Towards Optimum Demodulation of Bandwidth-Limited and Low SNR Square-Wave Subcarrier Signals

Y. Feria  
Communications Systems Research Section  

W. Hurd  
Radio Frequency and Microwave Subsystems Section

The optimum phase detector is presented for tracking square-wave subcarriers that have been bandwidth limited to a finite number of harmonics. The phase detector is optimum in the sense that the loop signal-to-noise ratio (SNR) is maximized and, hence, the rms phase tracking error is minimized. The optimum phase detector is easy to implement and achieves substantial improvement. Also presented are the optimum weights to combine the signals demodulated from each of the harmonics. The optimum weighting provides SNR improvement of 0.1 to 0.15 dB when the subcarrier loop SNR is low (15 dB) and the number of harmonics is high (8 to 16).

I. Introduction

This work was motivated by the need for near-optimum demodulation of the extremely weak signal received from the Galileo spacecraft. This demonstration is accomplished in the buffered telemetry demodulator (BTD). Since the BTD is a software demodulator, it is practical to tailor the processing more closely to the Galileo signal conditions than would be practical in other systems, such as the Block V Receiver.

A limitation of the BTD is that the input signal has been recorded by the full spectrum recorder and contains only the first four harmonics of the originally transmitted square-wave subcarrier. The subcarrier phase detector initially implemented in the BTD uses a windowing technique similar to that used in the Advanced Receiver II and the Block V Receiver [1] but modified for the four-harmonic case [3]. There is a parameter, $W_{sc}$, that is analogous to the fractional window width in a square-wave subcarrier phase detector. As shown in Fig. 1, this phase detector results in a degradation (loss in symbol signal-to-noise ratio (SNR) due to harmonic truncation and phase tracking error), which does not monotonically decrease as the number of harmonics is increased.\(^1\) In fact, when the tracking error is large, and when the harmonics are combined using the usual $1/n$ weighting for the $n$th harmonic, it is sometimes better to use only four harmonics than to use all harmonics. This suggests two things: First, it tells us that

\(^{1}\)Based on work by D. Rogstad, Tracking Systems and Applications Section, and Y. Feria, Communications Systems Research Section, Jet Propulsion Laboratory, Pasadena, California, October 1994.
the phase detector may not be using the harmonics optimally. Second, it indicates that the demodulated harmonics may not be optimally combined.

The phase detector used in [3] is derived from a window used on a square-wave subcarrier loop. This phase detector may not be the optimum for a finite-harmonic subcarrier. As a previous work [2] indicates, the higher harmonics get larger phase noise jitters. Therefore, the effective signal amplitude on the \( n \)th harmonic is no longer \( 1/n \) but some number smaller than that. The optimum weights to combine the demodulated harmonics should account for the SNR losses due to the loop.

II. Optimum Phase Detector

Here we derive a phase detector (PD) that is optimum in the sense that the loop SNR is maximized. To show the derivation, let us first take a look at the current phase detector used in the BTD. The current phase detector is the product of the combined in-phase signals \( \sqrt{P_d d_k \cos \phi_c (8/\pi^2) \sum_{n=0}^{L-1} (1/(2n + 1))^2 \cos[(2n + 1)\phi_{sc}]} \) and the combined quadrature signals \( \sqrt{P_d d_k \cos \phi_c (8/\pi^2) \sum_{n=0}^{L-1} w_n (1/(2n + 1)) \sin[(2n + 1)\phi_{sc}]} \) where the \( w_n \) are the weights used to combine the quadrature signals and, in the current BTD, these weights are

\[
w_n = \frac{\sin[(2n + 1)(\pi/2)W_{sc}]}{2n + 1}
\]

The loop SNR using the current BTD is derived as\(^2\)

\[
\rho_{sc} = \frac{\alpha \beta^2}{\gamma B_{sc}} \frac{P_d}{N_0} \left( \alpha + \frac{1}{2E_s/N_0} \right)^{-1}
\]

where

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\(^2\) H. Tsou, personal communication, Communications Systems Research Section, Jet Propulsion Laboratory, Pasadena, California, October 1994.
\[ \alpha = \frac{8}{\pi^2} \sum_{n=0}^{L-1} \frac{1}{(2n+1)^2} \]

\[ \beta = \frac{8}{\pi^2} \sum_{n=0}^{L-1} w_n \]

\[ \gamma = \frac{8}{\pi^2} \sum_{n=0}^{L-1} w_n^2 \]

where \( L \) is the total number of harmonics used in the phase detector, \( P_d/N_0 \) is the data power-to-noise ratio, \( E_s/N_0 \) is the symbol SNR, and \( B_{sc} \) is the subcarrier loop bandwidth.

Now in order to maximize the subcarrier loop SNR, \( \rho_{sc} \), let \( w_k, k = 0, \ldots, L - 1 \), be unknown and \( \alpha \) be the same as before, and differentiate the loop SNR, \( \rho_{sc} \), with respect to \( w_k \) and set the expression to zero. We then have

\[
\frac{\partial \rho_{sc}}{\partial w_k} = \frac{2\beta \gamma - 2\beta^2 w_k}{\gamma^2} \frac{1}{B_{sc} N_0} \frac{P_d}{\alpha + 1/(2E_s/N_0)} = 0
\]

(1)

Since \( P_d/N_0 \neq 0 \), \( \alpha \neq 0 \), and \( \gamma, B_{sc} \) are finite, the above is zero if and only if

\[ \gamma - \beta w_k = 0 \]

That is,

\[ \sum_{n=0}^{L-1} w_n^2 - \sum_{n=0}^{L-1} w_n w_k = 0 \]

or

\[ \sum_{n=0}^{L-1} w_n (w_n - w_k) = 0, \text{ for all } k \]

which implies that

\[ w_n = w_k, \text{ for all } n \text{ and } k \]

The conclusion is that the optimum weights to combine the quadrature signals in the phase detector are a constant for all (finite) harmonics. Note that, for infinite harmonics, the parameters \( \beta \) and \( \gamma \) do not converge; therefore, the above weights cannot be used for square waves. When the optimum weights are used in the phase detector, the loop SNR becomes
\[ \rho_{sc} = \frac{L}{B_{sc}} \frac{P_d}{N_0} \frac{\alpha}{\alpha + 1/(2E_s/N_0)} \] (2)

Using the optimum weights in the phase detector (called the optimum phase detector), we can improve the loop SNR by 9.5 dB over the current BTD with window size = 1, and by 1.1 dB over the current BTD with window size = 1/4 (see Fig. 2). The same figure also shows that, using the optimum phase detector, the loop SNR obtained by using only one harmonic is higher than that using the current BTD with the window size being either 1 or 1/2. Note that when we use only one harmonic in the optimum phase detector, we may still use all the available harmonics to demodulate the subcarrier.

![Fig. 2. Comparison in loop SNR using the optimum phase detector and the current BTD.](image)

Degradations due to a finite-harmonic subcarrier loop can be computed using the expressions given in [2]. Degradations as a function of the number of harmonics are shown in Fig. 3. Clearly, we can observe that, using the optimum phase detector, we obtained a lower degradation with more harmonics. This agrees with our intuition.

With the increase of the loop SNR, that is, with the increase of the number of harmonics, the linear region shrinks. See the normalized S-curves shown in Fig. 4. As the number of harmonics approaches infinity, the linear region of the S-curve approaches zero. In other words, this optimum phase detector is only for a finite number of harmonics.

**III. Optimum Combining Weights in Demodulation**

The demodulated harmonics are currently combined with the weight 1/n for the nth harmonic. These weights are optimum if each of the harmonics of the subcarrier is demodulated with the same phase jitter. In our case, however, we know that if the first harmonic has a phase jitter with a variance of \(\sigma^2\), then the nth harmonic would have a variance of \((n\sigma)^2\). The weight 1/n is no longer optimum.
To derive the optimum combining weights, we assume that the harmonics are combined using unknown weights $b_n$. We then express the SNR in terms of the weights. Differentiating the SNR with respect to the weights and setting it to zero, we should obtain the optimum weights.

The optimum weight to combine the demodulated $(2n + 1)$th harmonics is derived in the Appendix as

$$b_n = \frac{\cos[(2n + 1)\phi_{sc}]}{\cos \phi_{sc}} \frac{1}{2n + 1}$$  \hspace{1cm} (3)

When $\phi_{sc}$ is assumed to have a Tikhonov distribution,

$$\cos(2n + 1)\phi_{sc} = \int_0^{\pi} \frac{\exp[(1/4)\rho_{sc} \cos \phi_{sc}]}{\pi I_0(\rho_{sc}/4)} \cos \left[ \frac{2n + 1}{2} \phi_{sc} \right] d\phi_{sc}$$

Assuming that we have 4, 8, and 16 harmonics, the degradations in symbol SNR versus the subcarrier loop SNR, using the optimum weights and the usual $1/n$ weights, are compared in Figs. 5 through 7.

**IV. Approximated Optimum Combining Weights in Demodulation**

Since the cosine function is “smooth” in the vicinity of zero, for small phase jitters, $n\phi_{sc}$, the expected value of $\cos(n\phi_{sc})$ can be approximated by

$$\mathcal{E}\{\cos(n\phi_{sc})\} \approx 1 - n^2 \frac{\sigma^2}{2}$$  \hspace{1cm} (4)

The approximated optimum weights are
Note that this approximation is valid only when $\pi \varphi_{sc}$ is small. Using the approximated optimum weights for four harmonics, the symbol SNR degradation is only slightly more than that using the optimum weight as shown in Fig. 5.
V. Conclusion

We presented an optimum way of tracking and demodulating a finite-harmonic subcarrier. We found an optimum phase detector in the sense that the loop SNR is maximized. The more harmonics used, the higher the loop SNR we obtain. However, the linear region of the phase error signal shrinks with the increase of the number of harmonics. Therefore, this optimum phase detector is only appropriate for a finite number of harmonics. Using the optimum phase detector, the loop SNR is about 9.5 dB higher than that of the current BTD using window size 1, and is about 1 dB higher than that of the current BTD with window size 1/4.

For demodulation, we found the optimum combining weights that account for the losses due to the phase jitter. Compared to using the usual $1/n$ combining weights, the use of the optimum combining weights can improve the symbol SNR by 0.1 to 0.15 dB at a low loop SNR (15 dB) and a high number of harmonics (8 to 16).

References


Appendix

Derivation of the Optimum Combining Weights in Demodulation

After each of the harmonics of the subcarrier is demodulated, the signals from each harmonic demodulation need to be combined. Assume that the combining weight for the \((2n + 1)\)th harmonic is \(b_n\); the signal amplitude at the \(l\)th symbol is

\[
s = \sqrt{P_d} d_l \cos \phi_c \frac{2}{\pi} \sum_{n=0}^{L-1} b_n \frac{1}{2n+1} \cos[(2n + 1)\phi_{sc}]
\]

(A-1)

where \(P_d\) is the data power, and \(\phi_c\) and \(\phi_{sc}\) are the phase offsets of the carrier and subcarrier, respectively.

The noise variance is

\[
\sigma^2 = \sum_{n=0}^{L-1} b_n^2 N_0 R_{sym}^\frac{1}{2}
\]

(A-2)

Taking the ratio of the average signal power and the noise variance, we have the average symbol SNR of the combined signal:

\[
SNR = \frac{\mathcal{E}\{s^2\}}{2\sigma^2}
\]

\[
= \frac{\mathcal{E}\{4\pi^2 P_d \cos^2 \phi_c (\sum_{n=0}^{L-1} b_n \cos[(2n + 1)\phi_{sc}])^2/(2n + 1)^2\}}{\sum_{n=0}^{L-1} b_n^2 N_0 R_{sym}}
\]

(A-3)

Differentiating the symbol SNR with respect to \(b_k\), \(k = 0, \cdots, L - 1\), we have

\[
\frac{\partial (SNR)}{\partial b_k} = \frac{P_d \cos^2 \phi_c (4\pi^2)}{(\sum_{n=0}^{L-1} b_n^2 N_0 R_{sym})^2} \mathcal{E} \left\{ 2 \sum_{n=0}^{L-1} b_n \cos[(2n + 1)\phi_{sc}]/(2n + 1) \cos[(2k + 1)\phi_{sc}]/(2k + 1) \right\} \sum_{n=1}^{L-1} b_n^2 N_0 R_{sym}
\]

\[
- \left[ \sum_{n=0}^{L-1} b_n \cos[(2n + 1)\phi_{sc}]/(2n + 1) \right]^2 2b_k N_0 R_{sym}
\]

(A-4)

Simplifying the above equation, we have

\[
\mathcal{E} \left\{ \cos[(2k + 1)\phi_{sc}]/(2k + 1) \sum_{n=0}^{L-1} b_n^2 - \sum_{n=0}^{L-1} b_n \cos[(2n + 1)\phi_{sc}] b_k \right\} = 0
\]

(A-5)
Let $k = 0$ and $b_0 = 1$; we have

$$
\cos \phi_{sc} \sum_{n=0}^{L-1} b_n^2 - \sum_{n=0}^{L-1} b_n \frac{\cos[(2n + 1)\phi_{sc}]}{2n + 1} = 0 \quad (A-6)
$$

That is,

$$
\sum_{n=0}^{L-1} b_n \left[ \cos \phi_{sc} b_n - \frac{\cos[(2n + 1)\phi_{sc}]}{2n + 1} \right] = 0 \quad (A-7)
$$

Finally, solving for $b_n$, we have the optimum combining weights,

$$
b_n = \frac{\cos[(2n + 1)\phi_{sc}]}{\cos \phi_{sc}} \frac{1}{2n + 1} \quad (A-8)
$$