Performance of Residual Carrier Array-Feed Combining in Correlated Noise

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An array feed combining system for the recovery of signal-to-noise ratio (SNR) loss due to antenna reflector deformation has been implemented and is currently being evaluated on the Jet Propulsion Laboratory's 34-meter DSS-13 antenna. In this system, the defocused signal field captured by a focal plane array feed is recovered using real-time signal-processing and signal-combining techniques. The current signal-processing and signal-combining algorithms are optimum under the assumption that the white Gaussian noise processes in the received signals from different array elements are mutually uncorrelated. Experimental data at DSS 13 indicate that these noise processes are indeed mutually correlated. The main result of this article is an analytical derivation of the actual SNR performance of the current suboptimal signal-combining algorithm in this correlated-noise environment. The analysis here shows that the combined signal SNR can either be improved or degraded depending on the relation between the array signal and noise correlation coefficient phases. Further performance improvement will require the development of signal-combining methods that take into account the correlated noises.

I. Introduction

Operation of deep-space communication networks at higher carrier frequencies has the advantage of greater antenna gains as well as increased bandwidths for enhancing telemetry capabilities. However, the use of higher frequencies also has certain disadvantages. These include more stringent antenna pointing requirements and larger receiving antenna signal-to-noise ratio (SNR) losses due to mechanical deformations of large reflector surfaces. These SNR losses become more significant at higher frequencies when carrier wavelengths become smaller than the mechanical imperfections of the reflector. This is the case in the Jet Propulsion Laboratory's Deep Space Network plan to employ Ka-band (32-GHz) communications using 34- and 70-meter receiving antennas.

An array feed combining system for the recovery of the SNR loss due to antenna reflector deformation has been proposed and analyzed in [1]. In this system, a focal plane feed array is used to collect the defocused signal fields. All the signal power captured by the feed array is then recovered using real-time signal-processing and signal-combining techniques. In phase and quadrature, baseband signal samples are obtained from the downconverted received signal of each of the array feed elements and then are recombined after application of combiner weights. The optimum combiner weights that maximize the combined signal SNR were derived in [1] under the assumption that the white Gaussian noise processes in the received signals from different array elements are mutually uncorrelated. These optimum weights depend on unknown signal and noise parameters that need to be estimated. The work in [1] proposed to estimate the optimum weights from the observed residual carrier received signal samples using a maximum likelihood (ML) estimation of these unknown parameters. The actual combined signal SNR in this uncorrelated-noise environment was also derived in [1] when the estimated weights were used in place of the optimum weight coefficients.

The array feed combining system is currently being evaluated at the JPL DSS-13 34-meter antenna. Although the work in [1] assumed mutually uncorrelated-noise processes, experimental data [2] indicate that the white noise processes in the received signals from different feed elements are indeed correlated, with correlation coefficients of the order of 0.01 under clear-sky conditions. Since the noise in each of the array feed element signals consists of receiver white noise plus noise due to background radiation, this small correlation is conjectured to be caused by near-field atmospheric background noise. Although the observed correlation in [2] is quite small in the current array feed combining system, future planned improvements in the the receiver noise temperature could magnify the effect of atmospheric background noise and result in considerably higher amounts of correlation. Thus, it is important to determine the performance of the signal-combining system proposed in [1] when the white Gaussian noise processes in the signals from different array elements are mutually correlated. That is the objective of this article, which provides an exact analysis of the combined signal SNR performance in this correlated-noise environment.

The performance analysis here considers only the signal combining algorithm proposed in [1], which was designed to operate in the environment where the white Gaussian noise processes in the signals from different array elements are mutually uncorrelated. The effect of the correlation is twofold. First, the optimum combining weights developed in [1] are no longer optimal in this correlated-noise environment. The other effect of this correlation is on the resulting combined-signal SNR performance. The analysis here shows that the combined-signal SNR can be either improved or degraded depending on the relation between the array signal and noise correlation coefficient phases. Further performance improvement will require effective combining systems that take into account the correlations between the array feed element noise processes. Our work on this problem is still in progress.

II. Array Feed Signals and Combining Algorithm

Consider a K-element array and the NASA Deep Space Network standard residual carrier modulation with a binary phase shift key (BPSK)-modulated square-wave subcarrier [3]. The received signal from each array element is downconverted to baseband and sampled. The combining system proposed in [1] uses only the residual carrier portion of the received signal spectrum to estimate the unknown parameters in the combiner weights. The full spectrum modulated signals from the array elements, which contain both the modulated sidebands as well as the residual carrier spectrum, are subsequently combined. In this system [1], the higher-bandwidth primitive baseband signal samples are low-pass filtered by averaging successive blocks of M_B samples to yield a full-spectrum signal stream B for each array element. Additive white Gaussian noise is assumed to be present in the primitive baseband signal sequences from each of the array elements. Let

$$y_k(i_B) = V_k[\cos\delta + j\, s(i_B)\, \sin\delta] + n_k(i_B), \quad i_B = 1, 2, \cdots \tag{1}$$

denote the stream B signal samples from the kth array element. The complex signal parameters

$$V_k = |V_k|\, \mathrm{e}^{j\,\theta_k} \tag{2}$$

represent the unknown signal amplitude and phase parameters induced by the antenna reflector deformation. Moreover, δ is the modulation index, $s(i_B) = \pm 1$ is the transmitted data, and $\{n_k(i_B)\}$ is the zero-mean white Gaussian noise corruption in the stream B signal samples from the kth array element. The primitive baseband signal samples are also more narrowly low-pass filtered by averaging successive blocks of M_A samples to yield a residual carrier signal stream A for each array element. Clearly $M_A > M_B$, and $\eta = M_A/M_B$ is the ratio of the bandwidth of stream B to stream A. Let

$$u_k(i_A) = V_k \cos \delta + m_k(i_A), \quad i_A = 1, 2, \cdots \tag{3}$$

denote the stream A signal samples from the kth array element. Here $\{m_k(i_A)\}$ is the zero-mean white Gaussian noise corruption in the stream A signal samples from the kth array element.

Let $\underline{A}^T$ and $\underline{A}^\dagger$ denote the transpose and complex conjugate transpose of the matrix $\underline{A}$, respectively. The white noise sequences corresponding to different array elements are assumed to be correlated. To specify these correlations, consider

$$\begin{aligned} \underline{n}(i_B) &= (n_1(i_B), \cdots, n_K(i_B))^T \\ \underline{m}(i_A) &= (m_1(i_A), \cdots, m_K(i_A))^T \end{aligned}$$

Then $\{\underline{n}(i_B)\}$ and $\{\underline{m}(i_A)\}$ are each sequences of independent identically distributed (i.i.d.) zero-mean complex Gaussian random vectors of dimension K. The respective covariance matrices

$$\begin{aligned} \underline{R}_B &= \{r_{Bkj}\} = \mathrm{E}\left[\underline{n}(i_B)\underline{n}(i_B)^\dagger\right] \\ \underline{R}_A &= \{r_{Akj}\} = \mathrm{E}\left[\underline{m}(i_A)\underline{m}(i_A)^\dagger\right] \end{aligned}$$

of $\underline{n}(i_B)$ and $\underline{m}(i_A)$ then specify the mutual correlations between the white noises in the signal streams from different array elements. For example, r_{Bkj} is the correlation between the noise variables $n_k(i_B)$ and $n_j(i_B)$ in the stream B signals from the kth and jth array elements, respectively. Moreover, define

$$\rho_{Bkj} = \frac{r_{Bkj}}{\sqrt{r_{Bkk} r_{Bjj}}} = |\rho_{Bkj}|\, \mathrm{e}^{j\,\varphi_{Bkj}} \tag{4}$$

to be the correlation coefficient between the noise samples $n_k(i_B)$ and $n_j(i_B)$. We shall assume as in [1] that the complex Gaussian noise samples $n_k(i_B)$ and $m_k(i_A)$ each has statistically independent real and imaginary parts of equal variance. This assumption is not required for the following analysis, but is made to maintain consistency with the results reported in [1]. So, $2\sigma_{Bk}^2 = r_{Bkk}$ and $2\sigma_{Ak}^2 = r_{Akk}$ are the respective variances of $n_k(i_B)$ and $m_k(i_A)$, where σ_{Bk}^2 and σ_{Ak}^2 are the respective variances of the real or imaginary parts. Because of the different averaging rates in streams A and B on the primitive baseband signals, it follows that $\underline{R}_B = \eta \underline{R}_A$. Finally, these different averaging rates also imply that $\underline{m}(i_A)$ is independent of $\underline{n}(i_B)$ provided that $i_A < i_B$ and the samples averaged to yield $\underline{m}(i_A)$ occurred prior to the samples averaged to yield $\underline{n}(i_B)$.

The complex combining weight coefficients w_k, $1 \le k \le K$, given by

$$w_k = \frac{V_k^*}{2\eta\sigma_{Ak}^2} = \frac{V_k^*}{2\sigma_{Bk}^2} \tag{5}$$

were shown in [1] to maximize the SNR of the combiner output in the uncorrelated-noise case, resulting in a maximum possible SNR equal to

$$\gamma = \sum_{k=1}^{K} \frac{|V_k|^2}{2\eta\sigma_{Ak}^2} = \sum_{k=1}^{K} \frac{|V_k|^2}{2\sigma_{Bk}^2} \tag{6}$$

That is, the optimum attainable SNR in the uncorrelated-noise case is equal to the sum of the SNRs of each of the feed array element outputs. The signal parameters V_k and the noise variances σ_{Ak}^2 are unknown parameters that need to be estimated to obtain an estimate of the optimum weight coefficients.

Assume that these unknown parameters are not random. The estimates for V_k and σ_{Ak}^2 developed in [1] are univariate sampling estimates based on the stream A residual carrier signal samples $\{u_k(i_A)\}$. In the uncorrelated-noise case, the stream A signal samples from different array elements are statistically independent. Hence, estimates of the weight coefficients w_k based on these estimates of V_k and σ_{Ak}^2 are also mutually independent. However, in the correlated-noise environment, these signal streams are no longer mutually independent and, hence, the resulting estimates for w_k are also no longer independent. In order to put this dependence in the proper perspective for the SNR performance analysis below, we will describe the estimation techniques developed in [1] in terms of multivariate sampling estimates based on the vector of stream A signal samples $\{\underline{u}(i_A)\}$ where

$$\underline{u}(i_A) = (u_1(i_A), \cdots, u_K(i_A))^T$$

Instead of estimating V_k directly, consider estimating $X_k = V_k \cos\delta$. Define

$$\underline{X} = (X_1, \cdots, X_K)^T$$

Then it follows from Eq. (3) that $\{\underline{u}(i_A)\}$ is an i.i.d. sequence of complex Gaussian random vectors with mean $\underline{X}$ and covariance matrix $\underline{R}_A$. It follows from multivariate statistical analysis [4,5] that, based on observations $\{\underline{u}(i_A - 1), \cdots, \underline{u}(i_A - L)\}$,

$$\underline{\hat{X}}(i_A) = \left(\hat{X}_1(i_A), \cdots, \hat{X}_K(i_A)\right)^T = \frac{1}{L} \sum_{l=i_A-L}^{i_A-1} \underline{u}(l) \tag{7}$$

is the ML sample mean estimate of $\underline{X}$ and

$$\underline{\hat{R}}_A(i_A) = \frac{1}{L-2} \sum_{l=i_A-L}^{i_A-1} \left[\underline{u}(l) - \underline{\hat{X}}(i_A)\right] \left[\underline{u}(l) - \underline{\hat{X}}(i_A)\right]^\dagger \tag{8}$$

is equal to $(L-1)/(L-2)$ times the corresponding sample covariance estimate of $\underline{R}_A$. The approach in [1] uses $\hat{X}_k(i_A)$ as the estimate of X_k and consequently $\hat{V}_k(i_A) = \hat{X}_k(i_A)/\cos\delta$ as the estimate of V_k. Moreover, the kth diagonal element $2\hat{\sigma}_{Ak}^2(i_A)$ of $\underline{\hat{R}}_A(i_A)$ is used in [1] as the estimate of $2\sigma_{Ak}^2$, which is the kth diagonal element of $\underline{R}_A$. Finally, the estimate given by

$$\hat{w}_k(i_A) = \frac{\hat{V}_k^*(i_A)}{2\eta\hat{\sigma}_{Ak}^2(i_A)} = \frac{\hat{X}_k^*(i_A)}{2\eta\cos\delta\,\hat{\sigma}_{Ak}^2(i_A)} \tag{9}$$

was shown in [1] to be an unbiased estimate of the optimum combining weight coefficient w_k given by Eq. (5) in the uncorrelated-noise case. These weight coefficient estimates are used in a sliding window structure to produce the following combiner output sequence:

$$z(i_B) = \sum_{k=1}^{K} \hat{w}_k \left(\tilde{i}_A\right) y_k(i_B) \tag{10}$$

where $\tilde{i}_A$ is the largest integer less than i_B, so that the residual carrier signal samples $\{u_k(\tilde{i}_A - 1), \ldots, u_k(\tilde{i}_A - L)\}$ used for estimating $\hat{w}_k(\tilde{i}_A)$ occur before the full-spectrum signal sample $y_k(i_B)$.

III. SNR Performance Analysis

The objective is to determine the actual SNR of the combiner output in the correlated-noise environment. From Eqs. (1) and (10), the combiner output can be written as

$$z(i_B) = s_c(i_B) + n_c(i_B) \tag{11}$$

where

$$s_c(i_B) = \sum_{k=1}^{K} \hat{w}_k \left(\tilde{i}_A\right) V_k \, \mathrm{e}^{j\, s(i_B)\, \delta} \tag{12}$$

and

$$n_c(i_B) = \sum_{k=1}^{K} \hat{w}_k \left(\tilde{i}_A\right) n_k(i_B) \tag{13}$$

are the signal and noise components, respectively. Since the residual carrier signal samples used for the estimates $\hat{w}_k(\tilde{i}_A)$ occur prior to the full spectrum signal samples $y_k(i_B)$, and since $\{m_k(i_A)\}$ and $\{n_j(i_B)\}$ are i.i.d. sequences, it follows that $\hat{w}_k(\tilde{i}_A)$ and $n_j(i_B)$ are uncorrelated random variables for every k and j. Each $n_j(i_B)$ has zero mean. It then follows from Eqs. (13) and (12) that $n_c(i_B)$ also has zero mean and is, moreover, uncorrelated with $s_c(i_B)$. Let $\mathrm{Var}[Z] = \mathrm{E}\left[|Z - \mathrm{E}[Z]|^2\right]$ denote the variance of a complex random variable Z. Thus it follows from Eq. (11) that the actual SNR of the combiner signal output $z(i_B)$ given by Eq. (10) can be written as

$$\gamma_{ML} = \frac{|\mathrm{E}[z(i_B)]|^2}{\mathrm{Var}[z(i_B)]} = \frac{|\mathrm{E}[s_c(i_B)]|^2}{\mathrm{Var}[s_c(i_B)] + \mathrm{Var}[n_c(i_B)]} \tag{14}$$

It is well known [4,5], that $\underline{\hat{X}}(i_A)$ and $\underline{\hat{R}}_A(i_A)$ are statistically independent and that $2(L-2)\hat{\sigma}^2_{Ak}(i_A)/\sigma^2_{Ak}$ has a chi-square distribution with $2(L-1)$ degrees of freedom. As a result of these properties, it follows from Eq. (9) in a derivation similar to that in [1] that, for $1 \le k \le K$,

$$\mathrm{E}\left[\hat{w}_k(\tilde{i}_A)\right] = w_k \tag{15}$$

where $\{w_k\}$ are the optimal combining weights given by Eq. (5). That is, the estimated weight coefficients are unbiased as in the uncorrelated-noise case [1]. It then follows from Eqs. (12), (15), (5), and (6) that for both the correlated- and uncorrelated-noise cases,

$$|\mathrm{E}[s_c(i_B)]| = |\mathrm{e}^{j\,s(i_B)\,\delta} \sum_{k=1}^{K} \mathrm{E}[\hat{w}_k(\tilde{i}_A)]V_k| = \gamma \tag{16}$$

Consider next the variances of $s_c(i_B)$ and $n_c(i_B)$ in Eq. (14). Using Eqs. (12) and (15), we have

$$\mathrm{Var}[s_c(i_B)] = \sum_{k=1}^{K}\sum_{j=1}^{K} \mathrm{E}\left[(\hat{w}_k(\tilde{i}_A) - w_k)(\hat{w}_j(\tilde{i}_A) - w_j)^* V_k V_j^*\right] \tag{17}$$

where w_k is given by Eq. (5). Consider first the case when the Gaussian noise processes in the signals from different array elements are mutually uncorrelated. Since $\hat{w}_k(\tilde{i}_A)$ and $\hat{w}_j(\tilde{i}_A)$ are pairwise independent for $k \neq j$ in this case, the variance of $s_c(i_B)$ can be written as

$$\mathrm{Var}_U[s_c(i_B)] = \sum_{k=1}^{K} \mathrm{Var}\left[\hat{w}_k\left(\tilde{i}_A\right)\right] |V_k|^2 \tag{18}$$

Let

$$\beta_1 = 2\,\mathcal{R}e\left\{\sum_{k=1}^{K}\sum_{j=k+1}^{K} V_k V_j^* \left\{\mathrm{E}\left[\hat{w}_k\left(\tilde{i}_A\right)\hat{w}_j^*\left(\tilde{i}_A\right)\right] - w_k w_j^*\right\}\right\} \tag{19}$$

Combining Eqs. (17), (18), and (19) then yields

$$\mathrm{Var}[s_c(i_B)] = \mathrm{Var}_U[s_c(i_B)] + \beta_1 \tag{20}$$

Recall that $n_c(i_B)$ has zero mean and $\hat{w}_k(\tilde{i}_A)$ is statistically independent of $n_j(i_B)$ for all k and j. Then, similar to the derivation leading to Eq. (20), we can write

$$\mathrm{Var}[n_c(i_B)] = \sum_{k=1}^{K}\sum_{j=1}^{K} \mathrm{E}\left[\hat{w}_k\left(\tilde{i}_A\right)\hat{w}_j^*\left(\tilde{i}_A\right)\right]\, \mathrm{E}\left[n_k(i_B)n_j^*(i_B)\right] = \mathrm{Var}_U[n_c(i_B)] + \beta_2 \tag{21}$$

where

$$\beta_2 = 2\,\mathcal{R}e\left\{\sum_{k=1}^{K}\sum_{j=k+1}^{K} \mathrm{E}\left[\hat{w}_k\left(\tilde{i}_A\right)\hat{w}_j^*\left(\tilde{i}_A\right)\right]\, \mathrm{E}\left[n_k\left(i_B\right)n_j^*\left(i_B\right)\right]\right\} \tag{22}$$

and where

$$\mathrm{Var}_U[n_c(i_B)] = \sum_{k=1}^{K} \mathrm{E}\left[\left| \hat{w}_k\left(\tilde{i}_A\right) \right|^2 \right] \mathrm{Var}\left[\left| n_k(i_B) \right|^2 \right]$$

is the variance of $n_c(i_B)$ in the uncorrelated-noise case. It then follows from Eqs. (14) and (16) that the actual SNR of the combiner output in the uncorrelated-noise case is given by

$$\gamma_{ML}^{U} = \frac{\gamma^2}{\mathrm{Var}_U[s_c(i_B)] + \mathrm{Var}_U[n_c(i_B)]} \tag{23}$$

So it follows from Eqs. (14), (16), (20), (21), and (23) that

$$\gamma_{ML} = \gamma_{ML}^{U} \left(\frac{1}{1+d} \right) \tag{24}$$

where

$$d = \frac{\beta_1 + \beta_2}{\mathrm{Var}_U[s_c(i_B)] + \mathrm{Var}_U[n_c(i_B)]} \tag{25}$$

The factor $1/(1+d)$ in Eq. (24) represents the improvement in SNR caused by the correlation between the noises in the signals received from different array elements. Note in particular that β_1 and β_2 can be either positive or negative in value. Hence, an SNR improvement is obtained when d is negative and a degradation is obtained otherwise.

Expressions for $\mathrm{Var}_U[s_c(i_B)]$ and $\mathrm{Var}_U[n_c(i_B)]$ are given in [1]. Thus, we need only determine β_1 and β_2 to obtain d and thereby obtain an expression for γ_{ML} from Eq. (24). In order to do this, we need only obtain an expression for $\mathrm{E}[\hat{w}_k(\tilde{i}_A)\hat{w}_j^*(\tilde{i}_A)]$ when $k \neq j$. Using the property that $\underline{\hat{X}}(i_A)$ is statistically independent of $\underline{\hat{R}}_A(i_A)$, it then follows from Eqs. (9), (7), and (8) that, for $k \neq j$,

$$\mathrm{E}\left[\hat{w}_k\left(\tilde{i}_A\right) \hat{w}_j^*\left(\tilde{i}_A\right) \right] = \frac{1}{4\eta^2 \cos^2 \delta} \mathrm{E}\left[\hat{X}_k^*\left(\tilde{i}_A\right) \hat{X}_j\left(\tilde{i}_A\right) \right] \mathrm{E}\left[\frac{1}{\hat{\sigma}_{Ak}^2(\tilde{i}_A)\hat{\sigma}_{Aj}^2(\tilde{i}_A)} \right] \tag{26}$$

Since $\underline{\hat{X}}(i_A)$ has mean $\underline{X}$ and covariance matrix $\frac{1}{L}\underline{R}_A = \frac{1}{\eta L}\underline{R}_B$ [5], it follows that

$$\mathrm{E}\left[\hat{X}_k^*\left(\tilde{i}_A\right) \hat{X}_j\left(\tilde{i}_A\right) \right] = \frac{1}{\eta L} r_{Bkj}^* + X_k^* X_j \tag{27}$$

Recall that $2(L-2)\hat{\sigma}_{Ak}^2(\tilde{i}_A)$ is the kth diagonal element of the matrix $(L-2)\underline{\hat{R}}_A(\tilde{i}_A)$. Let

$$\underline{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}$$

be a 2×2 matrix where A_{11} and A_{22} are the kth and jth diagonal elements, respectively, and A_{12} is the element in the kth row and jth column of $(L-2)\underline{\hat{R}}_A(\tilde{i}_A)$. So we have

$$\mathrm{E}\left[\frac{1}{\hat{\sigma}_{Ak}^2(\tilde{i}_A)\hat{\sigma}_{Aj}^2(\tilde{i}_A)}\right] = 4(L-2)^2\mathrm{E}\left[\frac{1}{A_{11}A_{22}}\right] \tag{28}$$

Complex multivariate statistical sampling theory [4,5] has shown that $\underline{A}$ has the same distribution as that of $\sum_{i=1}^{L-1} \underline{Z}_i\,\underline{Z}_i^\dagger$ where $\{\underline{Z}_i\}$ is a sequence of i.i.d. zero-mean complex Gaussian random vectors with covariance matrix $\underline{\Sigma}$ given by

$$\underline{\Sigma} = \begin{bmatrix} r_{Akk} & r_{Akj} \\ r_{Akj}^* & r_{Ajj} \end{bmatrix} = \frac{1}{\eta}\begin{bmatrix} r_{Bkk} & r_{Bkj} \\ r_{Bkj}^* & r_{Bjj} \end{bmatrix} \tag{29}$$

This type of distribution is called a complex Wishart distribution [4,5], with parameters $\underline{\Sigma}$ and $(L-1)$. Denote the determinant and trace of a matrix $\underline{A}$ by $|\underline{A}|$ and $\mathrm{tr}(\underline{A})$, respectively. Then if $L \geq 4$, the joint Wishart probability density of (A_{11}, A_{22}, A_{12}) is given by [4]

$$p(A_{11}, A_{22}, A_{12}) = \frac{\left(A_{11}A_{22} - |A_{12}|^2\right)^{L-3}}{\pi\Gamma(L-1)\Gamma(L-2)|\underline{\Sigma}|^{L-1}} \exp\left[-\mathrm{tr}(\underline{\Sigma}^{-1}\underline{A})\right] \tag{30}$$

for $A_{11}, A_{22} \geq 0$ and $|A_{12}|^2 \leq A_{11}A_{22}$, where $\Gamma(x)$ is the gamma function. The derivation in Appendix A obtains the expression given by Eq. (A-5) for $\mathrm{E}[1/A_{11}A_{22}]$ starting from Eq. (30). Define for $L \geq 4$ and $0 \leq x < 1$,

$$f_L(x) = (L-2)(1-x)^{L-3}\sum_{k=0}^{\infty}\binom{k+L-3}{k}\frac{x^k}{k+L-2} \tag{31}$$

Assume that the correlation coefficients between noise components of the kth and jth array element outputs ρ_{Bkj} given by Eq. (4) are always less than one in magnitude. Then, by using Eqs. (28), (49), (31), and (27), Eq. (26) can be written as

$$\mathrm{E}\left[\hat{w}_k\left(\tilde{i}_A\right)\hat{w}_j^*\left(\tilde{i}_A\right)\right] = f_L\left(|\rho_{Bkj}|^2\right)\left[\frac{1}{\eta L\cos^2\delta}\,\frac{\rho_{Bkj}^*}{2\sigma_{Bk}\sigma_{Bj}} + \frac{V_k^*V_j}{4\sigma_{Bk}^2\sigma_{Bj}^2}\right] \tag{32}$$

When $|\rho_{Bkj}| < 1$ and $L \geq 4$, we obtain, by using Eqs. (2), (4), (5), and (32) in Eqs. (19) and (22),

$$\begin{aligned}\beta_1 + \beta_2 &= 2\sum_{k=1}^{K}\sum_{j=k+1}^{K}\Bigg\{f_L\left(|\rho_{Bkj}|^2\right)\Bigg[\frac{|\rho_{Bkj}|^2}{\eta L\cos^2\delta} + \frac{|V_k|^2|V_j|^2}{4\sigma_{Bk}^2\sigma_{Bj}^2} \\ &\quad + \left(1 + \frac{1}{\eta L\cos^2\delta}\right)\frac{|V_k||V_j|}{2\sigma_{Bk}\sigma_{Bj}}\,|\rho_{Bkj}|\cos\left(\vartheta_{kj} - \varphi_{Bkj}\right)\Bigg] - \frac{|V_k|^2|V_j|^2}{4\sigma_{Bk}^2\sigma_{Bj}^2}\Bigg\}\end{aligned} \tag{33}$$

where φ_{Bkj} is the phase of the correlation coefficient ρ_{Bkj} between $n_k(i_B)$ and $n_k(i_B)$ and where $\vartheta_{kj} = \theta_k - \theta_j$ is the phase difference between the signal components of the kth and jth array elements. Finally, by using Eqs. (44) and (48) of [1] for $\mathrm{Var}_U[s_c(i_B)]$ and $\mathrm{Var}_U[n_c(i_B)]$, respectively, Eq. (25) can be written as

$$d = \frac{\beta_1 + \beta_2}{\left(\dfrac{L-2}{L-3}\right)[\gamma + (\gamma + K)/\eta L \cos^2 \delta] + \left(\dfrac{1}{L-3}\right)\sum_{k=1}^{K}\left(\dfrac{|V_k|^2}{2\sigma_{Bk}^2}\right)^2} \tag{34}$$

where $\beta_1 + \beta_2$ is given by Eq. (33) and γ is given by Eq. (6). In order to arrive at an explicit expression for γ_{ML}, we note that Eqs. (44) and (48) of [1] in Eq. (23) give

$$\gamma_{ML}^U = \frac{\gamma^2}{\left(\dfrac{L-2}{L-3}\right)[\gamma + (\gamma + K)/\eta L \cos^2 \delta] + \left(\dfrac{1}{L-3}\right)\sum_{k=1}^{K}\left(\dfrac{|V_k|^2}{2\sigma_{Bk}^2}\right)^2} \tag{35}$$

So the actual SNR of the combiner output in the correlated-noise case can be determined from Eqs. (24), (34), and (35) when $L \geq 4$ and $|\rho_{Bkj}| < 1$. The two measures of particular interest in understanding the SNR performance are $1/(1+d)$ and γ_{ML}/γ. The measure $1/(1+d)$ represents the gain in SNR caused by the correlation between the array element noises and will be referred to as the *correlation gain*. In the uncorrelated-noise case, γ/γ_{ML} represents the loss in SNR due to the combining algorithm since γ is the maximum possible achievable SNR. We shall adopt the same measure here and define γ_{ML}/γ as the *combining gain* for ease of comparison with the uncorrelated-noise case. The combining gain also represents the gain in SNR over the sum of SNRs of the individual array element outputs.

Let us examine the characteristics of the SNR performance. In the uncorrelated-noise case, the actual SNR performance γ_{ML}^U converges to the maximum possible SNR achievable γ as the number of samples L approaches infinity. It is interesting to also examine the combining gain in the correlated-noise case as the number of samples approaches infinity. It is shown in Appendix B that $f_L(x) \to 1$ as $L \to \infty$ for $0 \leq x < 1$. Assume that the pairwise noise correlation coefficients ρ_{Bkj} are all less than one in magnitude. Then, taking the limit as $L \to \infty$ in Eqs. (34) and (33) yields

$$\lim_{L\to\infty} d = \frac{2}{\gamma}\sum_{k=1}^{K}\sum_{j=k+1}^{K}\frac{|V_k||V_j|}{2\sigma_{Bk}\sigma_{Bj}}|\rho_{Bkj}|\cos(\vartheta_{kj} - \varphi_{Bkj}) \tag{36}$$

So the limiting value of d can also be of either sign, positive or negative. In fact, the limiting value is always negative if $\vartheta_{kj} - \varphi_{Bkj} = \pi$ for all $k \neq j$, and always positive if $\vartheta_{kj} - \varphi_{Bkj} = 0$ for all $k \neq j$. It then follows from Eq. (24) that as $L \to \infty$, the limiting value of the actual SNR performance γ_{ML} in the correlated-noise case can be either greater or smaller than the maximum possible SNR γ in the uncorrelated-noise case, depending on the relation between the signal and noise correlation phases. This is not really that surprising, since the maximum possible SNR performance in the correlated-noise case is generally not equal to γ.

Bounds on the actual SNR performance γ_{ML} that depend on a fewer number of parameters than the exact expression are also useful. We shall derive upper and lower bounds that depend only on the maximum magnitude of the noise correlation coefficients and on γ, the sum of the SNRs of the individual array element outputs. We first note the following inequalities derived in [1] for this purpose:

$$\frac{\gamma^2}{K} = \frac{1}{K}\left(\sum_{k=1}^{K}\frac{|V_k|^2}{2\sigma_{Bk}^2}\right)^2 \leq \sum_{k=1}^{K}\left(\frac{|V_k|^2}{2\sigma_{Bk}^2}\right)^2 \leq \gamma^2 \tag{37}$$

Similar to the left-hand inequality of Eq. (37), we have

$$\left(\sum_{k=1}^{K} \frac{|V_k|}{\sqrt{2\sigma_{Bk}^2}} \right)^2 \leq K \left(\sum_{k=1}^{K} \frac{|V_k|^2}{2\sigma_{Bk}^2} \right) = K\gamma \tag{38}$$

Applying the left-hand inequality of Eq. (37), the inequality of Eq. (38) gets the following upper bounds:

$$2 \sum_{k=1}^{K} \sum_{j=k+1}^{K} \frac{|V_k|^2 |V_j|^2}{4\sigma_{Bk}^2 \sigma_{Bj}^2} \leq \gamma^2 \left(1 - \frac{1}{K}\right) \tag{39}$$

and

$$2 \sum_{k=1}^{K} \sum_{j=k+1}^{K} \frac{|V_k||V_j|}{2\sigma_{Bk}\sigma_{Bj}} \leq (K-1)\gamma \tag{40}$$

Let

$$\rho_{max} = \max_{k \neq j} |\rho_{Bkj}|$$

be the maximum magnitude of the correlation coefficients between array element noise components. Note from Eq. (33) that the worst-case phase resulting in the largest possible d occurs when $\vartheta_{kj} - \varphi_{Bkj} = 0$ for all $k \neq j$. Hence, application of the left-hand inequality in Eq. (37), the inequalities of Eqs. (39) and (40), and the bounds of Eq. (B-7) on $f_L(x)$ given in Appendix B yields the following upper bound on the worst-case d:

$$d \leq \frac{(L-2)(K-1)\rho_{max}\left[\gamma + (K\rho_{max} + \gamma)/\eta L \cos^2\delta\right] + \gamma^2(1-1/K)}{(L-2)\left[\gamma + (\gamma + K)/\eta L \cos^2\delta\right] + \gamma^2/K} \tag{41}$$

Similarly, since the best-case phase resulting in the most negative possible d occurs when $\vartheta_{kj} - \varphi_{Bkj} = \pi$ for all $k \neq j$, the following lower bound on the best-case d can be obtained:

$$d \geq -\frac{(L-2)(K-1)\rho_{max}\gamma\left[1 + 1/\eta L \cos^2\delta\right]}{(L-2)\left[\gamma + (\gamma + K)/\eta L \cos^2\delta\right] + \gamma^2/K} \tag{42}$$

Finally, using the inequalities of Eq. (37) in Eq. (35) yields the following bounds on the actual SNR performance γ_{ML}^U in the uncorrelated-noise case:

$$\gamma_{ML}^U \leq \frac{(L-3)\gamma^2}{(L-2)\left[\gamma + (\gamma + K)/\eta L \cos^2\delta\right] + \gamma^2/K} \tag{43}$$

and

$$\gamma_{ML}^U \geq \frac{(L-3)\gamma^2}{(L-2)\left[\gamma + (\gamma + K)/\eta L \cos^2\delta\right] + \gamma^2} \tag{44}$$

An upper bound on the actual SNR performance γ_{ML} is obtained by using the lower bound of Eq. (42) on d and the upper bound of Eq. (43) on γ_{ML}^U in Eq. (24). Similarly, a lower bound on γ_{ML} is obtained by using, instead, the upper bound of Eq. (41) on d and the lower bound of Eq. (44) on γ_{ML}^U.

IV. Numerical Example

We consider here the numerical example in [1] of using a $K = 7$ element array feed in the JPL Deep Space Network. In this example, a modulation index $\delta = 80$ deg and a primitive sample period $T_0 = 2.5 \times 10^{-8}$ s are assumed. The full-spectrum modulation signal is assumed to be of bandwidth 2×10^6 Hz, which yields $M_B = 20$. Moreover, the ratio of the full-spectrum bandwidth to the residual carrier bandwidth $\eta = M_A/M_B = 200$. Nominal P_T/N_0 of 55 and 65 dB-Hz are considered with corresponding $\gamma = (P_T/N_0)M_B T_0$. Upper and lower bounds on the combining gain γ_{ML}/γ are shown in Fig. 1 as a function of the number of samples L averaged to obtain the weight estimates. Here $P_T/N_0 = 55$ dB-Hz, and maximum correlation coefficient magnitudes ρ_{max} of 0.01 and 0.02 are considered. Convergence of these bounds to within 0.01 dB of their limiting values occurs at about $L = 3000$ samples. This corresponds to an averaging time of $M_A T_0 L = 0.3$ s and supports real-time operations for antenna deformation compensation. The limiting upper bounds on the combining gain are about 0.26 and 0.56 dB for ρ_{max} equal to 0.01 and 0.02, respectively. The corresponding lower bounds on the combining gain are -0.26 and -0.50 dB, respectively. The actual limiting value for the combining gain, which is given by Eq. (36), will fall between these bounds. Similar results are shown in Fig. 2 for $P_T/N_0 = 65$ dB-Hz, where convergence of the bounds occurs at smaller values of L to virtually the same limiting values as the $P_T/N_0 = 55$ dB-Hz case.

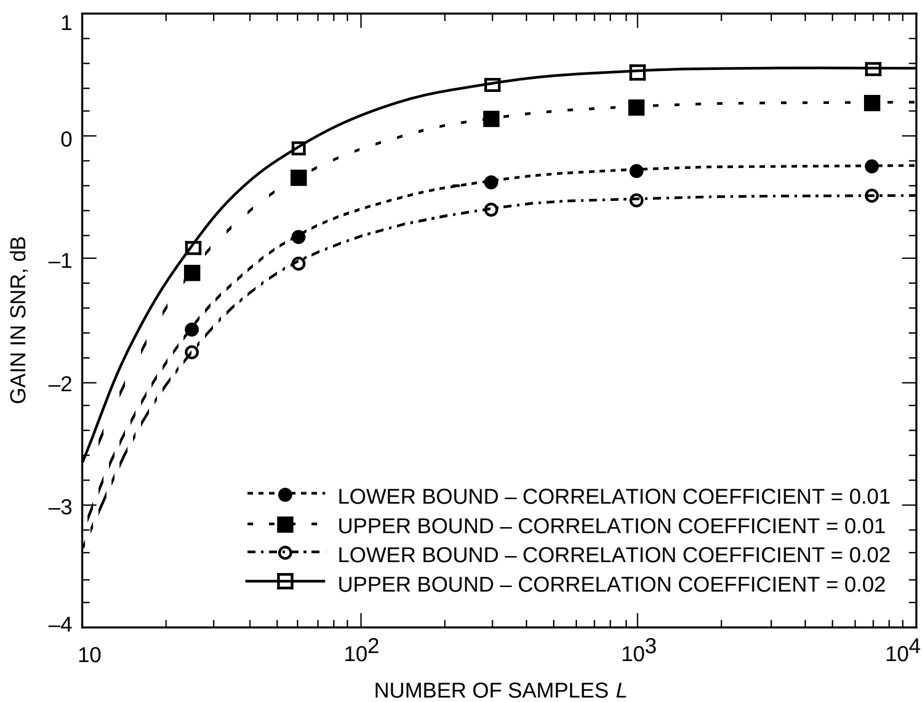

Fig. 1. Combining gain versus L for P_T/N_0 = 55 dB-Hz.

Figures 3 and 4 plot upper and lower bounds on the correlation gain $1/(1 + d)$ for ρ_{max} equal to 0.01 and 0.02. Figure 3 considers $P_T/N_0 = 55$ dB-Hz and Fig. 4 considers $P_T/N_0 = 65$ dB-Hz. The limiting values of these bounds are identical to the limiting values of the corresponding bounds on the combining gain. The differences between the behavior of the lower bounds at $P_T/N_0 = 55$ dB-Hz and those at $P_T/N_0 = 65$ dB-Hz are due to the looseness of these lower bounds at small values of L. For a large number L of samples, the upper and lower bounds on the combining gain diverge as the maximum correlation coefficient magnitude increases. This can be seen from Fig. 5, which shows the upper and lower bounds on combining gain for $P_T/N_0 = 55$ dB-Hz at $L = 5000$ samples as ρ_{max} increases from 0.01 to 0.1. The upper bound increases from 0.26 to 3.96 dB and the lower bound decreases from -0.26 to -2.05 dB in this range of ρ_{max}. The observed correlation coefficients of 0.01 magnitude in [2] were

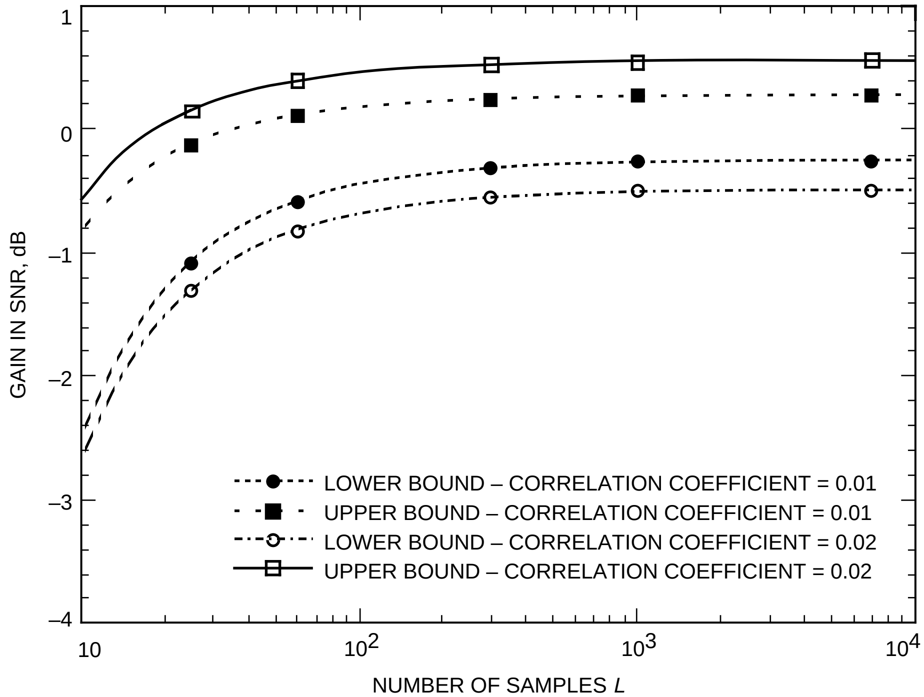

Fig. 2. Combining gain versus *L* for P_T/N_0 = 65 dB-Hz.

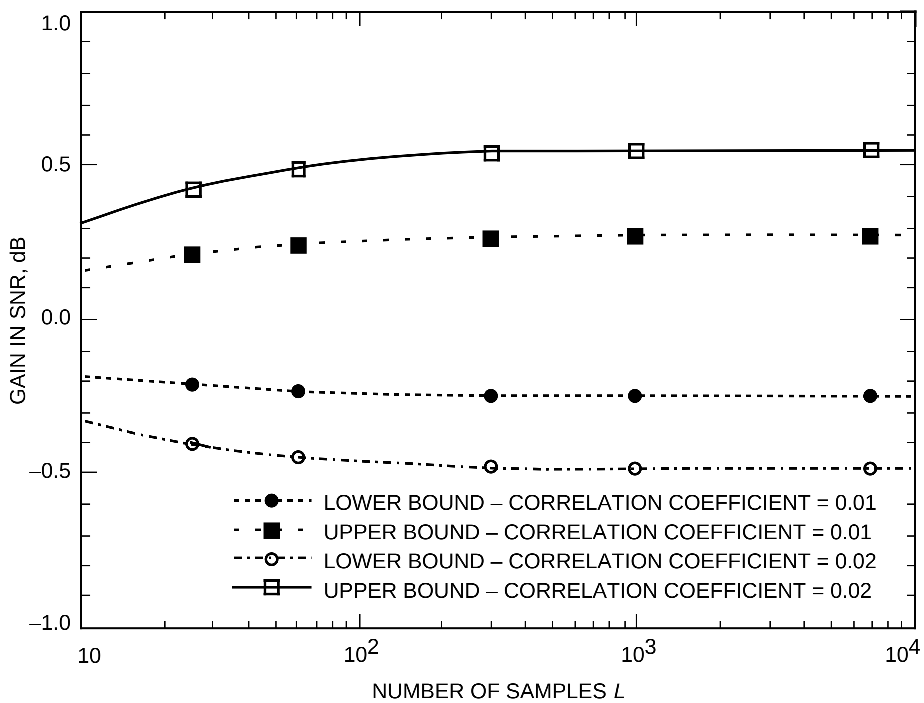

Fig. 3. Correlation gain versus *L* for P_T/N_0 = 55 dB-Hz.

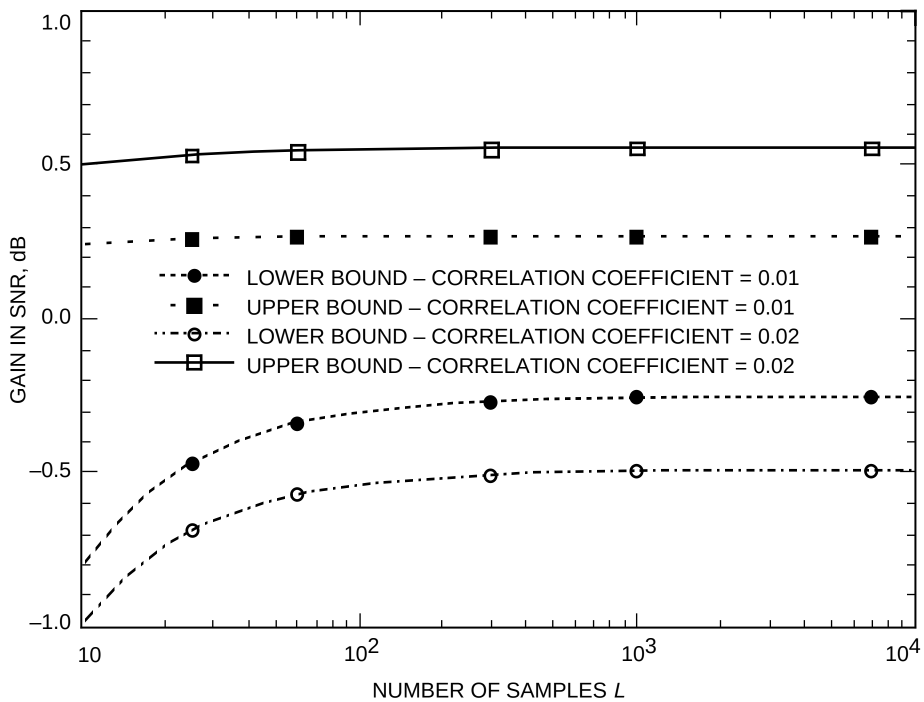

Fig. 4. Correlation gain versus L for P_T/N_0 = 65 dB-Hz.

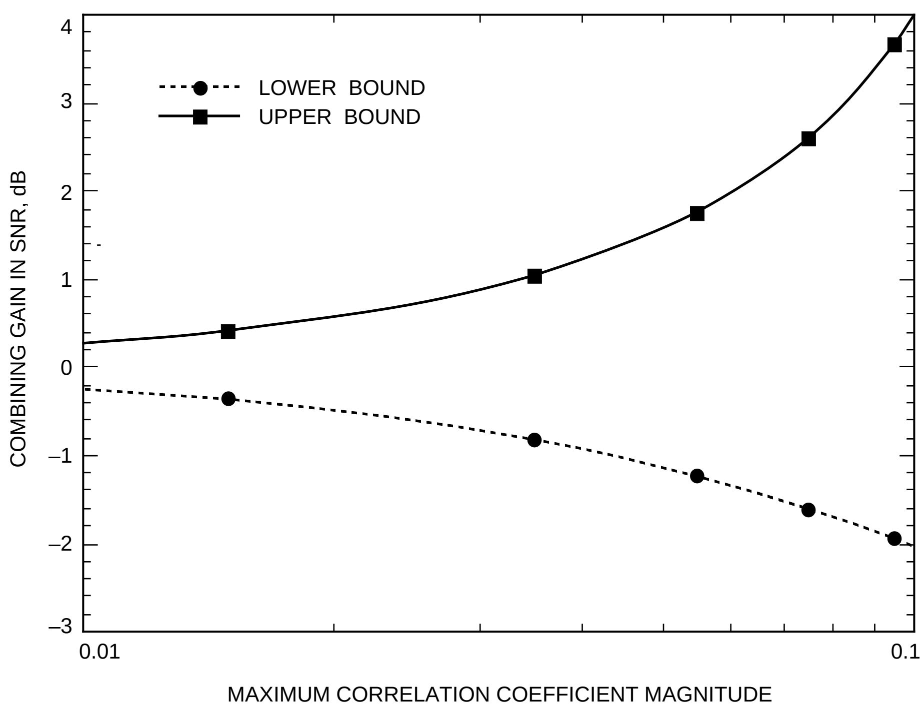

Fig. 5. Combining gain versus ρ_{max} for P_T/N_0 = 55 dB-Hz and L = 5000.

obtained in clear-sky conditions with a receiver noise temperature of 90 K and a system noise temperature of 120 K. Improvement of the receiver noise temperature to 25 K will increase the correlation coefficient magnitude to about 0.02. As noted above, a maximum possible improvement of 0.56 dB and a maximum possible degradation of −0.50 dB results. Preliminary measurements at DSS 13 indicate that even larger amounts of correlation occur under adverse weather conditions. This will result in even larger potential improvement or degradation of SNR performance relative to the uncorrelated-noise case.

V. Conclusion

An array feed combiner system for the recovery of SNR loss due to antenna reflector deformation has been implemented and is currently being evaluated on the Jet Propulsion Laboratory 34-meter DSS-13 antenna. The current signal-combining algorithms are optimum under the assumption that the white Gaussian noise processes in the received signals from different array elements are uncorrelated. Experimental data at DSS 13 indicate that these noise processes are indeed mutually correlated. The main result of this article is an analytical derivation of the actual SNR performance of the current suboptimal signal-combining algorithm in this correlated-noise environment. The analysis here shows that the combined-signal SNR can be either improved or degraded depending on the relation between the array signal and noise correlation coefficient phases. Further performance improvement will require the development of effective combining systems that take into account the correlations between the array feed element noise processes.

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Appendix A

Derivation of $E\left[\frac{1}{A_{11}A_{22}}\right]$

We first obtain the joint probability density function $p(A_{11}, A_{22})$ of (A_{11}, A_{22}) by integrating Eq. (30) over the complex region $\mathcal{S} = \{A_{12} : |A_{12}| < \sqrt{A_{11}A_{22}}\}$ of values taken on by A_{12}. Let $\underline{G} = \{G_{ij}\} = \underline{\Sigma}^{-1}$ and convert the variables G_{12} and A_{12} into polar coordinates: $G_{12} = |G_{12}|e^{j\psi}$ and $A_{12} = re^{j\phi}$. Then it follows from Eq. (30) that

$$
\begin{aligned}
p(A_{11}, A_{22}) &= \frac{e^{-(G_{11}A_{11}+G_{22}A_{22})}}{\pi\Gamma(L-1)\Gamma(L-2)|\underline{\Sigma}|^{L-1}} \int_{\mathcal{S}} \left(A_{11}A_{22} - |A_{12}|^2\right)^{L-3} e^{-2\mathcal{R}e(G_{12}^* A_{12})}\, dA_{12} \\
&= \frac{e^{-(G_{11}A_{11}+G_{22}A_{22})}}{\pi\Gamma(L-1)\Gamma(L-2)|\underline{\Sigma}|^{L-1}} \int_0^{\sqrt{A_{11}A_{22}}} r\left(A_{11}A_{22} - r^2\right)^{L-3} \left[\int_0^{2\pi} e^{-2r|G_{12}|\cos(\phi-\psi)}\, d\phi\right] dr \\
&= \frac{2e^{-(G_{11}A_{11}+G_{22}A_{22})}}{\Gamma(L-1)\Gamma(L-2)|\underline{\Sigma}|^{L-1}} \int_0^{\sqrt{A_{11}A_{22}}} r\left(A_{11}A_{22} - r^2\right)^{L-3} I_0(2r|G_{12}|)\, dr \qquad \text{(A-1)}
\end{aligned}
$$

where $I_0(x)$ is the zero-order modified Bessel function of the first kind, which has series representation

$$
I_0(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k}(k!)^2} \qquad \text{(A-2)}
$$

By making a change of the variable of integration using the series of Eq. (A-2) and the integral relation (3.251) of [6], the integral in Eq. (A-1) can be written as

$$
\begin{aligned}
\int_0^{\sqrt{A_{11}A_{22}}} r\left(A_{11}A_{22} - r^2\right)^{L-3} I_0(2r|G_{12}|)\, dr &= (A_{11}A_{22})^{L-2} \sum_{k=0}^{\infty} \frac{\left(A_{11}A_{22}|G_{12}|^2\right)^k}{k!} \int_0^1 \left(1-s^2\right)^{L-3} s^{2k+1}\, ds \\
&= (A_{11}A_{22})^{L-2} \sum_{k=0}^{\infty} \frac{\left(A_{11}A_{22}|G_{12}|^2\right)^k}{k!} \left[\frac{\Gamma(k+1)\Gamma(L-2)}{2\Gamma(k+L-1)}\right] \\
&\qquad \text{(A-3)}
\end{aligned}
$$

Substituting Eq. (A-3) into Eq. (A-1) and using the fact that $\Gamma(n) = (n-1)!$ for integer n, we obtain

$$
p(A_{11}, A_{22}) = \frac{(A_{11}A_{22})^{L-2}\, e^{-(G_{11}A_{11}+G_{22}A_{22})}}{(L-2)!\,|\underline{\Sigma}|^{L-1}} \sum_{k=0}^{\infty} \frac{(A_{11}A_{22}|G_{12}|^2)^k}{k!(k+L-2)!} \qquad \text{(A-4)}
$$

Using Eq. (A-4), integrating term by term in the series, and obtaining $\underline{G} = \underline{\Sigma}^{-1}$ and $|\underline{\Sigma}|$ directly from Eq. (29) in terms of ρ_{Bkj}, r_{Bkk}, and r_{Bjj}, we have

$$\mathrm{E}\left[\frac{1}{A_{11}A_{22}}\right] = \frac{1}{(L-2)|\underline{\Sigma}|^{L-1}(G_{11}G_{22})^{L-2}} \sum_{k=0}^{\infty} \binom{k+L-3}{k} \frac{\left(\frac{|G_{12}|^2}{G_{11}G_{22}}\right)^k}{k+L-2}$$

$$= \frac{\eta^2(1-|\rho_{Bkj}|^2)^{L-3}}{(L-2)r_{Bkk}r_{Bjj}} \sum_{k=0}^{\infty} \binom{k+L-3}{k} \frac{|\rho_{Bkj}|^{2k}}{k+L-2} \tag{A-5}$$

The series in Eq. (A-5) can be shown to converge by using the ratio convergence test whenever $|\rho_{Bkj}| < 1$.

Appendix B

Bounds on $f_L(x)$

Let $L \geq 4$ and $0 \leq x < 1$. We will obtain upper and lower bounds on $f_L(x)$ that are asymptotically tight in the limit as $L \to \infty$. First note that

$$\frac{L-2}{k+L-2} = \left(\frac{L-3}{k+L-3}\right)\left(\frac{L-2}{L-3}\right)\left(\frac{k+L-3}{k+L-2}\right) \tag{B-1}$$

and that for $k \geq 0$,

$$\frac{L-3}{L-2} \leq \frac{k+L-3}{k+L-2} \leq 1 \tag{B-2}$$

Using these bounds of Eq. (B-2) in Eq. (B-1), we have

$$\frac{L-3}{k+L-3} \leq \frac{L-2}{k+L-2} \leq \left(\frac{L-2}{L-3}\right)\frac{L-3}{k+L-3} \tag{B-3}$$

Next, by using the bounds of Eq. (B-3) in Eq. (31), we get the following:

$$(1-x)^{L-3} \sum_{k=0}^{\infty} \binom{k+L-4}{k} x^k \quad \leq \quad f_L(x) \tag{B-4}$$

$$f_L(x) \quad \leq \quad \left(\frac{L-2}{L-3}\right)(1-x)^{L-3} \sum_{k=0}^{\infty} \binom{k+L-4}{k} x^k \tag{B-5}$$

It can be shown in [7] that for $0 \leq x < 1$,

$$\sum_{k=0}^{\infty} \binom{k+L-4}{k} x^k \;=\; \left(\frac{1}{1-x}\right)^{L-3} \tag{B-6}$$

Using Eq. (B-6) in Eqs. (B-4) and (B-5), we then obtain, for $0 \leq x < 1$,

$$1 \leq f_L(x) \leq \frac{L-2}{L-3} \tag{B-7}$$

The upper and lower bounds given in Eq. (B-7) are both asymptotically tight in the limit as $L \to \infty$. So we can conclude that for $0 \leq x < 1$, $f_L(x) \to 1$ as $L \to \infty$, where the convergence is uniform in x.