

Level Sets of Real Functions on the Unit Square

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Suppose f is a real-valued continuous function on the unit square. The problem of finding a level curve of f which joins opposite sides of the square is investigated. It is shown that while f need not have such a level curve, it at least always has a level connected set with the desired property. This problem is connected with the problem of minimizing the bandwidth of a certain matrix.

I. Introduction

Suppose we are given a real-valued function f on the unit square I^2 , $f: I^2 \rightarrow I$. Viewed as a topographic map on the square, we seek to find a level path from one side of the square to the opposite side. We don't specify which pair of opposite sides are to be used, but only require that it must be somehow crossed at a single level. The original problem asks if this is always possible for, say, continuous functions f , or if not, whether suitable restriction of f (e.g., C^∞ , analytic, PL = piecewise linear, etc.) will do the trick. This problem was suggested by Professor J. Franklin in connection with the problem of minimizing the "bandwidth" of the incidence matrix of the graph of an $n \times n$ square grid (Ref. 1).

We first give a C^∞ function having no solution, and then a proof of the appropriate restatement of the theorem.

II. A C^∞ Counterexample

(1) We let the square I^2 be represented by the set of (x, y) with $|x \pm y| \leq 2$ (Fig. 1). The vertical strip between the dotted lines is given by $|x| \leq 1$.

We define f as follows:

$$f = \begin{cases} 0 & \text{if } |x| \geq 1 \text{ (the shaded triangles)} \\ e^{-w}(y - \sin w), & \text{where } w = \frac{1}{1-x^2} \end{cases} \text{ if } |x| \leq 1$$

f is C^∞ in the whole x, y plane: e^{-w} approaches 0 very fast as $|x| \rightarrow 1$, as well as all the derivatives, and $(y - \sin w)$ is bounded (for any y); this is the usual "smooth joining" technique.

(2) f is zero on the two shaded triangles (closed) as well as on the infinitely oscillating curve $y = \sin w$ ($|x| \leq 1$). Above the sine curve f is > 0 , below the sine curve f is < 0 ; thus there is no hope of traversing the square at any other level except $f = 0$. But to traverse at level zero, we clearly must start at one of the shaded triangles and cross to the other, *via* the sine curve. This cannot be done since the sine curve is not *pathwise* connected to either triangle.

(3) We note that the 0-level set, although not pathwise connected, is however connected; we have just the standard example of a connected non-path-connected space.

Thus we might expect the proper generalization of the theorem to be: if $f: I^2 \rightarrow I$ is continuous there exists a connected level set containing points of two opposite sides.

III. Proof of the General Theorem

We shall show that if $f: I^2 \rightarrow I$ is continuous then there exists a level set connecting a pair of opposite sides.

(1) We assume that m is the minimum value such that $A_m = f^{-1}[0, m]$ connects two opposite sides, assumed top and bottom (E_1 & E_0 resp., with $E = E_0 \cup E_1$). There must be such an m ; in fact $A_1 = I^2$ certainly connects opposite sides, and it is easy to see that the infimum of numbers having this property also has it.

(2) Let μ be any arc, disjoint from E , connecting the left and right vertical sides. Then μ must intersect A_m , for otherwise μ would separate A_m into two disjoint open sets containing E_0 & E_1 , respectively (i.e., the part below μ and the part above μ) contrary to the definition of A_m connecting E . Now since μ is an interval, its image $f(\mu)$ is an interval, and $f(\mu) \cap [0, m]$ is not empty. But f cannot be always less than m on μ , for then μ would connect two

sides at a lower value than m , contrary to choice of m . Thus we must have some points of μ with *exactly* the value m , i.e., if $L = f^{-1}(m)$ then $\mu \cap L$ is not empty.

(3) Now suppose that L does not connect E . Then we may choose two disjoint open sets V_0, V_1 , containing E_0 and E_1 , respectively, and such that $L \subset V_0 \cup V_1$. If we divide I^2 into a small enough mesh of squares, we may replace V_i by the set W_i of all those squares which intersect $L_i = V_i \cap L$, and in this way get polyhedral neighborhoods W_i separating $L \cup E$.

(4) Next look at the component K of W_0 which contains E_0 . Being PL and connected, its boundary is a PL circle (part of which is E_0 itself). As we follow its boundary around, starting at, e.g., the left side, it may leave the left side and return again, but eventually it must leave the left side and travel directly to the right. This is depicted in Fig. 2.

Thus the traversal gives us an arc μ which is outside the interior of W_0 and hence does not intersect L , a contradiction which proves that L does in fact connect E , and proves the theorem.

Reference

1. Franklin, J., *Matrix Theory*, Prentice-Hall Inc., Englewood Cliffs, N. J., 1968.

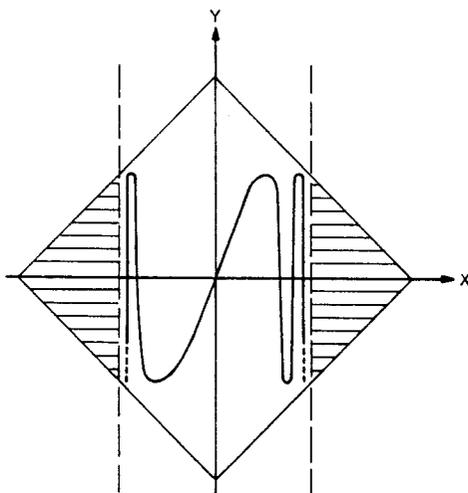


Fig. 1. The function f

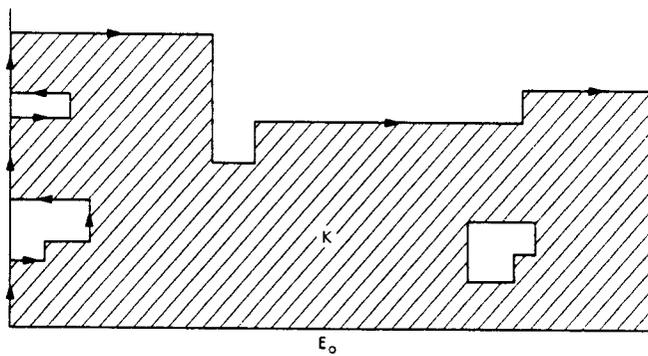


Fig. 2. E_0 component