Channel Assignments and Array Gain Bounds for the Ka-Band Array Feed Compensation System

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The performance of a real-time digital combining system for use with array feeds has been considered in previous articles. The purpose of the combining operation is to recover signal-to-noise ratio (SNR) losses due to antenna deformations and atmospheric effects. Previously, arbitrary signal powers and noise variances were assumed, but no attempt was made to match the receiver channels to the available signal powers. Here it is shown that for any signal power and noise variance distribution, a "best" channel assignment exists that maximizes the combined SNR in the limit of vanishingly small combining losses. This limit can be approached in practice by observing sufficiently many samples. Specific signal power and noise variance distributions are considered, and it is shown that even relatively "noisy" channels can be used effectively to recover SNR losses resulting from signals diverted out of a "high-quality" channel by antenna deformations.

I. Introduction

The potential benefits of array feed combining for recovering losses due to mechanical antenna distortions at high frequencies (32 GHz or higher) have been described in [1,2]. A conceptual block diagram of the real-time combining system considered here is shown in Fig. 1, where the dashed curve near the primary reflector represents mechanical distortions, while near the feed array it indicates the spreading of the signal field distribution induced by reflector distortions. The feedhorn outputs are downconverted to baseband, sampled at the Nyquist rate, and the samples are processed digitally to obtain the combined complex baseband samples. The objective of the combining operation is to maximize the signal-to-noise ratio (SNR) of the combined samples, thus recovering some of the losses induced by the distorted antenna. The seven-element feed array geometry is of particular interest, as it has been shown in [3] that most of the losses incurred by large Deep Space Network (DSN) antennas can be recovered by seven elements arranged in a maximally compact pattern, as in Fig. 1. However, the analysis and results apply for arbitrary K-element arrays, regardless of the array geometry.

A functional block diagram of the real-time compensation system is given in Fig. 2. The observables consist of K complex sample-streams, corrupted by independent complex noise samples in each channel. The corrupted samples are input to a parameter estimator, which estimates the complex weights that should be applied to each stream in order to maximize the SNR of the combined samples.

II. System Description

An exact expression for the signal-to-noise ratio ρ_{ML} of the combined sample-stream has been derived in [2] for arbitrary SNR in each channel, assuming use of a "maximum-likelihood" (ML) parameter estimator to obtain the combining weights. The combined SNR was found to depend on the number of channels K, the total number of observed samples L, the ratio of data to estimation stream bandwidths η, the modulation index δ, the sum of the channel SNRs, and the sum of squares of the channel SNRs. The complex weights that maximize the combined SNR are

$$\tilde{w}_k = \frac{\tilde{V}_k^*}{2\sigma_k^2} \tag{1a}$$

in which case the combined SNR is given by

$$\rho = \sum_{k=1}^{K} |\tilde{V}_k|^2/2\sigma_k^2 \tag{1b}$$

where $\tilde{V}_k$ is the complex voltage due to signal, and $2\sigma_k^2$ is the variance of the complex noise in the kth channel. However, in the presence of noise one must rely on estimates of the combining weights, with inadvertent errors. If maximum-likelihood estimates are employed, the combined SNR becomes

$$\rho_{ML} = \rho\left\{1 + \frac{1}{L-3} + \frac{L-2}{\eta L(L-3)\cos^2\delta}\left(1+\frac{K}{\rho}\right) + \frac{1}{(L-3)\rho}\sum_{k=1}^{K}\frac{|\tilde{V}_k|^4}{4\sigma_k^4}\right\}^{-1} \tag{2}$$

Note that the combined SNR with maximum-likelihood weights is always less than ρ, approaching that limit as L approaches infinity. This is reasonable, since the maximum-likelihood weight estimates approach the true weights as the number of observed samples approaches infinity (the estimator is "consistent"). Note that for $L < 4$, this expression is not defined, since not enough samples have been observed to make all of the required estimates.

It is reasonable to define the "combining loss" γ as the inverse of the ratio of the actual SNR to its limiting value:

$$\gamma \triangleq \frac{\rho}{\rho_{ML}} \tag{3}$$

All losses due to imperfect weight estimates can be attributed to γ, where its inverse γ^{-1} is simply the variance of the combined signal normalized by ρ. This quantity can be decomposed into loss components due to signal γ_s and noise γ_n,

$$\gamma = \gamma_s + \gamma_n \tag{4a}$$

where

$$\gamma_s = \frac{L-2}{\eta L(L-3)\cos^2\delta} + \frac{1}{(L-3)\rho}\sum_{k=1}^{K}\frac{|\tilde{V}_k|^4}{4\sigma_k^4} \tag{4b}$$

$$\gamma_n = 1 + \frac{1}{L-3} + \frac{K(L-2)}{\eta L(L-3)\rho\cos^2\delta} \tag{4c}$$

The behavior of the combining loss as a function of L is examined in Section V.

III. Receiver Channel Assignment

Suppose the total signal power captured by the array is P_T watts, with the kth horn contributing P_k watts to the total, $P_k = |V_k|^2$, so that

$$P_T = \sum_{k=1}^{K} P_k \tag{5}$$

Assume the ordering $P_1 \geq P_2 \geq \cdots \geq P_K$, so that horn number 1 contributes the greatest signal component, horn number 2 the second greatest, and so on. For want of a better term, this can be called a "standard ordering" of the array feeds (in case of equalities, ordering becomes irrelevant). Suppose that each receiver channel adds independent noise components to the signal, with the kth channel contributing variance $2\sigma_k^2$. If receiver channels could be assigned to array elements in any order, how should the receivers be assigned to the feeds in order to maximize the SNR of the combined signal ρ_{ML}? To answer this question, one first maximizes the ideal SNR ρ.

A. Channel Assignment to Maximize ρ

Index the receiver channels according to the variance of the noise in that channel, with the least noisy channel called channel number 1, the second "quietest" channel number 2, etc., so that $\sigma_1^2 \leq \sigma_2^2 \leq \cdots \leq \sigma_K^2$. Assigning the receiver channels to the array feeds according to this indexing scheme leads to the sum ρ, which is called "standard channel assignment" to be consistent with the above terminology. Then

$$2\rho = \sum_{k=1}^{K} P_K/\sigma_K^2 = \frac{P_1}{\sigma_1^2} + \frac{P_2}{\sigma_2^2} + \cdots + \frac{P_K}{\sigma_K^2}$$

$$= \left(\sum_{\substack{k=1 \\ k \neq i \\ k \neq j}}^{K} \frac{P_k}{\sigma_k^2} \right) + \left(\frac{P_i}{\sigma_i^2} + \frac{P_j}{\sigma_j^2} \right) \tag{6}$$

Here the last equality focuses on the ith and jth channels. With $i < j$, let

$$\sigma_j^2 = \sigma_i^2 + \Delta_{ij} , \qquad \Delta_{ij} \geq 0 \tag{7a}$$

$$P_i = P_j + \delta_{ij} , \qquad \delta_{ij} \geq 0 \tag{7b}$$

hence, the sum of the ith and jth channel SNRs may be written as

$$\left(\frac{P_i}{\sigma_i^2} + \frac{P_j}{\sigma_j^2} \right) = \frac{P_i}{\sigma_i^2} + \frac{(P_i - \delta_{ij})}{(\sigma_i^2 + \Delta_{ij})} \tag{8a}$$

Exchange the ith and jth channels, assigning the jth channel to the ith horn and the ith channel to the jth horn, obtaining in place of Eq. (8a)

$$\left(\frac{P_i}{\sigma_j^2} + \frac{P_j}{\sigma_i^2} \right) = \frac{P_i}{(\sigma_i^2 + \Delta_{ij})} + \frac{(P_i - \delta_{ij})}{\sigma_i^2} \tag{8b}$$

This pairwise exchange of a lower index channel with a higher index channel shall be termed an "unsorting exchange," for reasons that will soon become apparent. A comparison of the initial and exchanged pairs shows that the sum of the SNRs in Eq. (8b) is less than that in Eq. (8a) by an amount

$$l(i,j;i,j) = \frac{\Delta_{ij}\delta_{ij}}{\sigma_i^2 (\sigma_i^2 + \Delta_{ij})} \geq 0 \tag{9}$$

Because a pairwise exchange does not affect the rest of the channel assignments, it follows that the total SNR also decreases by exactly this amount as a result of the exchange.

Next, allow an arbitrary channel assignment (that is, not the "standard assignment" defined above), and perform an unsorting exchange on this configuration. This again leads to a decrease in the sum, now by an amount

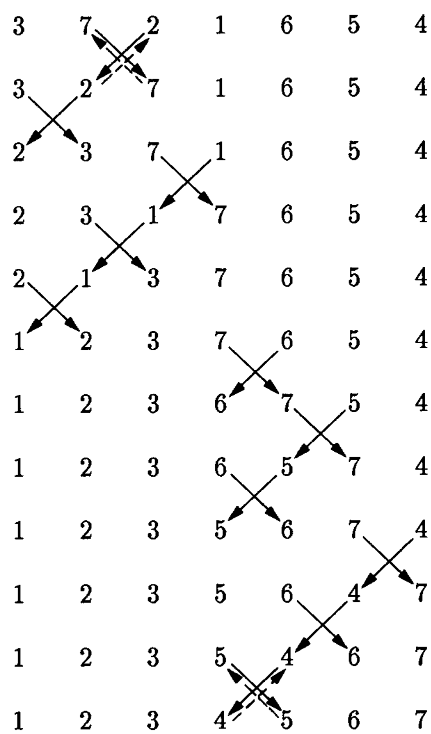
$$l(i,j;m,n) = \frac{\Delta_{mn}\delta_{ij}}{\sigma_m^2 (\sigma_m^2 + \Delta_{nm})} \geq 0 \tag{10}$$

where m and n, $m < n$, are the indices of the exchanged channels (as before, i and j are the indices of the horns, in order of decreasing signal power). Thus, regardless of the state of the system, an unsorting exchange always leads to a decrease in the achievable combined SNR. If it can be shown that any channel assignment can be reached from the standard assignment by performing only unsorting exchanges, then it will have been proven that the standard assignment achieves the highest possible combined SNR for the given set of signal powers and noise variances.

This property can be shown in an interesting way, using the following (rather inefficient) sorting concept. The algorithm sorts a sequence of numbers by sequentially comparing nearest neighbors, starting at the left end of the sequence, and performing a pairwise "sorting" exchange if and only if the value of a given element exceeds that of its neighbor to the right (thus, for this algorithm, the indices i, j, and m, n are always consecutive integers). After each exchange, the algorithm restarts at the beginning of the sequence and continues to compare each element to its rightmost neighbor until no further exchanges are possible. At this point, the sequence is sorted with the smallest element appearing at the left. Retracing the exchange path from the sorted sequence to its initial state shows that the "unsorting" operation requires only unsorting exchanges. The following example illustrates the procedure. Consider a 7-horn array with standard ordering, and let the channel assignment be denoted by some arbitrary index sequence, say

$$3 \quad 7 \quad 2 \quad 1 \quad 6 \quad 5 \quad 4$$

(that is, the channel with the third lowest noise variance assigned to horn number 1, with the greatest noise to horn number 2, and so on). Applying the above sorting algorithm to this sequence yields



where the downward pointing arrows indicate the pairwise "sorting" exchanges performed by the algorithm to arrive at the standard channel assignment. When these exchanges are applied in reverse, as indicated by the upward pointing arrows, it is clear that only unsorting exchanges were used to unsort the standard assignment (hence the terminology). Since the above algorithm sorts any sequence, it follows that one can arrive at any channel assignment by performing only unsorting exchanges on the receiver channels. Since each such exchange decreases the resulting combined SNR, it follows that the standard channel assignment achieves the highest combined SNR possible with any set of signal power and noise variances distributions.

B. Channel Assignment to Maximize ρ_{ML}

The standard ordering does not always maximize ρ_{ML}. This is demonstrated by the following example. Suppose that $\eta = 1000$, $\delta = 60$ deg, $K = 7$, and $L = 4$, so that the smallest number of samples is observed for which Eq. (1) is valid. Further suppose that the power levels in the first, second, ..., seventh horns are

$$10 \quad 8 \quad 7 \quad 6 \quad 6 \quad 5 \quad 5$$

while the corresponding noise variance distribution is

$$0.5 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1$$

which implies a standard channel assignment. Direct substitution into Eq. (1b) and Eq. (2) yields

$$\rho = 28.5000 \qquad \rho_{ML} = 3.7635$$

Exchanging channel 1 with channel 2 yields the noise variance distribution

$$1 \quad 0.5 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1$$

Recomputing Eq. (1b) and Eq. (2), one obtains

$$\rho = 27.5000 \qquad \rho_{ML} = 4.0480$$

which shows that ρ_{ML} increased while ρ decreased as a result of the exchange. Observe that this is not a preferred region of operation, since the combining losses are unacceptably high for such small L. However, the standard ordering does maximize ρ_{ML} in the limit as the number of observed samples grows without bound.

IV. Some Special Cases of Interest

Next, consider situations where the signal or noise distributions, or both, obey simple rules due to symmetries in the underlying model, namely:

(1) the noise variance is identical in $(K-1)$ of K channels (constrained channel noise model)

(2) the signal power is identical in $(K-1)$ of K channels (constrained signal distribution model)

(3) both the channel noise and the signal distribution are constrained according to (1) and (2) in the same $(K-1)$ channels (jointly constrained model)

The above conditions might not be strictly satisfied in practice, but the simplicity of the model may nevertheless provide useful approximations to actual operating conditions.

It will be assumed that independent estimates are made of the required parameters in each channel, without taking advantage of the special relations that exist when the above models hold. This is equivalent to admitting that the above special conditions are not known to exist a priori. If these conditions were known to exist a priori, the estimation algorithms could be matched to the unusually favorable distributions, resulting in improved performance. However, rather than pursue that idea, this article focuses instead on the performance of the "unmatched" independent channel estimators described in [1] and [2].

A. Constrained Channel Noise Model

Consider a model where the noise variance in one of the channels is considerably less than in the surrounding channels, all of which have the same (higher) noise variance. This situation could arise, for example, when a low-noise cooled maser is employed for reception and is surrounded by a ring of higher noise high-electron-mobility transistor (HEMT) low-noise amplifier (LNA) channels for possible gain improvement. The outer channels are intended for use at extreme antenna elevation angles, where mechanical distortions tend to spread the signal power in the focal plane.

Ignoring minor noise variations among the outer channels, the difference in channel noise variances can be accounted for by specifying the ratio of "low" to "high" noise temperatures: call this ratio α and let $0 < \alpha < 1$. Thus, the central channel is modeled as having the lowest noise temperature, while the surrounding channels are modeled as having the same higher noise temperature, each a factor $(1/\alpha)$ times that of the central channel. The two-sided spectral level of the thermal noise in the kth channel can be obtained from the corresponding noise temperature T_k as

$$\frac{N_{ok}}{2} = \frac{\kappa T_k}{2} \tag{11a}$$

so that for τ-sec sample averaging, the sample variance becomes

$$\sigma_k^2 = \frac{N_{ok}}{2\tau} \tag{11b}$$

(here κ is Boltzmann's constant). This model is sufficiently accurate to provide insight into the expected behavior of the combining system.

Although the central channel may have significantly less noise than the outer channels, it does not follow that only the central channel should be observed: there could be situations where the outer channels contain significant signal components that, when properly combined, could improve the overall SNR of the system. This situation may be due to distortions of the main reflector induced by gravity, wind gusts, defocusing of the subreflector, or even a simple pointing error. In each case, signal collected by the outer channels may improve the system SNR, as well as provide real-time pointing error information to the system.

Denote the sum of the channel SNRs by ρ, as before, and let ζ denote the sum of squares of channel SNRs. Thus, for the special case under consideration,

$$\rho = \sum_{k=1}^{K} P_k/2\sigma_k^2 = \frac{P_1}{2\sigma_1^2} + \frac{\alpha}{2\sigma_1^2}\sum_{k=2}^{K} P_k \tag{12a}$$

$$\zeta = \sum_{k=1}^{K} P_k^2/4\sigma_k^4 = \frac{P_1^2}{4\sigma_1^4} + \frac{\alpha^2}{4\sigma_1^4}\sum_{k=2}^{K} P_k^2 \tag{12b}$$

Letting Q_k denote the square of the kth signal power and Q_T denote their sum, Eqs. (12a) and (12b) can be rewritten as

$$\rho = \frac{\alpha P_T}{2\sigma_1^2} + (1-\alpha)\frac{P_1}{2\sigma_1^2} \tag{13a}$$

$$\zeta = \frac{\alpha^2 Q_T}{4\sigma_1^4} + (1-\alpha^2)\frac{Q_1}{4\sigma_1^4} \tag{13b}$$

which shows that ρ is a weighted average of P_1 and P_T. Thus, the effective SNR ranges from $P_1/2\sigma_1^2$ as α approaches zero (i.e., as the outer channels become infinitely noisy), to $P_T/2\sigma_1^2$ as α approaches 1.

B. Constrained Signal Distribution Model

Next, consider the signal distribution model. Let β be the fraction of the total received signal power intercepted by the central horn, $0 < \beta < 1$, and let the remaining signal power be distributed uniformly among the outer $(K-1)$ horns. Thus, if the total signal power intercepted by the array is P_T watts, and if the central horn is designated horn number 1 while the outer horns are labeled number 2–number K, the power in the central horn P_1 is

$$P_1 = \beta P_T \tag{14a}$$

while the power in any of the outer horns is

$$P_k = \frac{(1-\beta)}{(K-1)} P_T \tag{14b}$$

This model assumes a symmetrical signal distribution, which is generally valid for a focusing error, and may often be used to approximate the effects of gravity-induced mechanical distortions as well.

For this type of signal distribution (with arbitrary noise in each channel), one obtains

$$\rho = \frac{\beta P_T}{2\sigma_1^2} + \frac{(1-\beta)P_T}{2(K-1)} \sum_{k=2}^{K} \sigma_k^{-2} \tag{15a}$$

$$\zeta = \left(\frac{\beta P_T}{2\sigma_1^2}\right)^2 + \left[\frac{(1-\beta)P_T}{2(K-1)}\right]^2 \sum_{k=2}^{K} \sigma_k^{-4} \tag{15b}$$

The first term in Eq. (15a) is simply the SNR of the central channel. The combined SNR is augmented by an equivalent second channel with total signal power $\frac{(1-\beta)P_T}{K-1}$ and effective noise variance $\left(\sum_{k=2}^{K} \sigma_k^{-2}/2\right)^{-1}$

C. Jointly Constrained Model

When the above constraints on the noise variance and the signal distribution are simultaneously satisfied over the same channels, the expressions for ρ and ζ can be put into a particularly simple form:

$$\rho = \frac{P_T}{2\sigma_1^2} \left[\beta + \alpha(1-\beta)\right] \tag{16a}$$

$$\zeta = \left(\frac{P_T}{2\sigma_1^2}\right)^2 \left[\beta^2 + \frac{\alpha^2(1-\beta)^2}{(K-1)}\right] \tag{16b}$$

For the jointly constrained problem, there is only one nontrivial switching operation. It is instructive to examine the behavior of ρ and ρ_{ML} for standard and switched channel assignments in this case: let the subscript sw denote switched channel assignments. The behavior of these quantities, as well as their unswitched counterparts, is shown in Fig. 3, for the signal power distribution

100 10 10 10 10 10 10

and noise variance distribution

0.5 1 1 1 1 1 1

with $K = 7$, $\eta = 1000$, and $\delta = 80$ deg. Clearly, $P_T = 160$, $\beta = 10/16$, and $\alpha = 0.5$. Both ρ_{ML} and ${}_{sw}\rho_{ML}$ approach their limiting values for large L, but drop off rapidly below some critical number which, in this case, appears to be between 100 and 1000 samples. For even lower values of L, note that ${}_{sw}\rho_{ML} > \rho_{ML}$, indicating that the switching operation actually improves the combined SNR in this region. This behavior is attributed to the fact that in this region the switching operation generates so large a decrease in ζ that the value of the denominator in Eq. (2) decreases more than ρ, resulting in a net increase in ρ_{ML}. Although this unusual behavior is not expected to occur during normal operating conditions, it does point out a need to verify the optimality of the standard assignment in practice. The combining loss γ as a function of L corresponding to the standard assignment in the above example is displayed in Fig. 4. For $L > 1000$, the combining loss rapidly becomes negligibly small, reaching a value of less than 0.1 dB when about 3500 samples are observed.

V. Array Gain

Finally, consider the potential gain in SNR that could be achieved over a single feed horn by using an array of feeds in the focal plane. Using the notation of Eqs. (1) and (2), the array gain G_A is defined as the ratio of the combined SNR ρ_{ML} divided by the SNR in channel number 1:

$$G_A \triangleq \rho_{ML}/\left(P_1/2\sigma_1^2\right) = \rho/\gamma\left(P_1/2\sigma_1^2\right) \tag{17}$$

Since the combining loss γ is never less than 1, the array gain can be bounded with the ratio

$$G_A^u \triangleq \rho / (P_1/2\sigma_1^2) \quad (18)$$

which is recognized as the theoretical maximum of the array gain and achieved only with perfect weight estimates. The array gain and its upper bound may be expressed in decibels as

$$G_A(\text{dB}) = \rho(\text{dB}) - \left(\frac{P_1}{2\sigma_1^2}\right)(\text{dB}) - \gamma(\text{dB})$$

$$= G_A^u(\text{dB}) - \gamma(\text{dB}) \quad (19)$$

This expression shows that if the gain bound and the combining loss are known, the actual array gain can always be determined. To illustrate this point, the array gain, its upper bound, and the associated combining loss are shown in Fig. 5, for the example treated in Figs. 3 and 4. Although this example shows the behavior of G_A, G_A^u, and γ with increasing L, it corresponds to but a single point in (α,β) space. Greater appreciation for the benefits of array-feed combining may be obtained by examining the array gain as a function of the fractional signal power β for various noise temperature ratios α. This behavior is illustrated in Fig. 6. Since for large L the array gain approaches the gain bound G_A^u, only the gain bound is examined here, keeping in mind that the actual gain can always be obtained from these bounds by computing the combining loss and applying it to Eq. (19).

It is immediately apparent from Fig. 6 that G_A^u increases with decreasing β, which simply means that the importance of the outer ring increases as more signal power is diverted to it. It is also clear that for any β, the array gain increases with increasing α; that is, as the relative noise temperature of the outer ring decreases. This behavior is consistent with intuition, reaffirming that array feed combining is most effective when the signal power is spread over an area that is large as compared with the effective area of a single feed, and when similar quality channels are used to recover the signal. However, note also that considerable improvements are possible even in intermediate cases, for example, when $\alpha = \beta = 1/2$. This assignment refers to a situation where half the received signal power is diverted out of the central feed into the outer ring, while the noise variance in the central channel is half that of the outer channels (approximating the model for a hybrid maser/HEMT system operating with a distorted DSN antenna). The resulting array gain is seen to be close to 2 dB, which clearly justifies the use of an array combining system.

VI. Conclusions

The generalized results presented in [1,2] have been extended by allowing an additional maximization over the signal power and noise variance distributions in an array-feed combining system. It was shown that for any given signal power and noise variance distributions, a unique channel assignment exists that achieves the greatest combined SNR in the limit of vanishingly small combining losses, and that this limit can be approached in practice by observing a sufficient number of samples. Some special signal and noise distributions were examined, which approximated the signal power distribution characteristic of distorted antennas, observed by a low-noise channel augmented by a ring of higher noise receivers. It was shown that considerable gain improvements are possible by using such a "hybrid" arrangement. For example, even if half of the received signal power (3 dB) is diverted out of the low-noise channel into the outer ring, as much as 2 dB of SNR may be recovered by using receiver channels with twice the noise temperature of the low-noise receiver. This approach may prove to be a cost-effective way to improve the performance of low-noise (hence, high-cost) receivers degraded by distorted antennas operating at Ka-band (32 GHz) or higher carrier frequencies.

References

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[3] P. W. Cramer, "Initial Studies of Array Feeds for the 70-Meter Antenna at 32 GHz," *TDA Progress Report 42-104*, vol. October–December 1990, Jet Propulsion Laboratory, Pasadena, California, pp. 50–67, February 15, 1991.

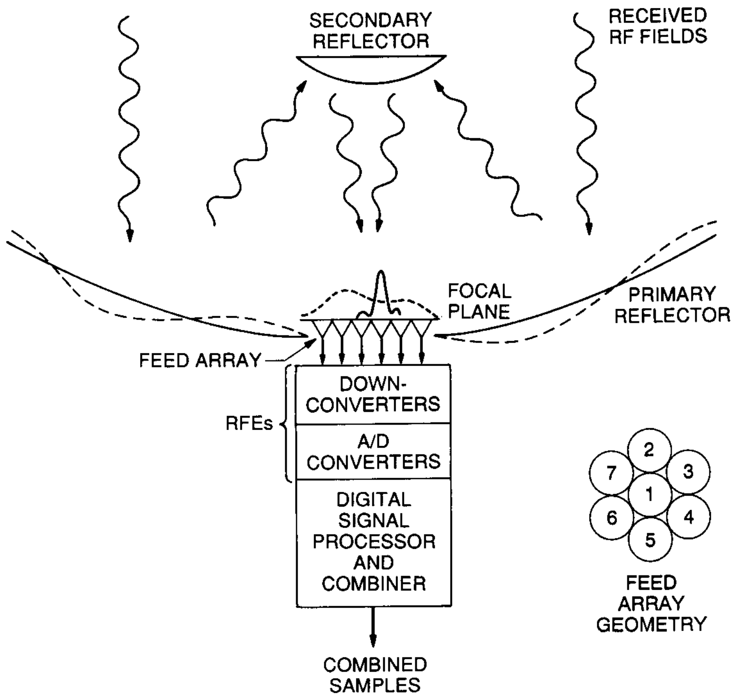

Fig. 1. Real-time antenna compensation system conceptual design.

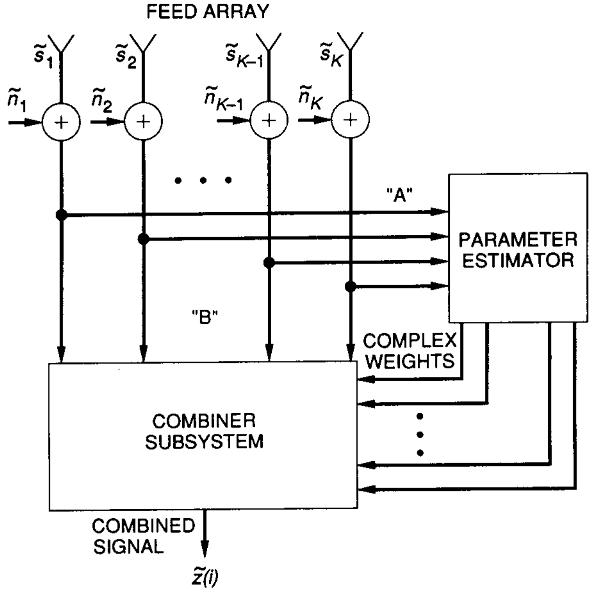

Fig. 2. Signal-combining system (baseband model) block diagram.

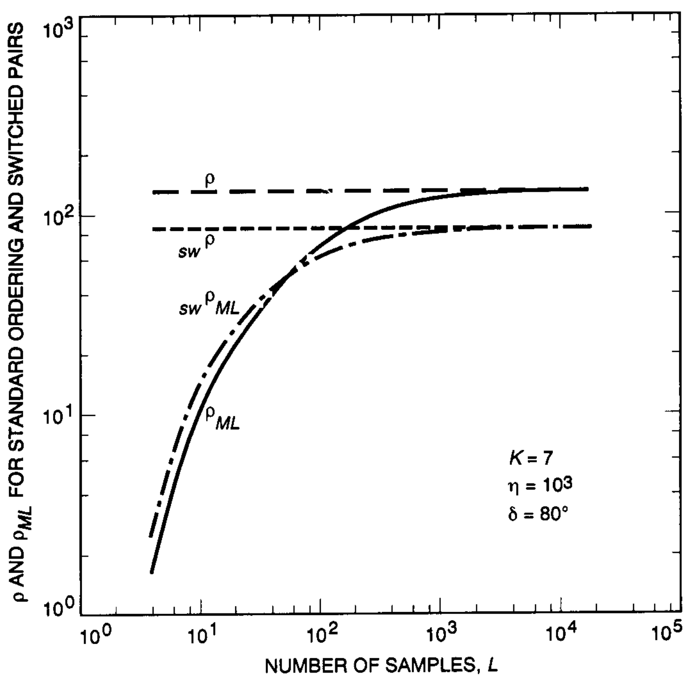

Fig. 3. Signal-to-noise ratios for standard and switched assignments versus L.

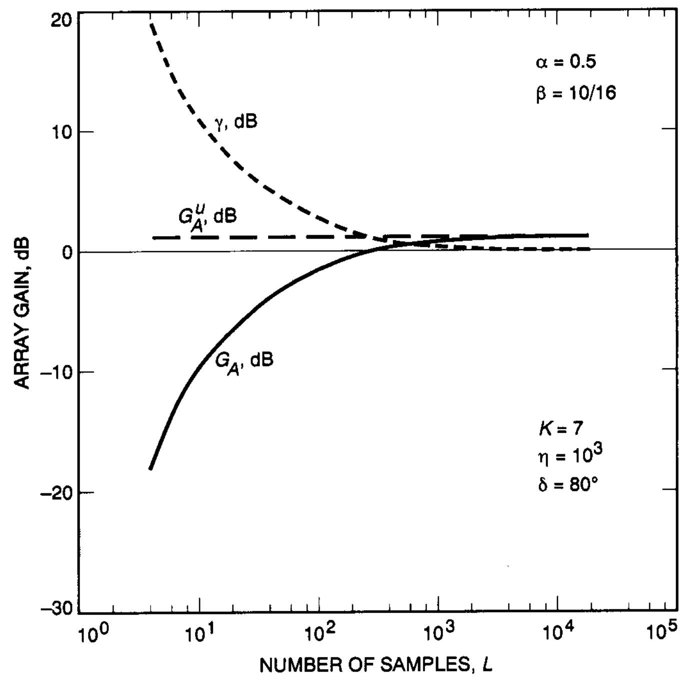

Fig. 5. Array gain and bound as a function of L.

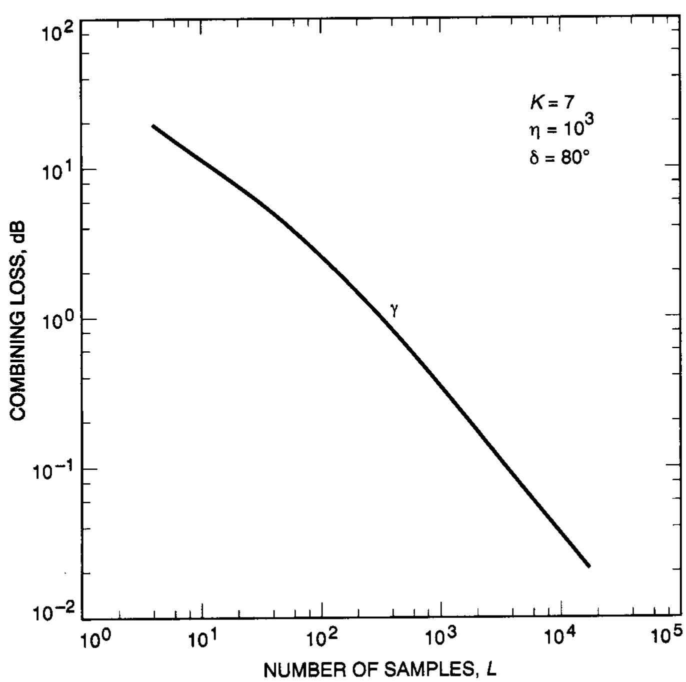

Fig. 4. Combining loss versus L.

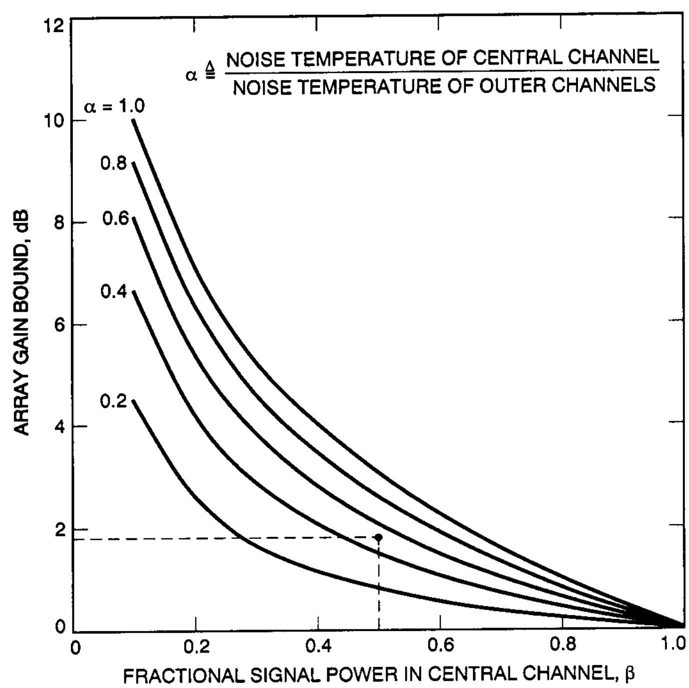

Fig. 6. Array gain bound as a function of α and β.