Carrier-to-Noise Power Estimation for the Block V Receiver

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Two possible algorithms for the carrier-to-noise power (P_c/N_0) estimation in the Block V Receiver are analyzed and their performances compared. The expected value and the variance of each estimator algorithm are derived. The two algorithms examined here are known as the I-arm estimator, which relies on samples from only the in-phase arm of the digital phase-lock loop, and the IQ-arm estimator, which uses both in-phase and quadrature-phase arm signals. The IQ-arm algorithm is currently implemented in the Advanced Receiver II (ARX II). Both estimators are biased. The performance degradation due to phase jitter in the carrier tracking loop is taken into account. Curves of the expected value and the signal-to-noise ratio of the P_c/N_0 estimators versus actual P_c/N_0 are shown. From this, it is clear that the I-arm estimator performs better than the IQ-arm estimator when the data-to-noise power ratio (P_d/N_0) is high, i.e., at high P_c/N_0 values and a significant modulation index. When P_d/N_0 is low, the two estimators have essentially the same performance.

I. Introduction

In the Block V Receiver, P_c/N_0 estimates are made from accumulated samples in the carrier tracking loop. These estimates are used in various ways. Besides giving knowledge of the actual P_c/N_0 and as an indicator of when the loop is in-lock, the estimates can also be used in the Conscan process. (In Conscan, the pointing error of the Deep Space Network, DSN, ground antennas is reduced by conically scanning the antennas around a boresight. The present Conscan technique uses automatic gain control, AGC, samples of the ground receiver as a measure of the carrier power; but, if the noise power is fairly constant over several scan cycles, the P_c/N_0 estimates will serve the equivalent purpose.)

A simplified block diagram of the section of the digital phase-lock loop (DPLL) preceding the P_c/N_0 estimator is shown in Fig. 1. The sampling rate before the half-band filters (HBF's) is 40 MHz. The output of the HBF's has been decimated by two so that the new rate is one half of 40 MHz ($f_s = 1/T_s = 20$ MHz). The signals, i_n and q_n (shown below), are then accumulated over K samples, resulting in I_j and Q_j, which are the inputs to the P_c/N_0 estimator to be implemented in software. The sampling rate of I_j and Q_j is $f_s/K = 1/T_u$, where $T_u = KT_s$ is the carrier tracking loop update time, and can range from about 10 Hz to 10 kHz.

II. Estimator Algorithms

Assuming the carrier tracking loop is locked onto the received carrier frequency, the in-phase and quadrature-phase baseband signals (at sampling instant n) at the output of the HBFs (neglecting higher order terms that are filtered out by the HBF's) are, respectively [1],

$$i_n = \sqrt{P_c}\cos\phi_n + \sqrt{P_d}d_n \operatorname{Sin}(\omega_{sc}nT_s + \theta_n)\sin\phi_n + n_i(n) \tag{1}$$

and

$$q_n = \sqrt{P_c}\sin\phi_n + \sqrt{P_d}d_n \operatorname{Sin}(\omega_{sc}nT_s + \theta_n)\cos\phi_n + n_q(n) \tag{2}$$

where P_c is the carrier power, ϕ_n is the phase error in the carrier tracking loop (assumed to be constant over one estimation period for all calculations), P_d is the data power, d_n represents the independent data samples that equal ± 1 with equal probability, and $\operatorname{Sin}(\omega_{sc}nT_s + \theta_n)$ represents a sample of a square-wave subcarrier with arbitrary phase θ_n.

In this case, $n_i(n)$ and $n_q(n)$ are the noise terms from the in-phase and quadrature-phase channels, respectively, and are assumed to be statistically independent Gaussian random variables with a mean of zero and variance of $\sigma^2 = N_0/(2T_s)$.

As seen in Fig. 1, the i_n and q_n signals are accumulated to give I_j and Q_j, respectively. Thus,

$$I_j = \frac{1}{K}\sum_{n=jK}^{(j+1)K-1} \sqrt{P_c}\cos\phi + n_I(j) \tag{3}$$

and

$$Q_j = \frac{1}{K}\sum_{n=jK}^{(j+1)K-1} \sqrt{P_d}d_n \operatorname{Sin}(\omega_{sc}nT_s + \theta_n)\cos\phi + n_Q(j) \tag{4}$$

where the $\sin\phi_n$ terms have been neglected under the assumption that the tracking error is very small (i.e., the loop is in lock). In addition, ϕ_n and θ_n have been replaced by ϕ and θ, respectively, under the assumption of constant phase over an observation interval. Both $n_I(j)$ and $n_Q(j)$ are formed by averaging samples of $n_i(n)$ and $n_q(n)$ and are, therefore, also independent Gaussian random variables with a mean of zero and a variance of $\sigma_I^2 = \sigma_Q^2 = N_0/(2KT_s) = N_0/(2T_u)$.

The IQ-arm estimator algorithm is [1]

$$\widehat{R_{IQ}} = \frac{\left(\frac{1}{L}\sum_{j=0}^{L-1} I_j\right)^2}{2T_u\left(\frac{1}{L}\sum_{j=0}^{L-1} Q_j^2\right)} \tag{5}$$

where the random variable $\widehat{R_{IQ}}$ is the P_c/N_0 estimator and L equals the number of DPLL updates in the estimation period.

By inserting Eqs. (3) and (4) into Eq. (5), it can be seen that the numerator of Eq. (5) essentially works to average out the zero mean noise samples $n_I(n)$, resulting in an estimate of the carrier power (assuming a small phase error). In the denominator of Eq. (5), the Q-arm estimates are first squared, then averaged. Assuming that the subcarrier frequency ω_{sc} is large as compared with $1/T_s$ and the data power is not too high, the first term in Eq. (4) will be small and, thus, the denominator of Eq. (5) will be a scaled estimate of the sum of the squares of $n_Q(n)$, which equals the noise power. (In reality, the effect of the data power is important, as will be seen in the analysis below.)

The I-arm estimator algorithm [2] is

$$\widehat{R_I} = \frac{\left(\frac{1}{L}\sum_{j=0}^{L-1} I_j\right)^2}{\frac{2T_u}{L-1}\left[\sum_{j=0}^{L-1} I_j^2 - \frac{1}{L}\left(\sum_{j=0}^{L-1} I_j\right)^2\right]} \tag{6}$$

where $\widehat{R_I}$ is the P_c/N_0 estimator and L is defined as above.

By inserting Eq. (3) into Eq. (6), it can be seen that the numerator of Eq. (6) is the same as that of Eq. (5). However, the denominator, which reduces to an estimate of noise power, does not depend on the data power and, as shall be seen below, is the major advantage of the I-arm estimator over the IQ-arm estimator.

III. First and Second Moments of the IQ- and I-Arm Estimators

For simplicity, the following intermediate variables are defined:

$$X = \left(\frac{1}{L} \sum_{j=0}^{L-1} I_j \right)^2 \tag{7}$$

$$Y = \frac{1}{L} \sum_{j=0}^{L-1} Q_j^2 \tag{8}$$

$$Z = \sum_{j=0}^{L-1} I_j^2 \tag{9}$$

Using Eqs. (7), (8), and (9), Eqs. (5) and (6) can be written as

$$\widehat{R_{IQ}} = \frac{X}{2T_u Y} \tag{10}$$

and

$$\widehat{R_I} = \frac{X}{\frac{2T_u}{L-1}(Z - LX)} \tag{11}$$

The statistics of the estimators are not trivial, since in both cases the numerator and the denominator are correlated. (In the case of the IQ-arm estimator, the numerator and denominator are correlated, even though they are obtained from in-phase and quadrature-phase arms, since both arms have terms that contain the phase error ϕ.)

The method used here to find the statistics is to expand the expressions for $\widehat{R_{IQ}}$ and $\widehat{R_I}$ (each of which is a function of two random variables whose moments are shown below) in a two-dimensional Taylor series and then to find the expected value and variance of the expanded series. The higher order terms of the Taylor series expansion are ignored. Using this approach, the expected value of the IQ-arm estimator can be approximated by the following expression [3,4]:

$$\overline{\widehat{R_{IQ}}} = \widehat{R_{IQ}}\Big|_{\overline{X},\overline{Y}} + \frac{1}{2}\left[\frac{\partial^2 \widehat{R_{IQ}}}{\partial X^2}\operatorname{var}(X) + \frac{\partial^2 \widehat{R_{IQ}}}{\partial Y^2}\operatorname{var}(Y)\right]\Bigg|_{\overline{X},\overline{Y}} + \frac{\partial^2 \widehat{R_{IQ}}}{\partial X \partial Y}\operatorname{cov}(X,Y)\Bigg|_{\overline{X},\overline{Y}} \tag{12}$$

and the variance by

$$\operatorname{var}\left(\widehat{R_{IQ}}\right) = \left(\frac{\partial \widehat{R_{IQ}}}{\partial X}\right)^2\Bigg|_{\overline{X},\overline{Y}} \operatorname{var}(X) + \left(\frac{\partial \widehat{R_{IQ}}}{\partial Y}\right)^2\Bigg|_{\overline{X},\overline{Y}} \operatorname{var}(Y) + 2\,\frac{\partial \widehat{R_{IQ}}}{\partial X}\frac{\partial \widehat{R_{IQ}}}{\partial Y}\operatorname{cov}(X,Y)\Bigg|_{\overline{X},\overline{Y}} \tag{13}$$

where the overbar indicates the statistical average and the covariance of X and Y is defined as $\operatorname{cov}(X,Y) \triangleq \mathbf{E}\{(X-\overline{X})\,(Y-\overline{Y}\} = \overline{XY} - \overline{X}\,\overline{Y}$. The expected value operator is represented by $\mathbf{E}\{\cdot\}$. For the I-arm estimator, the equations are identical, except that Y is replaced with Z.

The moments of X, Y, and Z, which are necessary for the computation of Eqs. (12) and (13), are shown here (see the Appendix for the derivations).

Defining $g_2 \triangleq \overline{\cos^2\phi}$ and $g_4 \triangleq \overline{\cos^4\phi}$, the first-order moments are given by

$$\overline{X} = g_2 P_c + \frac{N_0}{2LT_u} \tag{14}$$

$$\overline{Y} = K^{-1} g_2 P_d + \frac{N_0}{2T_u} \tag{15}$$

$$\overline{Z} = L g_2 P_c + \frac{LN_0}{2T_u} \tag{16}$$

Eq. (14) shows that the numerators of both estimators, Eqs. (5) and (6), are, in fact, a positively biased estimate of the carrier power (neglecting phase error). Also, the estimate of noise power in the IQ-arm case, given by a scaled version of Eq. (15), is biased. The I-arm estimate of the noise power, however, is unbiased.

The second-order moments are given by

$$\overline{X^2} = g_4 P_c^2 + \frac{6N_0}{2LT_u} g_2 P_c + 3\left(\frac{N_0}{2LT_u}\right)^2 \tag{17}$$

$$\overline{Y^2} = \frac{3K-2}{K^3} g_4 P_d^2 + \frac{L+2}{LKT_u} N_0 g_2 P_d + \frac{L+2}{L}\left(\frac{N_0}{2T_u}\right)^2 \tag{18}$$

$$\overline{Z^2} = L^2 g_4 P_c^2 + L(L+2)\frac{N_0}{T_u} g_2 P_c + L(L+2)\left(\frac{N_0}{2T_u}\right)^2 \tag{19}$$

and the covariances by

$$\mathrm{cov}\,(X,Y) = K^{-1}\left(g_4 - g_2^2\right) P_d P_c \tag{20}$$

$$\mathrm{cov}\,(X,Z) = L\left(g_4 - g_2^2\right) P_c^2 + 2T_u^{-1} N_0 g_2 P_c + \frac{N_0^2}{2LT_u^2} \tag{21}$$

In the case of the Block V receiver, the DPLL is a third-order loop whose phase error has a probability density function characterized, approximately, by a Tikhonov distribution (for zero detuning) [5]. For this case, g_2 and g_4 are obtained by using the relation $\overline{\cos n\phi} = [I_n(\alpha)]/[I_0(\alpha)]$ [5] where, for high loop signal-to-noise ratio ($\rho = P_c/N_0 B_L$), α is approximated by ρ [5]. Thus,

$$\left.\begin{aligned} \overline{\cos^2\phi} &= \frac{1}{2} + \frac{I_2(\rho)}{2I_0(\rho)} \\ \overline{\cos^4\phi} &= \frac{1}{8}\left[3 + \frac{4I_2(\rho)}{I_0(\rho)} + \frac{I_4(\rho)}{I_0(\rho)}\right] \end{aligned}\right\} \tag{22}$$

By inserting Eqs. (14) through (22) into Eqs. (12) and (13), the mean for the IQ-arm estimator, after simplifying, is

$$\overline{\widehat{R_{IQ}}} = \frac{KA}{2LT_u B} + \frac{2T_u KA}{LB^3}\left\{\left[\left(3 - \frac{2}{K}\right) g_4 - g_2^2\right]\left(\frac{P_d}{N_0}\right)^2 + \frac{2Kg_2}{LT_u}\frac{P_d}{N_0} + \frac{K^2}{2LT_u^2}\right\} - \frac{2KT_u}{B^2}\left(g_4 - g_2^2\right)\frac{P_d}{N_0}\frac{P_c}{N_0} \tag{23}$$

and the variance is

$$\mathrm{var}\left(\widehat{R_{IQ}}\right) = \frac{K^2}{B^2}\left\{\left(g_4 - g_2^2\right)\left(\frac{P_c}{N_0}\right)^2 + \frac{2g_2}{LT_u}\frac{P_c}{N_0} + \frac{1}{2L^2T_u^2}\right\} + \frac{K^2A^2}{L^2B^4}\left\{\left[\left(3 - \frac{2}{K}\right) g_4 - g_2^2\right]\left(\frac{P_d}{N_0}\right)^2 + \frac{2Kg_2}{LT_u}\frac{P_d}{N_0} + \frac{K^2}{2LT_u^2}\right\} - \frac{2K^2A}{LB^3}\left(g_4 - g_2^2\right)\frac{P_d}{N_0}\frac{P_c}{N_0} \tag{24}$$

where $A = 2LT_u g_2(P_c/N_0)+1$ and $B = 2T_u g_2(P_d/N_0)+K$.

The mean for the I-arm estimator is

$$\overline{\widehat{R_I}} = \frac{L+1}{L-1}\left(g_2\frac{P_c}{N_0} + \frac{1}{2LT_u}\right) \tag{25}$$

and the variance is

$$\mathrm{var}\left(\widehat{R_I}\right) = \left(g_4 - \frac{L-3}{L-1}g_2^2\right)\left(\frac{P_c}{N_0}\right)^2 + \frac{2g_2}{T_u(L-1)}\frac{P_c}{N_0} + \frac{1}{2T_u^2 L(L-1)} \tag{26}$$

The signal-to-noise ratio (SNR) of the estimators is defined here as

$$SNR\left(\hat{R}\right) = \frac{\overline{\hat{R}}^2}{\mathrm{var}\left(\hat{R}\right)} \tag{27}$$

It should be noted that for P_d/N_0 equal to zero and $L \gg 1$, the IQ-arm estimators expected value and variance reduce almost exactly to that of the I-arm estimator.

IV. Discussion of the Results

The performance of the estimators is shown in Figs. (2) and (3). Figure 2 shows $\overline{\widehat{R_{IQ}}}$ and SNR $(\widehat{R_{IQ}})$ versus the carrier-to-noise power (P_c/N_0) in dB-Hz for a typical modulation index such as that used by the Galileo spacecraft, i.e., $\Delta = 76$ deg where $\tan^2\Delta = P_d/P_c$. Figure 3 depicts $\overline{\widehat{R_I}}$ and SNR $(\widehat{R_I})$ versus P_c/N_0 in dB-Hz. The effect of the data power P_d in the IQ-arm estimator becomes critical when the data power is large, which, for a $\Delta =$ 76 deg, occurs at high values of P_c/N_0. At this point, the mean of the IQ-arm estimate begins to diverge rapidly from the true value, in contrast to the mean of the I-arm estimate, which asymptotically approaches the true value. (This contrast is more clearly seen when comparing the SNR curves.) *At low* P_c/N_0 *the I- and IQ-arms are essentially identical.* The expected values of both estimators approach a positive asymptote at low values of P_c/N_0 as a result of the positive bias in the estimator, which decreases with increasing L. Also, the effect of the loop bandwidth B_L on both estimators can be seen. This is the result of tracking loop jitter at low P_c/N_0 caused by the g_2 and g_4 terms in Eqs. (23) through (26). (When B_L gets larger, the phase error variance increases causing g_2 and g_4 to decrease, thus, biasing the estimates.)

The I-arm estimator SNR approaches an asymptotic value at high P_c/N_0 values as a result of the bias in the estimator. This asymptotic value increases proportionately with the estimation length, L. (For $P_c/N_0 \gg 1$ and $L \gg 1$, $\mathrm{SNR}(\widehat{R_I}) \rightarrow L/2$). To illustrate the meaning of the SNR in Fig. 3, consider the point $P_c/N_0 = 40$ dB-Hz where $\mathrm{SNR}(\hat{R}) = 24$ dB. This means that the expected value of $\hat{R}$ is 12 dB above the standard deviation of $\hat{R}$ (or $\sigma_{\hat{R}} = 10^{-1.2}\overline{\hat{R}}$). Thus, the tolerance on the estimator reading is $10,000\ (1 \pm 10^{-1.2})$, which is approximately equal to 40 ± 0.27 dB.

V. Conclusions

In this article, the expected value and variance of two P_c/N_0 estimators were derived and plotted. The results show that the I-arm estimator performs better than the IQ-arm estimator when P_d/N_0 is high, which occurs for high values of P_c/N_0 and significant modulation index. (In Fig. 2, this happens when P_c/N_0 exceeds 38 dB-Hz.) Also the effect of the positive estimator bias *and* loop jitter, which increases for higher values of B_L, is to degrade the performance of both estimators at low values of P_c/N_0.

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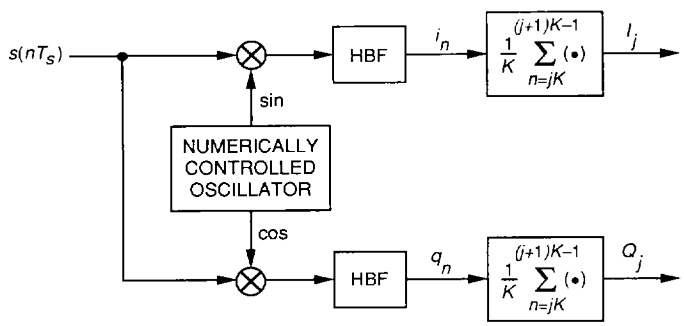

Fig. 1. Part of the DPLL preceding the P_c/N_0 estimator.

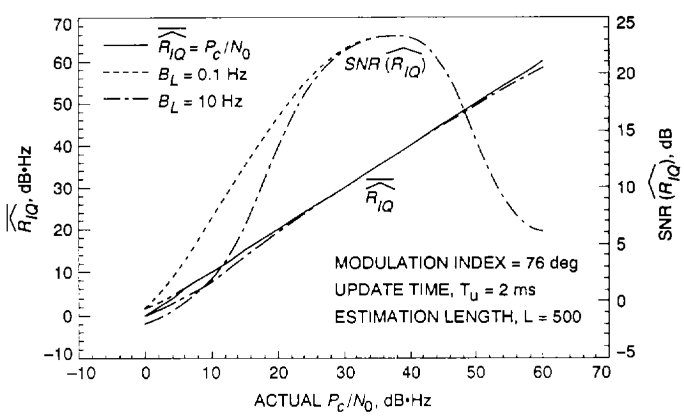

Fig. 2. Mean value and SNR of the IQ-arm estimator.

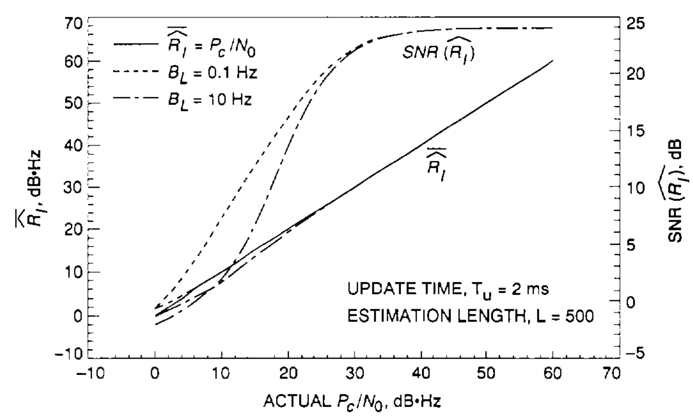

Fig. 3. Mean value and SNR of the I-arm estimator.

Appendix

Derivation of Equations

For ease of notation, define $g_n \stackrel{\Delta}{=} \overline{\cos^n \phi}$.

I. Evaluation of the First Moments

The first moments given in Eqs. (14) through (16) are derived, using Eqs. (3) and (7), as follows:

$$\overline{X} = \mathbf{E}\left\{ \frac{1}{L^2}\left[\sum_{j=0}^{LK-1} K^{-1}\sqrt{P_c}\cos\phi + \sum_{j=0}^{L-1} n_I(j) \right]^2 \right\}$$

$$= \mathbf{E}\left\{ P_c\cos^2\phi + \frac{2}{L}\sqrt{P_c}\cos\phi \sum_{j=0}^{L-1} n_I(j) + \frac{1}{L^2}\left[\sum_{j=0}^{L-1} n_I(j)\right]^2 \right\}$$

$$= P_c g_2 + \frac{N_0}{2LT_u} \tag{A-1}$$

The expression for $\overline{Y}$ contains products of d_n, $\mathsf{S}\mathrm{in}\,(\omega_{sc} nT_s + \theta)$ and $\cos\phi$. It is assumed that each of these terms is independent of the others. Here, $\mathsf{S}\mathrm{in}\,(\omega_{sc} nT_s + \theta)$ is represented by S_n (recall this is the square-wave subcarrier term). Thus, for example,

$$\mathbf{E}\left\{ d_n d_m \mathsf{S}_n S_m \cos^2\phi \right\} = \delta_{nm} g_2 \tag{A-2}$$

since $\mathsf{S}_n \mathsf{S}_n = 1$ and $\mathbf{E}\{d_n d_m\} = \delta_{nm}$, where δ_{nm} is the two-dimensional Kronecker delta function.

Using Eqs. (4) and (8) yields

$$\overline{Y} = \mathbf{E}\left\{ \frac{1}{L}\sum_{j=0}^{L-1} \left[\frac{P_d}{K^2} \sum_{n=jK}^{(j+1)K-1} \sum_{m=jK}^{(j+1)K-1} d_n d_m \mathsf{S}_n S_m \cos^2\phi \; + \frac{2\sqrt{P_d}}{K} \sum_{n=jK}^{(j+1)K-1} d_n \mathsf{S}_n \cos\phi\, n_Q(j) \; + n_Q^2(j) \right] \right\}$$

$$= \frac{P_d}{K} g_2 + \frac{1}{L}\sum_{j=0}^{L-1} \mathbf{E}\left\{ n_Q^2(j) \right\}$$

$$= \frac{P_d}{K} g_2 + \frac{N_0}{2T_u} \tag{A-3}$$

Using Eqs. (3) and (9) yields

$$\overline{Z} = \mathbf{E}\left\{\sum_{j=0}^{L-1}\left[\frac{1}{K}\sum_{n=jK}^{(j+1)K-1}\sqrt{P_c}\cos\phi + n_I(j)\right]^2\right\}$$

$$= \mathbf{E}\left\{LP_c\cos\phi + 2L\sqrt{P_c}\cos\phi\sum_{j=0}^{L-1}n_I(j) + \sum_{j=0}^{L-1}n_I^2(j)\right\}$$

$$= LP_c g_2 + \frac{LN_0}{2T_u} \tag{A-4}$$

II. Derivation of Second Moments

The second moment of X, using Eqs. (3) and (7), is:

$$\overline{X^2} = \mathbf{E}\left\{\frac{1}{L^4}\left[\sum_{j=0}^{LK-1}K^{-1}\sqrt{P_c}\cos\phi + \sum_{j=0}^{L-1}n_I(j)\right]^4\right\}$$

$$= g_4P_c^2 + 6P_c g_2\mathbf{E}\left\{\left[\frac{1}{L}\sum_{j=0}^{L-1}n_I(j)\right]^2\right\} + \mathbf{E}\left\{\frac{1}{L}\left[\sum_{j=0}^{L-1}n_I(j)\right]^4\right\}$$

$$= g_4P_c + \frac{6N_0}{2LT_u}g_2P_c + 3\left(\frac{N_0}{2LT_u}\right)^2 \tag{A-5}$$

Equations (A-6) through (A-11) are needed to compute $\mathbf{E}\{Y^2\}$.

$$\mathbf{E}\{d_n\} = 0 \tag{A-6}$$

$$\mathbf{E}\{d_n d_m d_p d_r\} = \begin{cases} 1 \text{ if } \begin{cases} n = m \neq p = r & \text{or} \\ n = p \neq m = r & \text{or} \\ n = r \neq m = r & \text{or} \\ n = m = p = r \end{cases} \\ 0 \text{ otherwise} \end{cases} \tag{A-7}$$

or, in a more mathematical form,

$$\mathbf{E}\{d_n d_m d_p d_r\} = (\delta_{nm}\delta_{pr} - \delta_{nmpr}) + (\delta_{np}\delta_{mr} - \delta_{nmpr}) + (\delta_{nr}\delta_{mr} - \delta_{nmpr}) + \delta_{nmpr}$$

$$= \delta_{nm}\delta_{pr} + \delta_{np}\delta_{mr} + \delta_{nr}\delta_{mr} - 2\delta_{nmpr} \tag{A-8}$$

where δ_{nmpr} is the four-dimensional Kronecker delta function, which equals unity only when $n = m = p = r$.

Using Eq. (A-8),

$$\sum_{n=jK}^{(j+1)K-1} \sum_{m=jK}^{(j+1)K-1} \sum_{n=jK}^{(j+1)K-1} \sum_{n=jK}^{(j+1)K-1} \mathbf{E}\{d_n d_m d_p d_r\} = K^2 + K^2 + K^2 - 2K$$

$$= 3K^2 - 2K \tag{A-9}$$

If n_i represents independent, Gaussian, zero mean random variables, then

$$\mathbf{E}\left\{n_1 n_2 n_3 n_4\right\} = \mathbf{E}\left\{n_1 n_2\right\} \mathbf{E}\left\{n_3 n_4\right\} + \mathbf{E}\left\{n_1 n_3\right\} \mathbf{E}\left\{n_2 n_4\right\} + \mathbf{E}\left\{n_1 n_4\right\} \mathbf{E}\left\{n_2 n_3\right\} \tag{A-10}$$

Using Eq. (A-10),

$$\mathbf{E}\left\{\sum_{j=0}^{L-1}\sum_{l=0}^{L-1} n_Q^2(j) n_Q^2(l)\right\} = \sum_{j=0}^{L-1}\sum_{l=0}^{L-1} \mathbf{E}\left\{n_Q^2(j) n_Q^2(l)\right\}$$

$$= \sum_{j=0}^{L-1}\sum_{l=0}^{L-1} \mathbf{E}\left\{n_Q^2(j)\right\} \mathbf{E}\left\{n_Q^2(l)\right\} + 2\left[\mathbf{E}\left\{n_Q(j) n_Q(l)\right\}\right]^2$$

$$= \sum_{j=0}^{L-1}\sum_{l=0}^{L-1} \left(\sigma_I^4 + 2\left[\delta_{jl}\sigma_I^2\right]^2\right)$$

$$= \left(L^2 + 2L\right)\sigma_I^4 \tag{A-11}$$

Using Eqs. (4), (8), (A-9), and (A-11) yields

$$\overline{Y^2} = \mathbf{E}\left\{\frac{1}{L^2}\sum_{j=0}^{L-1}\sum_{l=0}^{L-1}\left\{\left[\frac{P_d}{K^2}\sum_{n=jK}^{(j+1)K-1}\sum_{m=jK}^{(j+1)K-1} d_n d_m S_n S_m \cos^2\phi + \frac{2\sqrt{P_d}}{K}\sum_{n=jK}^{(j+1)K-1} d_n S_n \cos\phi\, n_Q(j) + n_Q^2(j)\right]\right.\right.$$

$$\left.\left.\times\left[\frac{P_d}{K^2}\sum_{p=lK}^{(l+1)K-1}\sum_{r=lK}^{(l+1)K-1} d_p d_r S_p S_r \cos^2\phi + \frac{2\sqrt{P_d}}{K}\sum_{p=lK}^{(l+1)K-1} d_p S_p \cos\phi\, n_Q(l) + n_Q^2(l)\right]\right\}\right\}$$

$$= \frac{P_d^2}{K^4}\left(3K^2 - 2K\right) g_4 + 2\left(\frac{P_d}{K^2} K g_2 \frac{N_0}{2T_u}\right) + \frac{4}{K^2} P_d \frac{K}{L} g_2 \frac{N_0}{2T_u} + \frac{1}{L^2}\left(L^2 + 2L\right)\left(\frac{N_0}{2T_u}\right)^2$$

$$= \frac{3K - 2}{K^3} g_4 P_d^2 + \frac{L+2}{LK} g_2 \frac{N_0}{T_u} P_d + \frac{L+2}{L}\left(\frac{N_0}{2T_u}\right)^2 \tag{A-12}$$

The second moment of Z is given, from Eqs. (3) and (9), by

$$\overline{Z^2} = \mathbf{E}\left\{\sum_{j=0}^{L-1}\sum_{l=0}^{L-1}\left[\frac{1}{K}\sum_{n=jK}^{(j+1)K-1}\sqrt{P_c}\cos\phi + n_I(j)\right]^2\left[\frac{1}{K}\sum_{m=lK}^{(l+1)K-1}\sqrt{P_c}\cos\phi + n_I(l)\right]^2\right\}$$

$$= \mathbf{E}\left\{\sum_{j=0}^{L-1}\sum_{l=0}^{L-1}\left[P_c\cos^2\phi + 2\sqrt{P_c}\cos\phi\, n_I(j) + n_I^2(j)\right]\left[P_c\cos^2\phi + 2\sqrt{P_c}\cos\phi\, n_I(l) + n_I^2(l)\right]\right\}$$

$$= L^2P_c^2g_4 + 2\left(L^2P_cg_2\frac{N_0}{2T_u}\right) + 4LP_cg_2\frac{N_0}{2T_u} + \left(L^2+2L\right)\left(\frac{N_0}{2T_u}\right)^2 \qquad \text{(A-13)}$$

The cross-correlation of X and Z is found as follows, from Eqs. (3), (7), and (9):

$$XZ = \frac{1}{L^2}\left(\sum_{j=0}^{LK-1}K^{-1}\sqrt{P_c}\cos\phi + \sum_{j=0}^{L-1}n_I(j)\right)^2\sum_{j=0}^{L-1}\left[\frac{1}{K}\sum_{n=jK}^{(j+1)K-1}\sqrt{P_c}\cos\phi + n_I(j)\right]^2$$

$$= \left\{P_c\cos^2\phi + \frac{2}{L}\sqrt{P_c}\cos\phi\sum_{j=0}^{L-1}n_I(j) + \frac{1}{L^2}\left[\sum_{j=0}^{L-1}n_I(j)\right]^2\right\}$$

$$\times\left[LP_c\cos^2\phi + 2L\sqrt{P_c}\cos\phi\sum_{j=0}^{L-1}n_I(j) + \sum_{j=0}^{L-1}n_I^2(j)\right] \qquad \text{(A-14)}$$

Using Eq. (A-10), the expectation of the fourth-order noise cross-terms in Eq. (A-14) is

$$\mathbf{E}\left\{\left(\sum_{j=0}^{L-1}n_I(j)\right)^2\sum_{j=0}^{L-1}n_I^2(j)\right\} = \sum_{i=0}^{L-1}\sum_{j=0}^{L-1}\sum_{k=0}^{L-1}\mathbf{E}\left\{n_I(i)n_I(j)n_I^2(k)\right\}$$

$$= \sum_{i=0}^{L-1}\sum_{j=0}^{L-1}\sum_{k=0}^{L-1}\mathbf{E}\left\{n_I(i)n_I(j)\right\}\mathbf{E}\left\{n_I^2(k)\right\} + 2\mathbf{E}\left\{n_I(i)n_I(k)\right\}\mathbf{E}\left\{n_I(j)n_I(k)\right\}$$

$$= \sum_{i=0}^{L-1}\sum_{j=0}^{L-1}\sum_{k=0}^{L-1}\delta_{ij}\sigma_I^2\sigma_I^2 + 2\delta_{ik}\sigma_I^2\delta_{jk}\sigma_I^2$$

$$= L^2\sigma_I^4 + 2L\sigma_I^4 \qquad \text{(A-15)}$$

where $\sigma_I^2 = N_0/2T_u$.

Thus, using Eqs. (A-14) and (A-15),

$$\overline{XZ} = LP_c^2 g_4 + LP_c g_2 \frac{N_0}{2T_u} + \frac{4}{L} P_c g_2 \frac{LN_0}{2T_u} + \frac{1}{L} P_c g_2 \frac{LN_0}{2T_u} + \frac{1}{L^2}\mathbf{E}\left\{ \left[\sum_{j=0}^{L-1} n_I(j) \right]^2 \sum_{j=0}^{L-1} n_I^2(j) \right\}$$

$$= LP_c^2 g_4 + (L+5) P_c g_2 \frac{N_0}{2T_u} + \left(1 + \frac{2}{L}\right)\left(\frac{N_0}{2T_u}\right)^2 \tag{A-16}$$

The cross-correlation of X and Y is found in a similar way, by making use of Eqs. (3), (4), (7), (8), (A-2), and (A-10),

$$\overline{XY} = \mathbf{E}\left\{ \frac{1}{L^2}\left(\sum_{j=0}^{LK-1} K^{-1}\sqrt{P_c}\cos\phi + \sum_{j=0}^{L-1} n_I(j) \right)^2 \right.$$

$$\left. \times \frac{1}{L}\sum_{j=0}^{L-1}\left[\frac{P_d}{K^2} \sum_{n=jK}^{(j+1)K-1} \sum_{m=jK}^{(j+1)K-1} d_n d_m \varsigma_n S_m \cos^2\phi + \frac{2\sqrt{P_d}}{K} \sum_{n=jK}^{(j+1)K-1} d_n \varsigma_n \cos\phi \, n_Q(j) + n_Q^2(j) \right] \right\}$$

$$= \mathbf{E}\left\{ \left[P_c\cos^2\phi + \frac{2}{L}\sqrt{P_c}\cos\phi \sum_{j=0}^{L-1} n_I(j) + \frac{1}{L^2}\left(\sum_{j=0}^{L-1} n_I(j) \right)^2 \right] \right.$$

$$\left. \times \frac{1}{L}\sum_{j=0}^{L-1}\left[\frac{P_d}{K^2} \sum_{n=jK}^{(j+1)K-1} \sum_{m=jK}^{(j+1)K-1} d_n d_m \varsigma_n S_m \cos^2\phi + \frac{2\sqrt{P_d}}{K} \sum_{n=jK}^{(j+1)K-1} d_n \varsigma_n \cos\phi \, n_Q(j) + n_Q^2(j) \right] \right\}$$

$$= \frac{P_d P_c}{K} g_4 + P_c g_2 \frac{N_0}{2T_u} + \frac{P_d}{LK} g_2 \frac{N_0}{2T_u} + \frac{1}{L}\left(\frac{N_0}{2T_u}\right)^2 \tag{A-17}$$