Towards Optimum Demodulation of Bandwidth-Limited and Low SNR Square-Wave Subcarrier Signals

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The optimum phase detector is presented for tracking square-wave subcarriers that have been bandwidth limited to a finite number of harmonics. The phase detector is optimum in the sense that the loop signal-to-noise ratio (SNR) is maximized and, hence, the rms phase tracking error is minimized. The optimum phase detector is easy to implement and achieves substantial improvement. Also presented are the optimum weights to combine the signals demodulated from each of the harmonics. The optimum weighting provides SNR improvement of 0.1 to 0.15 dB when the subcarrier loop SNR is low (15 dB) and the number of harmonics is high (8 to 16).

I. Introduction

This work was motivated by the need for near-optimum demodulation of the extremely weak signal received from the Galileo spacecraft. This demonstration is accomplished in the buffered telemetry demodulator (BTD). Since the BTD is a software demodulator, it is practical to tailor the processing more closely to the Galileo signal conditions than would be practical in other systems, such as the Block V Receiver.

A limitation of the BTD is that the input signal has been recorded by the full spectrum recorder and contains only the first four harmonics of the originally transmitted square-wave subcarrier. The subcarrier phase detector initially implemented in the BTD uses a windowing technique similar to that used in the Advanced Receiver II and the Block V Receiver [1] but modified for the four-harmonic case [3]. There is a parameter, W_{sc}, that is analogous to the fractional window width in a square-wave subcarrier phase detector. As shown in Fig. 1, this phase detector results in a degradation (loss in symbol signal-to-noise ratio (SNR) due to harmonic truncation and phase tracking error), which does not monotonically decrease as the number of harmonics is increased.[1] In fact, when the tracking error is large, and when the harmonics are combined using the usual $1/n$ weighting for the nth harmonic, it is sometimes better to use only four harmonics than to use all harmonics. This suggests two things: First, it tells us that

[1] Based on work by D. Rogstad, Tracking Systems and Applications Section, and Y. Feria, Communications Systems Research Section, Jet Propulsion Laboratory, Pasadena, California, October 1994.

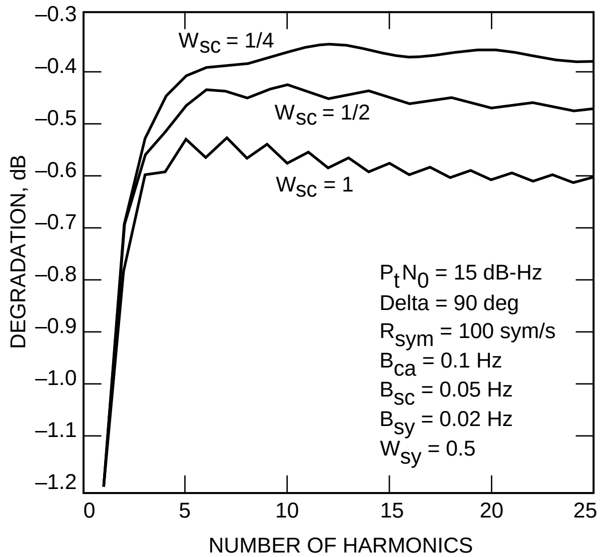

Fig. 1. Degradation as a function of the number of harmonics, using the current BTD.

the phase detector may not be using the harmonics optimally. Second, it indicates that the demodulated harmonics may not be optimally combined.

The phase detector used in [3] is derived from a window used on a square-wave subcarrier loop. This phase detector may not be the optimum for a finite-harmonic subcarrier. As a previous work [2] indicates, the higher harmonics get larger phase noise jitters. Therefore, the effective signal amplitude on the nth harmonic is no longer $1/n$ but some number smaller than that. The optimum weights to combine the demodulated harmonics should account for the SNR losses due to the loop.

II. Optimum Phase Detector

Here we derive a phase detector (PD) that is optimum in the sense that the loop SNR is maximized. To show the derivation, let us first take a look at the current phase detector used in the BTD. The current phase detector is the product of the combined in-phase signals $\sqrt{P_d} d_k \cos\phi_c (8/\pi^2) \sum_{n=0}^{L-1} (1/(2n + 1)^2) \cos[(2n+1)\phi_{sc}]$ and the combined quadrature signals $\sqrt{P_d} d_k \cos\phi_c (8/\pi^2) \sum_{n=0}^{L-1} w_n (1/(2n + 1)) \sin[(2n+1)\phi_{sc}]$ where the w_n are the weights used to combine the quadrature signals and, in the current BTD, these weights are

$$w_n = \frac{\sin[(2n+1)(\pi/2)W_{sc}]}{2n+1}$$

The loop SNR using the current BTD is derived as[2]

$$\rho_{sc} = \frac{\alpha\beta^2}{\gamma B_{sc}} \frac{P_d}{N_0} \left(\alpha + \frac{1}{2E_s/N_0}\right)^{-1}$$

where

[2] H. Tsou, personal communication, Communications Systems Research Section, Jet Propulsion Laboratory, Pasadena, California, October 1994.

$$\alpha = \frac{8}{\pi^2} \sum_{n=0}^{L-1} \frac{1}{(2n+1)^2}$$

$$\beta = \frac{8}{\pi^2} \sum_{n=0}^{L-1} w_n$$

$$\gamma = \frac{8}{\pi^2} \sum_{n=0}^{L-1} w_n^2$$

where L is the total number of harmonics used in the phase detector, P_d/N_0 is the data power-to-noise ratio, E_s/N_0 is the symbol SNR, and B_{sc} is the subcarrier loop bandwidth.

Now in order to maximize the subcarrier loop SNR, ρ_{sc}, let w_k, $k = 0, \cdots, L-1$, be unknown and α be the same as before, and differentiate the loop SNR, ρ_{sc}, with respect to w_k and set the expression to zero. We then have

$$\begin{aligned} \frac{\partial \rho_{sc}}{\partial w_k} &= \frac{2\beta\gamma - 2\beta^2 w_k}{\gamma^2} \frac{1}{B_{sc}} \frac{P_d}{N_0} \frac{\alpha}{\alpha + 1/(2E_s/N_0)} \\ &= 0 \end{aligned} \tag{1}$$

Since $P_d/N_o \neq 0$, $\alpha \neq 0$, and γ, B_{sc} are finite, the above is zero if and only if

$$\gamma - \beta w_k = 0$$

That is,

$$\sum_{n=0}^{L-1} w_n^2 - \sum_{n=0}^{L-1} w_n w_k = 0$$

or

$$\sum_{n=0}^{L-1} w_n(w_n - w_k) = 0, \text{ for all } k$$

which implies that

$$w_n = w_k, \text{ for all } n \text{ and } k$$

The conclusion is that the optimum weights to combine the quadrature signals in the phase detector are a constant for all (finite) harmonics. Note that, for infinite harmonics, the parameters β and γ do not converge; therefore, the above weights cannot be used for square waves. When the optimum weights are used in the phase detector, the loop SNR becomes

$$\rho_{sc} = \frac{L}{B_{sc}} \frac{P_d}{N_0} \frac{\alpha}{\alpha + 1/(2E_s/N_0)} \tag{2}$$

Using the optimum weights in the phase detector (called the optimum phase detector), we can improve the loop SNR by 9.5 dB over the current BTD with window size = 1, and by 1.1 dB over the current BTD with window size = 1/4 (see Fig. 2). The same figure also shows that, using the optimum phase detector, the loop SNR obtained by using only one harmonic is higher than that using the current BTD with the window size being either 1 or 1/2. Note that when we use only one harmonic in the optimum phase detector, we may still use all the available harmonics to demodulate the subcarrier.

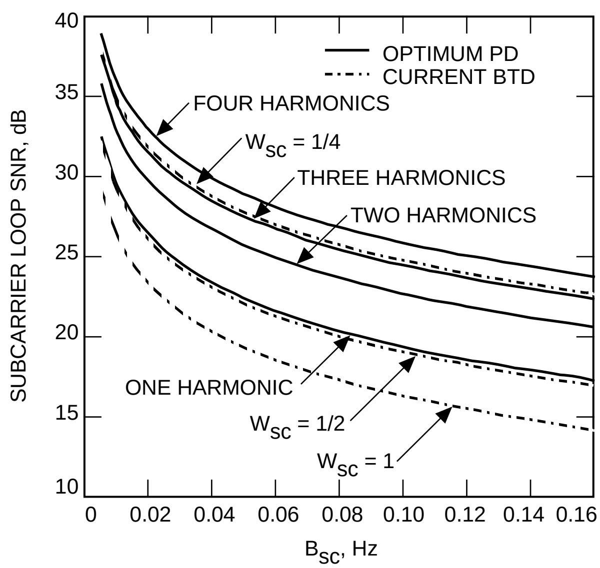

Fig. 2. Comparison in loop SNR using the optimum phase detector and the current BTD.

Degradations due to a finite-harmonic subcarrier loop can be computed using the expressions given in [2]. Degradations as a function of the number of harmonics are shown in Fig. 3. Clearly, we can observe that, using the optimum phase detector, we obtained a lower degradation with more harmonics. This agrees with our intuition.

With the increase of the loop SNR, that is, with the increase of the number of harmonics, the linear region shrinks. See the normalized S-curves shown in Fig. 4. As the number of harmonics approaches infinity, the linear region of the S-curve approaches zero. In other words, this optimum phase detector is only for a finite number of harmonics.

III. Optimum Combining Weights in Demodulation

The demodulated harmonics are currently combined with the weight $1/n$ for the nth harmonic. These weights are optimum if each of the harmonics of the subcarrier is demodulated with the same phase jitter. In our case, however, we know that if the first harmonic has a phase jitter with a variance of σ^2, then the nth harmonic would have a variance of $(n\sigma)^2$. The weight $1/n$ is no longer optimum.

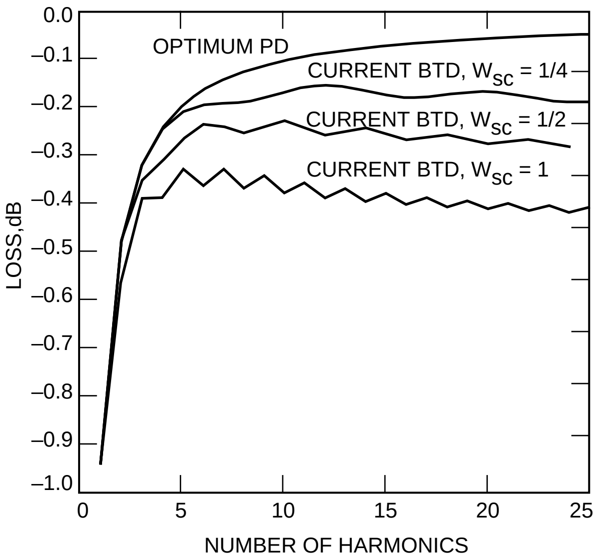

Fig. 3. Degradation as a function of the number of harmonics, using the optimum weights.

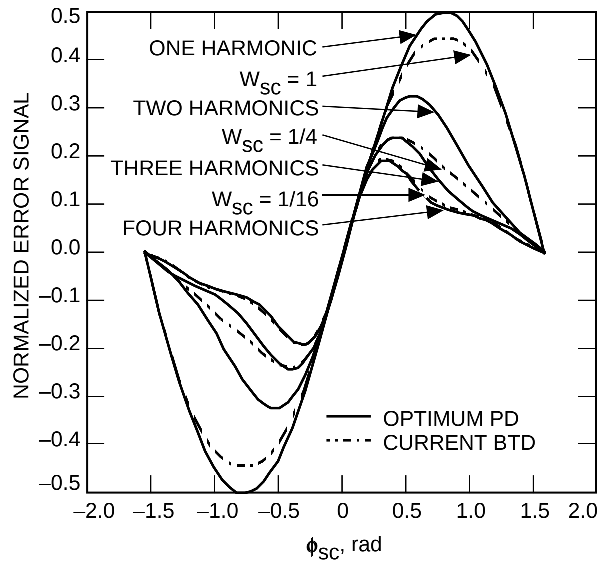

Fig. 4. Normalized S-curves.

To derive the optimum combining weights, we assume that the harmonics are combined using unknown weights b_n. We then express the SNR in terms of the weights. Differentiating the SNR with respect to the weights and setting it to zero, we should obtain the optimum weights.

The optimum weight to combine the demodulated $(2n+1)$th harmonics is derived in the Appendix as

$$b_n = \frac{\overline{\cos[(2n+1)\phi_{sc}]}}{\overline{\cos\phi_{sc}}} \, \frac{1}{2n+1} \tag{3}$$

When ϕ_{sc} is assumed to have a Tikhonov distribution,

$$\overline{\cos(2n+1)\phi_{sc}} = \int_0^\pi \frac{\exp[(1/4)\rho_{sc}\cos\phi_{sc}]}{\pi I_0(\rho_{sc}/4)} \cos\left[\frac{2n+1}{2}\phi_{sc}\right] d\phi_{sc}$$

Assuming that we have 4, 8, and 16 harmonics, the degradations in symbol SNR versus the subcarrier loop SNR, using the optimum weights and the usual $1/n$ weights, are compared in Figs. 5 through 7.

IV. Approximated Optimum Combining Weights in Demodulation

Since the cosine function is "smooth" in the vicinity of zero, for small phase jitters, $n\phi_{sc}$, the expected value of $\cos(n\phi_{sc})$ can be approximated by

$$\mathcal{E}\{\cos(n\phi_{sc})\} \approx 1 - n^2\frac{\sigma^2}{2} \tag{4}$$

The approximated optimum weights are

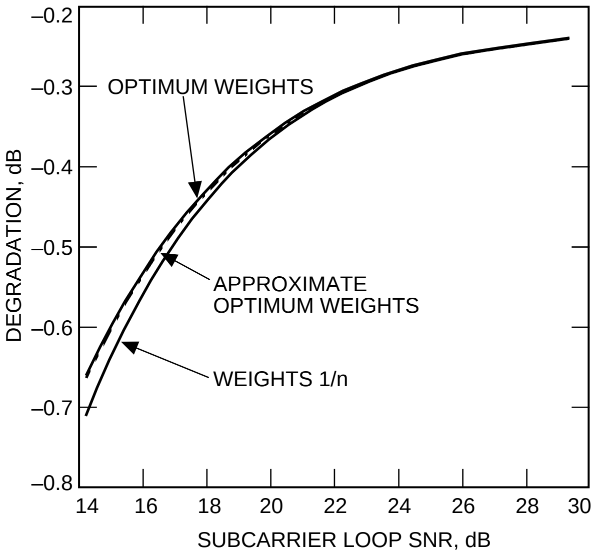

Fig. 5. Symbol SNR degradation when using optimum weights (four harmonics).

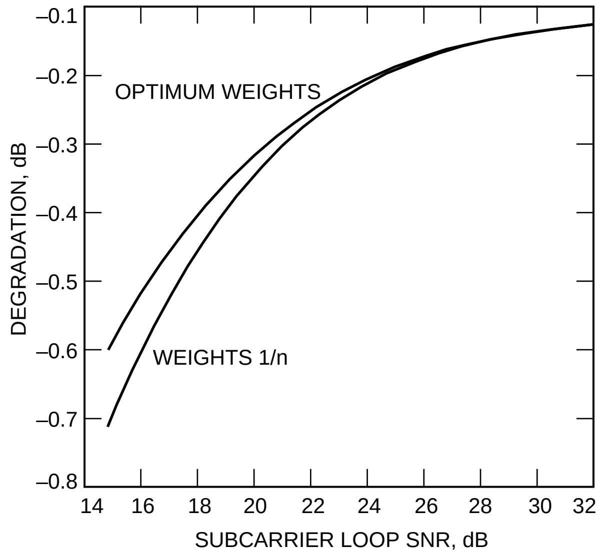

Fig. 6. Symbol SNR degradation when using optimum weights (eight harmonics).

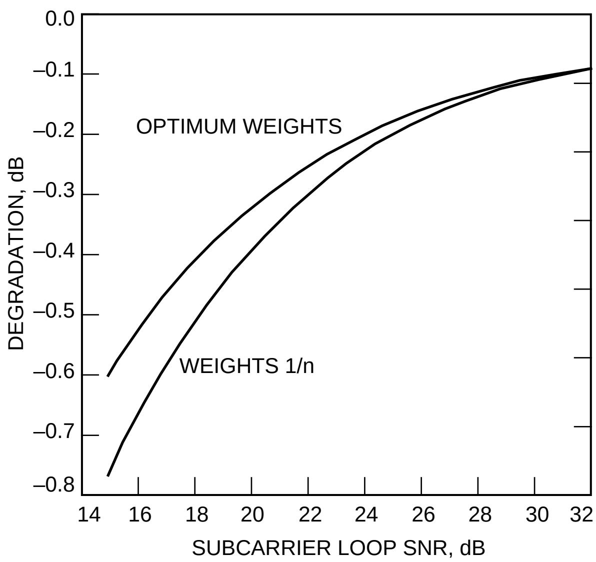

Fig. 7. Symbol SNR degradation when using optimum weights (16 harmonics).

$$b_n \approx \frac{1-(2n+1)^2\sigma^2}{1-\sigma^2/2}\,\frac{1}{2n+1} \tag{5}$$

Note that this approximation is valid only when $n\phi_{sc}$ is small. Using the approximated optimum weights for four harmonics, the symbol SNR degradation is only slightly more than that using the optimum weight as shown in Fig. 5.

V. Conclusion

We presented an optimum way of tracking and demodulating a finite-harmonic subcarrier. We found an optimum phase detector in the sense that the loop SNR is maximized. The more harmonics used, the higher the loop SNR we obtain. However, the linear region of the phase error signal shrinks with the increase of the number of harmonics. Therefore, this optimum phase detector is only appropriate for a finite number of harmonics. Using the optimum phase detector, the loop SNR is about 9.5 dB higher than that of the current BTD using window size 1, and is about 1 dB higher than that of the current BTD with window size 1/4.

For demodulation, we found the optimum combining weights that account for the losses due to the phase jitter. Compared to using the usual $1/n$ combining weights, the use of the optimum combining weights can improve the symbol SNR by 0.1 to 0.15 dB at a low loop SNR (15 dB) and a high number of harmonics (8 to 16).

References

[1] W. J. Hurd and S. Aguirre, "A Method to Dramatically Improve Subcarrier Tracking," *The Telecommunications and Data Acquisition Progress Report 42-86, April–June 1986*, Jet Propulsion Laboratory, Pasadena, California, pp. 103–110, August 15, 1986.

[2] Y. Feria, T. Pham, and S. Townes, "Degradation in Finite-Harmonic Subcarrier Demodulation," *The Telecommunications and Data Acquisition Progress Report 42-121, January–March 1995*, Jet Propulsion Laboratory, Pasadena, California, pp. 78–86, May 15, 1995.

[3] H. Tsou, B. Shah, R. Lee, and S. Hinedi, "A Functional Description of the Buffered Telemetry Demodulation (BTD)," *The Telecommunications and Data Acquisition Progress Report 42-112, October–December 1992*, Jet Propulsion Laboratory, Pasadena, California, pp. 50–73, February 15, 1993.

Appendix

Derivation of the Optimum Combining Weights in Demodulation

After each of the harmonics of the subcarrier is demodulated, the signals from each harmonic demodulation need to be combined. Assume that the combining weight for the $(2n+1)$th harmonic is b_n; the signal amplitude at the lth symbol is

$$s = \sqrt{P_d} d_l \cos\phi_c \frac{2}{\pi} \sum_{n=0}^{L-1} b_n \frac{1}{2n+1} \cos[(2n+1)\phi_{sc}] \tag{A-1}$$

where P_d is the data power, and ϕ_c and ϕ_{sc} are the phase offsets of the carrier and subcarrier, respectively. The noise variance is

$$\sigma^2 = \sum_{n=0}^{L-1} b_n^2 \frac{N_0}{2} R_{sym} \tag{A-2}$$

Taking the ratio of the average signal power and the noise variance, we have the average symbol SNR of the combined signal:

$$\begin{aligned} SNR &= \frac{\mathcal{E}\{s^2\}}{2\sigma^2} \\ &= \frac{\mathcal{E}\{(4/\pi^2) P_d \cos^2\phi_c (\sum_{n=0}^{L-1} b_n \cos[(2n+1)\phi_{sc}]/(2n+1))^2\}}{\sum_{n=0}^{L-1} b_n^2 N_0 R_{sym}} \end{aligned} \tag{A-3}$$

Differentiating the symbol SNR with respect to b_k, $k = 0, \cdots, L-1$, we have

$$\begin{aligned} \frac{\partial(SNR)}{\partial b_k} &= \frac{P_d \overline{\cos^2\phi_c}(4/\pi^2)}{(\sum_{n=0}^{L-1} b_n^2 N_0 R_{sym})^2} \mathcal{E}\left\{ \left[2 \sum_{n=0}^{L-1} b_n \frac{\cos[(2n+1)\phi_{sc}]}{2n+1} \; \frac{\cos[(2k+1)\phi_{sc}]}{2k+1} \right] \sum_{n=1}^{L-1} b_n^2 N_0 R_{sym} \right. \\ &\quad \left. - \left[\sum_{n=0}^{L-1} b_n \frac{\cos[(2n+1)\phi_{sc}]}{2n+1} \right]^2 2 b_k N_0 R_{sym} \right\} \\ &= 0 \end{aligned} \tag{A-4}$$

Simplifying the above equation, we have

$$\mathcal{E}\left\{ \frac{\cos[(2k+1)\phi_{sc}]}{2k+1} \sum_{n=0}^{L-1} b_n^2 - \sum_{n=0}^{L-1} b_n \frac{\cos[(2n+1)\phi_{sc}]}{2n+1} b_k \right\} = 0 \tag{A-5}$$

Let $k = 0$ and $b_0 = 1$; we have

$$\overline{\cos\phi_{sc}} \sum_{n=0}^{L-1} b_n^2 - \sum_{n=0}^{L-1} b_n \frac{\overline{\cos[(2n+1)\phi_{sc}]}}{2n+1} = 0 \tag{A-6}$$

That is,

$$\sum_{n=0}^{L-1} b_n \left[\overline{\cos\phi_{sc}} b_n - \frac{\overline{\cos[(2n+1)\phi_{sc}]}}{2n+1} \right] = 0 \tag{A-7}$$

Finally, solving for b_n, we have the optimum combining weights,

$$b_n = \frac{\overline{\cos[(2n+1)\phi_{sc}]}}{\overline{\cos\phi_{sc}}} \frac{1}{2n+1} \tag{A-8}$$