Computation of Vibration Mode Elastic-Rigid and Effective Weight Coefficients From Finite-Element Computer Program Output

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Post-processing algorithms are given to compute the vibratory elastic-rigid coupling matrices and the modal contributions to the rigid-body mass matrices and to the effective modal inertias and masses. Recomputation of the elastic-rigid coupling matrices for a change in origin is also described. A computational example is included. The algorithms can all be executed by using standard finite-element program eigenvalue analysis output with no changes to existing code or source programs.

I. Introduction

The concepts of effective vibratory modal mass [1] and the application to the combined structural-mechanical-control system model [2] are important tools in the analysis and simulation [3] of the dynamic response of antennas and other complex structures with stringent performance or safety [4] requirements. Effective modal masses or inertias and their components associated with particular coordinate axes also provide the analyst with insight into the characteristics of particular vibratory modes. Simplifications of the transient dynamic analysis procedures can be facilitated by the convenience of identifying modes with relatively insignificant masses or inertias associated with the coordinate axis of interest. Such modes are candidates for modal truncation and elimination from the analytical model and replacement by contributions to residual masses or inertias.

The following discussion will show how the key matrix to this development, the elastic-rigid coupling matrix, can be extracted from finite-element program eigenvalue analysis output, such as from the JPL-IDEAS [5], NASTRAN [6], or other typical finite-element analysis programs. Generation of the modal effective inertia matrix from the elastic-rigid coupling matrix follows readily. Finally, it will be shown how the effective inertia matrix with respect to one reference origin can be modified to relate a different origin by after-the-fact computations. Explicit example calculations are included.

II. Rigid-Body Mass Matrix and Modal Contributions

A finite-element structure model with N nodes and three translational and three rotational degrees of freedom at each node has a rigid-body transformation matrix, $\phi\mathbf{R}$, with $6N$ rows and 6 columns. The columns refer to the three rigid-body translational displacements in a Cartesian X-Y-Z coordinate system and the three corresponding rigid-body rotations about these axes with respect to a selected reference point origin. That is, each set of six rows of $\phi\mathbf{R}$ has the form:

$$\begin{matrix}\text{typical nodal}\\ \text{sextet of}\\ \text{rows of}\phi\mathbf{R}\end{matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & dZ & -dY \\ 0 & 1 & 0 & -dZ & 0 & dX \\ 0 & 0 & 1 & dY & -dX & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (1)$$

In the above, dX, dY, and dZ are the differences between the coordinates of the particular node and the reference point. Also, if, as in the JPL-IDEAS program, the finite-element model does not contain rotational degrees of freedom, only the first three rows are applicable and $\phi\mathbf{R}$ is a $3N$-by-6 matrix.

The elastic-rigid coupling matrix computation involves the finite-element symmetrical physical mass matrix, $\mathbf{M}$, and the eigenvector matrix, $\phi\mathbf{N}$. With three degrees of freedom per node, the mass matrix size is $3N$ by $3N$, and in the special case when the complete eigenvector matrix is available, this matrix is also the same size as the mass matrix. The elastic-rigid coupling matrix, **MER**, is computed as

$$\mathbf{MER} = \phi\mathbf{N}^t \ \mathbf{M} \ \phi\mathbf{R} \qquad (2)$$

Note that **MER** has either $3N$ or $6N$ rows and 6 columns.

A natural-vibration-mode generalized mass matrix, **MNN**, is diagonal (from orthogonality of the normal modes) and is computed from the eigenvectors and mass matrix as

$$\mathbf{MNN} = \phi\mathbf{N}^t \ \mathbf{M} \ \phi\mathbf{N} \qquad (3)$$

A rigid-body mass matrix, **MRR**, is ordinarily computed from the coordinate geometry and the mass matrix. However, it can also be computed from the previously mentioned expressions as

$$\mathbf{MRR} = \mathbf{MER}^t \ \mathbf{MNN}^{-1}\mathbf{MER} \qquad (4)$$

To show this, the eigenvector matrix must be complete. Therefore, the inverse exists and substituting Eqs. (2) and (3) in Eq. (4) leads to

$$\mathbf{MRR} = \phi\mathbf{R}^t \ \mathbf{M} \ \phi\mathbf{R} \qquad (5)$$

which is the rigid-body mass matrix by definition.

The rigid-body mass matrix is square, with six rows and columns. It is informative to partition **MRR** into four 3-by-3 matrices. If this is done, the upper left partition is diagonal and contains the sums of the nodal masses (usually all identical) in the coordinate axes' directions. The upper right partition (and its transpose in the lower left) contains the static (first) moments of the mass. Finally, the lower right partition is the inertia tensor, which has the mass moments of inertia about the respective coordinate axes on the diagonal and the products of inertia as off-diagonals. All dimensions implied in the computed terms of this matrix are with respect to the reference point.

Usually not all columns of the eigenvector matrix are available, or else, although available, some (or many) columns are truncated to condense the solution and calculations. The previous procedures will be applied to a particular retained column of eigenvectors. In the following, the subscript "j" refers to terms associated with a particular retained jth natural vibration mode. Thus, the jth mode elastic-rigid coupling matrix is the 1-by-6 row matrix $\mathbf{MER}_j$

$$\mathbf{MER}_j = \phi\mathbf{N}_j{}^t \ \mathbf{M} \ \phi\mathbf{R} \qquad (6)$$

and the contribution to the rigid-body mass matrix is the 6-by-6 matrix $\mathbf{MRR}_j$

$$\mathbf{MRR}_j = \frac{\mathbf{MER}_j{}^t\mathbf{MER}_j}{\mathbf{MNN}_j} \qquad (7)$$

In $\mathbf{MRR}_j$ above, the diagonals are the effective modal masses (1 to 3) and mass moments of inertia (4 to 6), which, when multiplied by the acceleration of gravity, are equivalent to the "Reduction to the Diagonals of the Residual Weight Matrix" in the JPL-IDEAS output. As in the discussion of the rigid-body mass matrix, the lower right-hand 3-by-3 partition is the contribution of this natural mode to the inertia tensor. If all these $\mathbf{MRR}_j$ matrices were computed and summed, the sum would (to within computer accuracy) be equal to the rigid-body mass matrix of Eq. (5).

III. Recovery of Elastic-Rigid Modal Coupling Matrices

In the JPL-IDEAS program formulation, the term "weight" is used in place of "mass" and "inertia," and implies weight moment of inertia, rather than mass moment of inertia. The conversion to the conventional mass formulation is made by dividing the respective weight terms by the acceleration of gravity. JPL-IDEAS directly provides a table of the elastic-rigid coupling weight matrix for each natural frequency mode under the heading "Un-Normalized Rigid Elastic Coupling Matrix." The six columns of the row matrix associated with each mode are tabulated under the headings "SUMX SUMY SUMZ SUMTX SUMTY SUMTZ." The first three headings refer to translational terms in the Cartesian axes' directions, and the last three refer to rotational terms about these axes.

In general, the elastic-rigid modal coupling matrices are not part of the usual output of finite-element programs. However, they can be recovered by post-processing the modal reactions. The computations use the jth mode reaction vector, $\mathbf{R}_j$, and the jth mode circular frequency, ω_j. Also, it is necessary to construct a rigid-body transformation matrix $\phi\mathbf{RR}$ for the reaction points. This matrix has the same format as $\phi\mathbf{R}$ in Eq. (1), except that the differences in coordinates are between the reaction points and the reference point. Consequently, from statics, the jth mode reaction forces produce the following set of forces and moments at the reference point:

$$\mathbf{FMR}_j = \phi\mathbf{RR}^t \ \mathbf{R}_j \tag{8}$$

On the other hand, forces and moments at the reference point for a natural mode of vibration can also be computed directly from the masses at the nodal points. That is, if $\mathbf{F}_j$ are the forces of acceleration in the jth vibration mode, then

$$\mathbf{F}_j = -\omega_j{}^2\mathbf{M}\phi N_j \tag{9}$$

At the reference point, these forces produce a force and moment vector FMN_j, which from statics is:

$$FMN_j = \phi\mathbf{R}^t \ \mathbf{F}_j \tag{10}$$

Then, setting Eqs. (8) and (10) equal, using Eq. (9) and transposing, then using Eq. (6) and solving for $\mathbf{MER}_j$ shows

$$\mathbf{MER}_j = \frac{\mathbf{R}_j{}^t\phi\mathbf{RR}}{\omega_j{}^2} \tag{11}$$

which is the algorithm that can be used to compute the elastic-rigid coupling matrix.

IV. Changing the Reference Origin

If, after completing the previously described computations, there is the need to use an alternative reference origin, it is not necessary to repeat all these computations. Naturally, it is possible to repeat the procedure to recover the $\mathbf{MER}_j$ from the reactions by changing $\phi\mathbf{RR}$ accordingly and repeating the computations of Eq. (11). However, if there are many components in the reaction matrix, the following alternative could prove to be simpler.

To illustrate, let dox, doy, and doz be the change in coordinates of the reference point, and $\mathbf{D}\phi\mathbf{RR}$ be the change in $\phi\mathbf{RR}$ corresponding to dox, doy, and doz. Then, from Eq. (11), the change in the elastic-rigid coupling matrix is

$$\mathbf{DMER}_j = \frac{-\mathbf{R}_j{}^t \ \mathbf{D}\phi\mathbf{RR}}{\omega_j{}^2} \tag{12}$$

It is evident from Eq. (1) that $\mathbf{D}\phi\mathbf{RR}$ will be null except for the last three columns of the first three rows, which will have the form:

$$\begin{array}{l}\text{typical nodal triad}\\ \text{for first three rows}\\ \text{and last three}\\ \text{columns of } \phi\mathbf{R}\end{array} = \begin{array}{c} \begin{array}{ccc}(4) & (5) & (6)\end{array} \\ \left\{\begin{array}{ccc} 0 & doz & -doy \\ doz & 0 & dox \\ doy & -dox & 0 \end{array}\right\}\end{array} \tag{13}$$

Therefore, it can be seen that only the fourth, fifth, and sixth columns of $\mathbf{MER}_j$ will be affected. Concentrating on the change in the fourth column of $\mathbf{DMER}_j$, Eq. (13) can be rewritten as

$$\begin{array}{l}\text{nodal change triad to}\\ \text{compute column 4}\\ \text{of } \mathbf{DMER}\end{array} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \left\{\begin{array}{c} dox \\ doy \\ -doz \end{array}\right\} \tag{14}$$

From Eqs. (12), (14), and the form of Eq. (1), and by identifying the components of $\mathbf{MER}$ as the row vector

$$\mathbf{MER}_j = \lfloor C1\ C2\ C3\ C4\ C5\ C6 \rfloor_j \qquad (15)$$

it can be seen that the fourth column of $\mathbf{DMER}_j$ can be represented as

$$\mathbf{DMER}_j(\text{column } 4) = \lfloor 0\ \ -C3\ C2 \rfloor \begin{Bmatrix} dox \\ doy \\ doz \end{Bmatrix} \qquad (16)$$

By repeating a similar procedure for the fifth and sixth columns and transposing, the result is

$$\mathbf{DMER}_j(\text{columns } 4,\ 5,\ 6) = \lfloor dox\ doy\ doz \rfloor \mathbf{C}_j \qquad (17)$$

where

$$\mathbf{C}_j = \begin{bmatrix} 0 & C3 & -C2 \\ -C3 & 0 & C1 \\ C2 & -C1 & 0 \end{bmatrix}_j \qquad (18)$$

V. Example Problem Computations

This example will illustrate computation of the elastic-rigid coupling matrices, the modal contributions to the rigid-body mass matrix and effective masses and inertias, and the computation of the changes in the elastic-rigid coupling matrix for a change in the reference origin.

The example structure is completely restrained in the X, Y, and Z coordinate directions at three foundation nodes. The analytical model represents only translational degrees of freedom, so that $\phi\mathbf{R}$ and $\phi\mathbf{RR}$ have three rows for each of their associated nodes. The computations will cover the first three natural modes.

Table 1 contains the needed information supplied by the finite-element program. This consists of the coordinates of the foundation nodes, the reference origin, and partial eigenvalue analysis output. The latter consists of the frequencies and generalized masses and the reactions for the first three natural modes. Note that for this problem, no information other than the eigenvalue analysis frequencies, generalized masses, and reactions is needed about the remainder of the finite-element model or its response.

Table 2 shows the solutions for the modal contributions to the rigid-body mass matrices. The diagonal elements of these 6-by-6 matrices are the effective masses and inertias. These provide a convenient, although partially qualitative, means to characterize the nature of the vibration mode. For example, in the first natural mode there will be almost no mass moving in the X and Z directions (masses = 4.5e−04 and 1.49e−02), nor moment of inertia with respect to rotations about the Y axis (inertia = 7.9e+01). The strong motion will be in the Y direction (mass = 4.0e+0) with rotations about the X and Z axes (inertias = 3.91e+04 and 1.37e+04).

Table 3 illustrates the computations for the changes in elastic-rigid coupling matrices when the reference origin is changed. Here it can be seen that a change in only the Z-coordinate of the origin produces changes only in the rotational terms associated with the X and Z axes.

VI. Summary

Recovery of the elastic-rigid coupling matrices and computations for the modal contributions to the rigid-body mass matrix and effective modal masses and inertias have been described. A method to recompute the elastic rigid coupling matrix for a change in origin is also given. A computational example is included to demonstrate the procedures. All the described computations are performed by post-processing of standard finite-element program eigenvalue analysis output and no modifications or code changes in existing programs are required.

References

[1] B. K. Wada, R. Bamford, and J. A. Garba, "Equivalent Spring-Mass System: A Physical Interpretation," *The Shock and Vibration Bulletin*, No. 42, Part 5, pp. 215–225, January 1972.

[2] R. Levy, "Condensed Antenna Structural Models for Dynamics Analysis," *TDA Progress Report 42-80*, vol. October–December 1984, Jet Propulsion Laboratory, Pasadena, California, pp. 40–61, February 15, 1985.

[3] R. E. Hill, "A New State Space Model for the NASA/JPL 70-Meter Antenna Servo Controls," *TDA Progress Report 42-91*, vol. July–September 1987, Jet Propulsion Laboratory, Pasadena, California, pp. 247–264, November 15, 1987.

[4] R. W. Clough, "Earthquake Response of Structures," in *Earthquake Engineering*, edited by R. L. Wiegel, New Jersey: Prentice-Hall, pp. 307–334, 1970.

[5] R. Levy and D. Strain, *JPL-IDEAS, Iterative Design of Antenna Structures*, COSMIC, NPO 17783, University of Georgia, Athens, Georgia, October 1988.

[6] *The NASTRAN Theoretical Manual*, COSMIC, NASA SP 221(06) University of Georgia, Athens, Georgia, January 1981.

Table 1. Example input data

Coordinates of the restrained nodes

X	Y	Z
50	0	30
−50	0	0
0	100	−20

Coordinates of the reference point

X	Y	Z
0	0	50

Eigenvalue analysis results

Mode		1	2	3
w		1.1920e+02	1.6000e+02	2.8570e+02
MNN		3.9327e+00	5.8179e+00	2.8151e+00
Reaction node 1	X	3.5228e+04	−8.8273e+04	−1.9411e+03
	Y	5.9308e+02	3.4942e+04	−8.4705e+04
	Z	−2.9574e+04	1.5995e+05	−1.9594e+05
Reaction node 2	X	−6.5662e+03	−7.4353e+04	−2.5214e+03
	Y	−6.9630e+03	−3.3808e+04	2.8558e+03
	Z	−6.8021e+04	−2.1592e+05	4.1937e+04
Reaction node 3	X	−2.9258e+04	2.5865e+04	7.9293e+04
	Y	−5.0125e+04	−1.8745e+04	−1.2541e+05
	Z	9.4154e+04	3.0557e+04	2.1629e+05

Table 2. Solutions for modal contributions

Relative coordinates of the restrained nodes

X	Y	Z
50	0	−20
−50	0	−50
0	100	−70

ϕ**RR** = [Eq. (1)]

Node 1	1	0	0	0	−20	0
	0	1	0	20	0	50
	0	0	1	0	−50	0
Node 2	1	0	0	0	−50	0
	0	1	0	50	0	−50
	0	0	1	0	50	0
Node 3	1	0	0	0	−70	−100
	0	1	0	70	0	0
	0	0	1	100	0	0

Columns of **MER**, Eq. (11)

Mode	C1	C2	C3	C4	C5	C6
1	4.1960e−02	3.9761e+00	2.4218e−01	−3.9204e+02	1.7633e+01	−2.3251e+02
2	5.3422e+00	6.8793e−01	9.9270e−01	−2.9375e+01	5.9066e+02	−3.3242e+01
3	−9.1677e−01	2.5392e+00	−7.6309e−01	−1.3843e+02	−7.9734e+01	1.5078e+02

Modal contributrions to rigid body mass matrix, Eq. (7)

$\mathbf{MRR}_1$ =

4.4771e−04	4.2424e−02	2.5840e−03	−4.1830e+00	1.8814e−01	−2.4808e+00
4.2424e−02	4.0200e+00	2.4485e−01	−3.9637e+02	1.7828e+01	−2.3507e+02
2.5840e−03	2.4485e−01	1.4913e−02	−2.4142e+01	1.0859e+00	−1.4318e+01
−4.1830e+00	−3.9637e+02	−2.4142e+01	3.9082e+04	−1.7578e+03	2.3178e+04
1.8814e−01	1.7828e+01	1.0859e+00	−1.7578e+03	7.9061e+01	−1.0425e+03
−2.4808e+00	−2.3507e+02	−1.4318e+01	2.3178e+04	−1.0425e+03	1.3746e+04

$\mathbf{MRR}_2$ =

4.9055e+00	6.3169e−01	9.1154e−01	−2.6973e+01	5.4237e+02	−3.0524e+01
6.3169e−01	8.1344e−02	1.1738e−01	−3.4734e+00	6.9843e+01	−3.9307e+00
9.1154e−01	1.1738e−01	1.6938e−01	−5.0122e+00	1.0078e+02	−5.6721e+00
−2.6973e+01	−3.4734e+00	−5.0122e+00	1.4831e+02	−2.9823e+03	1.6784e+02
5.4237e+02	6.9843e+01	1.0078e+02	−2.9823e+03	5.9967e+04	−3.3749e+03
−3.0524e+01	−3.9307e+00	−5.6721e+00	1.6784e+02	−3.3749e+03	1.8994e+02

$\mathbf{MRR}_3$ =

2.9855e−01	−8.2690e−01	2.4851e−01	4.5079e+01	2.5966e+01	−4.9102e+01
−8.2690e−01	2.2903e+00	−6.8829e−01	−1.2486e+02	−7.1918e+01	1.3600e+02
2.4851e−01	−6.8829e−01	2.0685e−01	3.7523e+01	2.1613e+01	−4.0872e+01
4.5079e+01	−1.2486e+02	3.7523e+01	6.8067e+03	3.9207e+03	−7.4142e+03
2.5966e+01	−7.1918e+01	2.1613e+01	3.9207e+03	2.2583e+03	−4.2706e+03
−4.9102e+01	1.3600e+02	−4.0872e+01	−7.4142e+03	−4.2706e+03	8.0758e+03

Table 3. Solutions for change of origin

Change in reference origin			
	X	Y	Z
Original	0	0	50
New	0	0	0
Change	0	0	−50

Mode 1

$$\mathbf{C} = \begin{bmatrix} 0 & 2.4218e-01 & -3.9761e+00 \\ -2.4218e-01 & 0 & 4.1960e-02 \\ 3.9761e+00 & -4.1960e-02 & 0 \end{bmatrix}$$

Eq. (18)

Eq. (17) $\mathbf{DMER} = [0 \quad 0 \quad 0 \quad -1.9880e+02 \quad 2.0980e+00 \quad 0]$

Mode 2

$$\mathbf{C} = \begin{bmatrix} 0 & 9.9270e-01 & -6.8793e-01 \\ -9.9270e-01 & 0 & 5.3422e+00 \\ 6.8793e-01 & -5.3422e+00 & 0 \end{bmatrix}$$

$\mathbf{DMER} = [0 \quad 0 \quad 0 \quad -3.4396e+01 \quad 2.6711e+02 \quad 0]$

Mode 3

$$\mathbf{C} = \begin{bmatrix} 0 & -7.6309e-01 & -2.5392e+00 \\ 7.6309e-01 & 0 & 9.1677e-01 \\ 2.5392e+00 & 9.1677e-01 & 0 \end{bmatrix}$$

$\mathbf{DMER} = [0 \quad 0 \quad 0 \quad -1.2696e+02 \quad -4.5838e+01 \quad 0]$