis process

i	j = 1	2	3	4	5	6
1	0.603023					
2	0.214635	0.512223				
3	0.301626E-1	0.241088	0.494406			
4	0.345089E-2	0.358003E-1	0.244953	0.491688		
5	0.384698E-3	0.412554E-2	0.366905E-1	0.245520	0.491287	
6	0.427600E-4	0.460283E-3	0.423277E-2	0.368209E-1	0.245599	0.491231

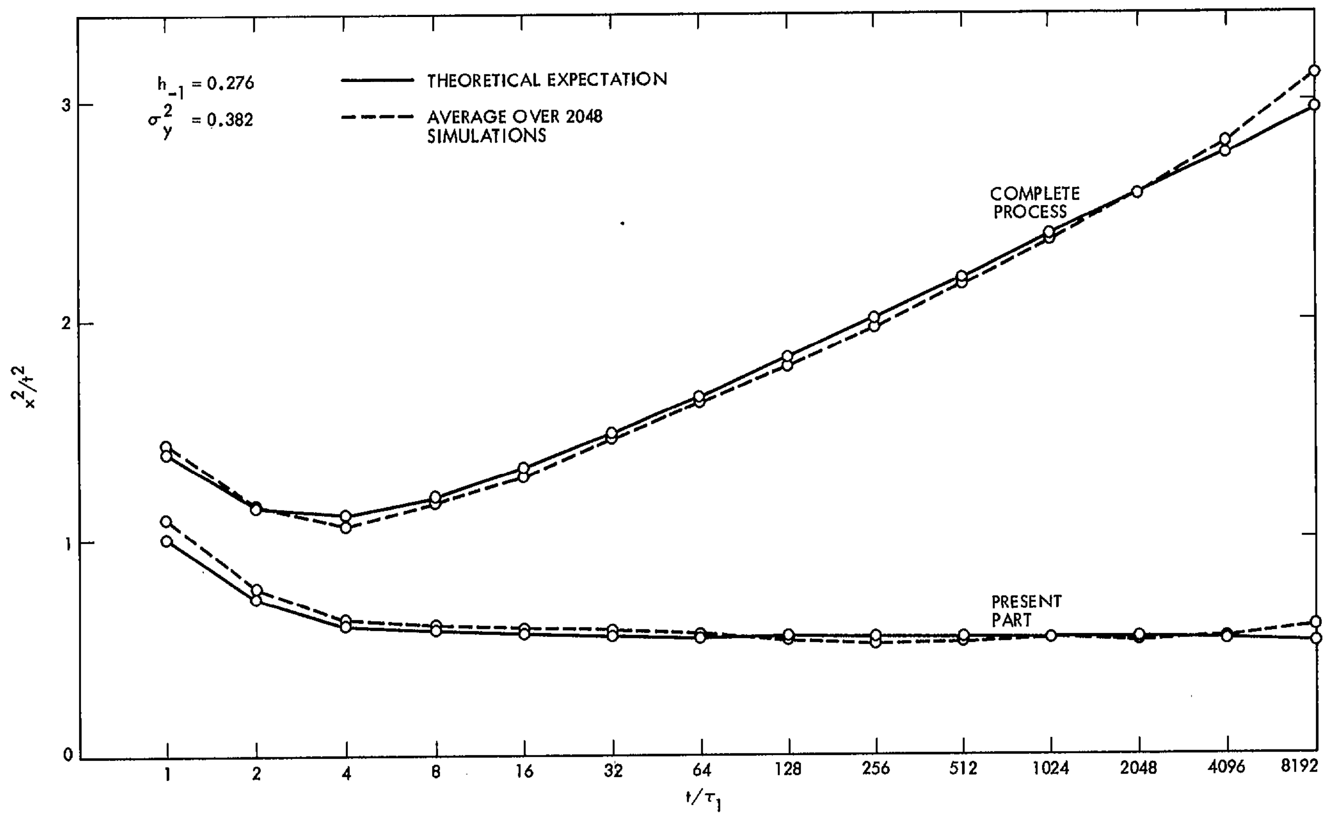

Fig. 1. Squared time interval error of five-stage Barnes-Jarvis Flicker FM model

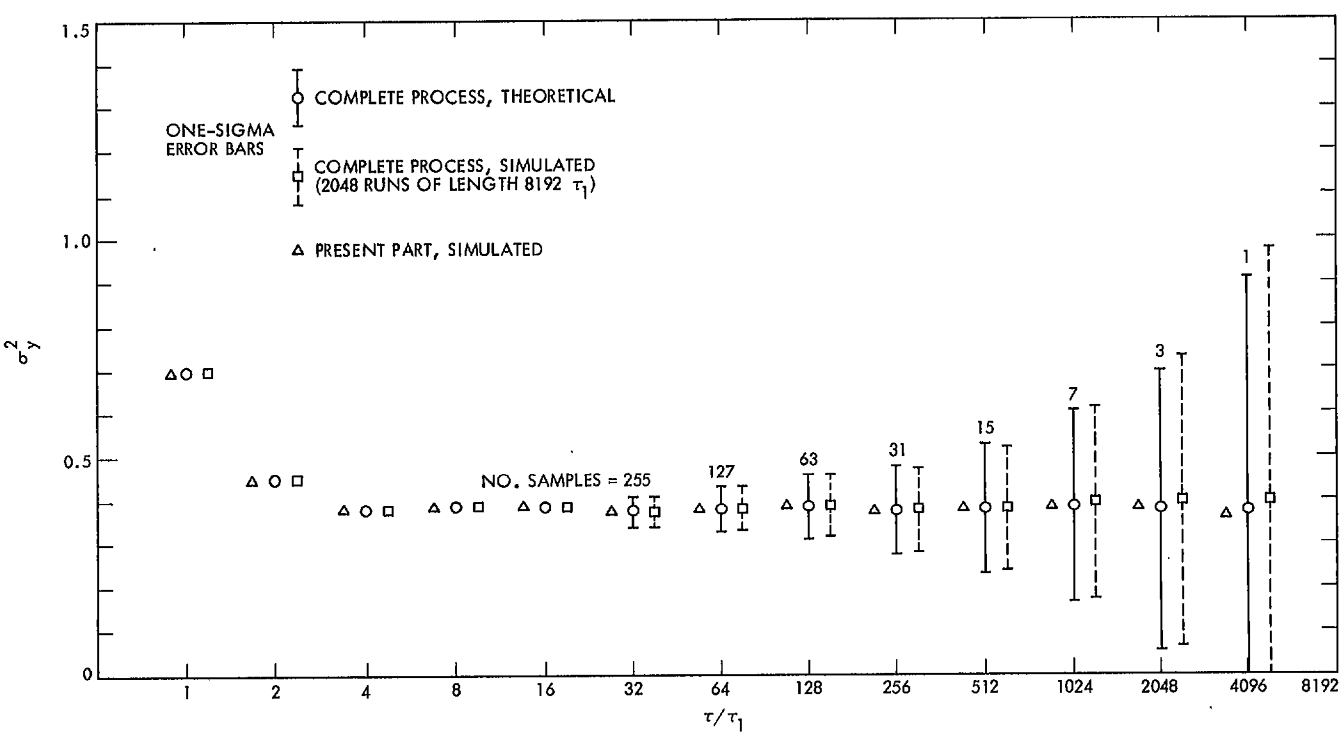

Fig. 2. Two-sample variance of five-stage Barnes-Jarvis Flicker FM model

Appendix

Derivation of Formulas for Mean-Square Tie

Given fixed numbers t, $\tau > 0$, define the second-order difference operator L by its action on a function $f(s)$:

$$Lf(s) = f(s+t) - \left(1 + \frac{t}{\tau}\right) f(s) + \frac{t}{\tau} f(s-\tau) \qquad \text{(A-1)}$$

where s is used as the time variable because t is fixed. The transfer function of L is

$$L(i\omega) = e^{i\omega t} - \left(1 + \frac{t}{\tau}\right) + \frac{t}{\tau} e^{-i\omega\tau} \qquad \text{(A-2)}$$

Then, with $Y_0 = \bar{y}(0,\tau)$, Eq. (3) can be written as

$$x(t) = Lx_m(0) \qquad \text{(A-3)}$$

Now assume that the second differences of the time deviation x_m are stationary. (The examples treated below satisfy this condition.) According to the structure theorem of Ref. 3, there is a function $D(t)$ (appearing as $2\,Re\,C(t)$ in Ref. 3) such that

$$\sigma_x^2(t) = Ex^2(t) = E\,Lx_m(0)\,Lx_m(0) = \Lambda D(0) \qquad \text{(A-4)}$$

where Λ is the fourth-order operator LL^*, and L^* is the operator with transfer function $L^*(i\omega) = L(i\omega)^*$. Hence the transfer function of Λ is

$$\begin{aligned}\Lambda(i\omega) = |L(i\omega)|^2 = {} & 2\left(1 + \frac{t}{\tau} + \frac{t^2}{\tau^2}\right) - \left(1 + \frac{t}{\tau}\right)\left(e^{i\omega t} + e^{-i\omega t}\right) \\ & - \frac{t}{\tau}\left(1 + \frac{t}{\tau}\right)\left(e^{i\omega\tau} + e^{-i\omega\tau}\right) \\ & + \frac{t}{\tau}\left(e^{i\omega(t+\tau)} + e^{-i\omega(t+\tau)}\right)\end{aligned} \qquad \text{(A-5)}$$

Thus, from Eq. (A-4),

$$\begin{aligned}\sigma_x^2(t) = {} & 2\left(1 + \frac{t}{\tau} + \frac{t^2}{\tau^2}\right) D(0) - \left(1 + \frac{t}{\tau}\right)\left(D(t) + D(-t)\right) \\ & - \frac{t}{\tau}\left(1 + \frac{t}{\tau}\right)\left(D(\tau) + D(-\tau)\right) + \frac{t}{\tau}\left(D(t+\tau) + D(-t-\tau)\right)\end{aligned} \qquad \text{(A-6)}$$

Each noise process has its own D-function to be used in Eq. (A-6).

I. Flicker FM

From Eq. (27) of Ref. 3,

$$D(t) = \frac{h_{-1}}{2} t^2 \ln|t| \qquad \text{(A-7)}$$

which, when substituted into Eq. (A-6), gives Eq. (8) with $\tau_1 = \tau$.

II. Random Walk FM

Since Λ is a fourth-order operator, one may add a third-degree polynomial to D. Thus, Eq. (28) of Ref. 3 is equivalent to

$$D(t) = \frac{h_{-2}\pi^2}{6} |t|^3 \qquad \text{(A-8)}$$

and Eq. (A-6) gives

$$\sigma_x^2(t) = h_{-2} \frac{2\pi^2}{3} t^2(t+\tau) \qquad \text{(A-9)}$$

As $\tau \to 0$, $\bar{y}(0,\tau) \to y(0)$ because y is a mean-continuous process (Brownian motion), and Eq. (A-9) reduces to Eq. (10). This does not happen for Flicker FM because it has too much power at high frequencies. Consequently, $\bar{y}(0,\tau)$ is well defined, but $y(0)$ is not.

III. White FM

Since $Y_0 = 0$, and x_m has stationary *first* differences, we redefine L as the first-order operator:

$$Lf(s) = f(s+t) - f(s) \qquad \text{(A-10)}$$

Then everything works as before, except that Λ is now a second-order operator. In place of Eq. (A-6), we have

$$\sigma_x^2(t) = 2D(0) - D(t) - D(-t) \qquad \text{(A-11)}$$

This time, a first-degree polynomial may be added to $D(t)$. Thus, Eq. (26) of Ref. 3 is equivalent to

$$D(t) = -\frac{h_0}{4}\,|t| \qquad \text{(A-12)}$$

which combines with Eq. (A-11) to give Eq. (7), itself a well-known fact.