Carrier Arraying—Revisited

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Depending on the basis used for comparison, it is possible to predict different gains when applying carrier arraying to a conventional coherent receiver. In the past, the specific performance comparisons were made between the arrayed and nonarrayed cases assuming a fixed carrier-tracking closed-loop bandwidth. While the notion of loop bandwidth for the nonarrayed loop with a single input is well defined and meaningful, the comparable notion for an arrayed loop with multiple inputs is not uniquely defined since it depends on the knowledge of the statistical relation, e.g., degree of correlation, among the various antenna inputs. To circumvent the need for such knowledge, which is often not completely available, we suggest in this article an alternate criterion for comparing arrayed and nonarrayed loops, namely, fixed steady-state phase error, which, in the arrayed case, is independent of the statistical relation among the antenna inputs. We show that, in this case, the gain derived from carrier arraying is quite different from that obtained under the assumption of fixed-loop bandwidth, which suggests that one should exercise care when comparing arrayed and unarrayed loops in that the criterion used for comparison should be directly applicable to the situation at hand.

I. Introduction

Carrier arraying is a technique for automatically aligning the phases of a multitude (say N) of received carriers possibly at geographically separated locations so as to allow them to be coherently combined, thereby improving the overall signal-to-noise ratio (SNR) performance of the system. Previous analysis [1] of such a carrier-arraying system focused on the tracking performance of the first receiver (where the carrier arraying is performed) as measured by its closed-loop rms phase jitter and radio loss associated with the data detection process. It was shown in [1] that when the loop bandwidths of stations 2, $3, \cdots, N$ are much narrower than that of the first station, then carrier arraying could offer a significant enhancement in station 1's loop SNR, thereby reducing its rms phase jitter and associated radio loss relative to the no carrier-arraying case.

The specific performance comparisons made in [1] between the arrayed and nonarrayed cases assumed a fixed carrier-tracking closed-loop bandwidth. As such, the authors of [1] first evaluated station 1's loop bandwidth after carrier arraying and then computed the loop SNR enhancement (reduction in rms phase jitter) with this bandwidth set equal to this same quantity prior to arraying. While the latter notion of loop bandwidth (i.e., for the nonarrayed loop with a single input) is well defined and meaningful (see any standard text on phase-locked loop (PLL) theory, e.g., [2]), the former notion of loop bandwidth (i.e., for an arrayed loop with multiple inputs) is not uniquely defined since it depends on the statistical relation among the antenna inputs. Thus, the performance comparisons derived and illustrated in [1] depend on

the particular definition of loop bandwidth in the presence of carrier arraying assumed in that reference. In what follows, we shall expand on this notion, starting out with an interpretation of the comparisons made in [1]. Following that, we shall suggest an alternate criterion for comparing arrayed and nonarrayed loops that has the advantage of being independent of the definition of loop bandwidth applied. We shall see that, in this case, the gain derived from carrier arraying is quite different from that obtained under the assumption of fixed-loop bandwidth. This suggests that one should exercise care when comparing arrayed and unarrayed loops in that the criterion used for comparison should be directly applicable to the situation at hand.

Although our results will be presented in a form that allows for any order loops, to keep matters simple and allow for comparisons that are easy to comprehend, we shall specifically consider the case of first-order loops.

II. Stochastic Differential Equation of Operation

Illustrated in Fig. 1 is the carrier-arraying system identified in [1] and in Fig. 2 its equivalent block diagram based on linear loop operation for all station receivers. The stochastic differential equation of operation of station 1's carrier-tracking loop (the one that is arrayed) is (see Eq. (A-2) of [1] with $\theta_m = 0$)[1]

$$\hat{\theta}_1 = \frac{K_1 F_1(p)}{p} \left[\sqrt{P_1} \beta_1 \left(\theta_1 - \hat{\theta}_1 \right) + \sum_{i=2}^{N} \sqrt{P_i} \beta_i \left(\theta_i - \hat{\theta}_i - \theta_1 \right) + \sum_{i=2}^{N} \beta_i N_i \right] \tag{1}$$

where P_i is the signal power of the received signal at station i; N_i is a zero-mean, variance $N_{0i}/2$ Gaussian random variable representing the additive noise in station i's receiver; K_1 is the total loop gain of station 1's carrier-tracking loop (prior to arraying); and $F_1(p)$ is the transfer function of the loop filter with p denoting the Heaviside operator. Similarly, the stochastic differential equation of loop operation for the remaining station receivers is given by (see Eq. (A-4) of [1] with $\theta_m = 0$)

$$\hat{\theta}_i = \frac{K_i F_i(p)}{p} \left[\sqrt{P_i} \left(\theta_i - \hat{\theta}_i - \theta_1 \right) + N_i \right], \quad i = 2, 3, \cdots, N \tag{2}$$

Substituting Eq. (2) into Eq. (1) and defining the gain ratio $\gamma_i = \sqrt{P_i/P_1}$, it is straightforward to show that

$$\hat{\theta}_1 = H_1(p) \left\{ \theta_1 + \frac{N_1}{\sqrt{P_1}} + \sum_{i=2}^{N} \left(\frac{\beta_i}{\beta_1} \right) \gamma_i \left(1 - H_i(p) \right) \left[\left(\theta_i - \hat{\theta}_1 \right) + \frac{N_i}{\sqrt{P_i}} \right] \right\} \tag{3}$$

where

$$\left. \begin{array}{rl} H_1(p) &= \dfrac{K_1 \sqrt{P_1} F_1(p) \beta_1}{p + K_1 \sqrt{P_1} F_1(p) \beta_1} \\[2em] H_i(p) &= \dfrac{K_1 \sqrt{P_i} F_i(p)}{p + K_i \sqrt{P_i} F_i(p)} \end{array} \right\} \tag{4}$$

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[1] To keep the normalization of the loop gains completely general, we do at this point set the combining weight β_1 equal to 1, as was done in [1].
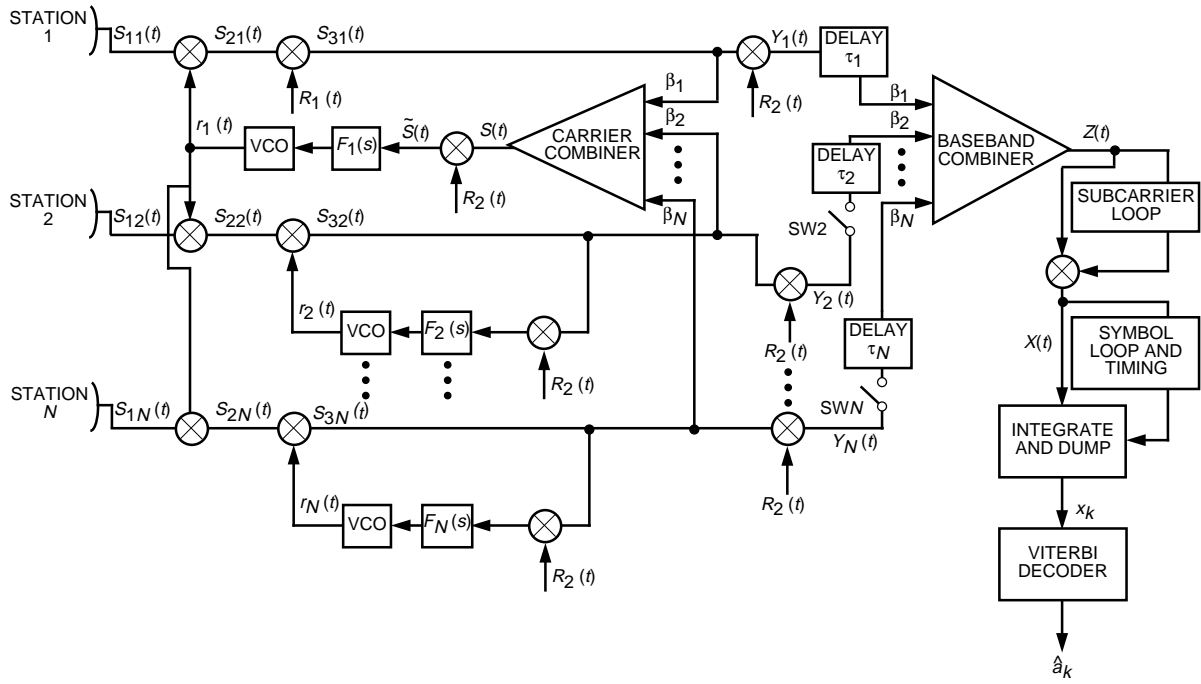
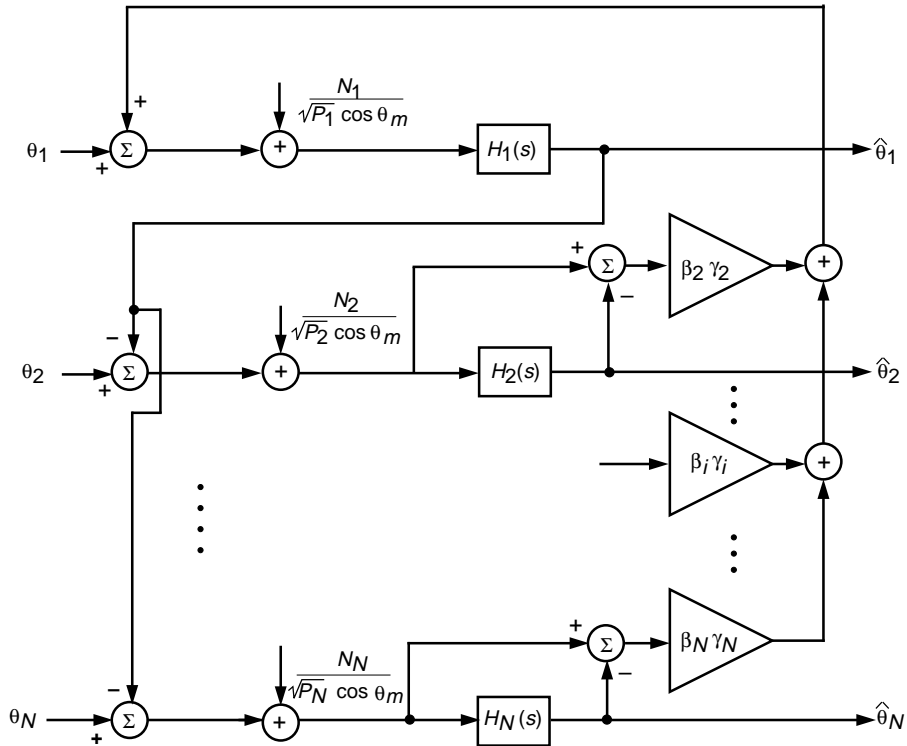Fig. 1. Configuration for arrayed network with carrier and baseband arraying.

Fig. 2. An equivalent linear representation of carrier arraying system for phase jitter analysis.

For the special case of two antennas ($N = 2$) with equal powers ($P_1 = P_2 = P$ or $\gamma_2 = 1$), Eq. (3) simplifies to

$$\hat{\theta}_1 = H_1(p) \left\{ \theta_1 + \frac{N_1}{\sqrt{P}} + \left(\frac{\beta_2}{\beta_1} \right) (1 - H_2(p)) \left[\left(\theta_2 - \hat{\theta}_1 \right) + \frac{N_2}{\sqrt{P}} \right] \right\}$$ (5)

Without loss in generality, we shall focus the remainder of our discussion on this special case since it is sufficient to illustrate the points that need to be made.

III. Defining the Loop Bandwidth

For a carrier-tracking loop with a single input, θ, and a single output, $\hat{\theta}$, the single-sided loop noise bandwidth, B_L, is defined by

$$2B_L \triangleq \frac{1}{2\pi j} \int\limits_{-j\infty}^{j\infty} |H(s)|^2 ds$$ (6)

where

$$H(s) \triangleq \frac{\hat{\theta}(s)}{\theta(s)}$$ (7)

with s denoting the LaPlace transform operator. For a first-order loop with $F(s) = 1$, Eq. (6) evaluates to the well-known result [2]

$$B_L = \frac{K\sqrt{P}}{4}$$ (8)

whereas for a second-order loop with imperfect integrating loop filter, $F(s) = (1 + s\tau_2)/(1 + s\tau_1)$, the result is

$$B_l = \frac{r + 1}{4\tau_2 \left(\dfrac{1 + \tau_2}{r\tau_1} \right)} \cong \frac{r + 1}{4\tau_2}$$

$$r = \frac{K\sqrt{P}\tau_2^2}{\tau_1}$$ (9)

where the approximation is valid for the usual case $r\tau_2 >> \tau_1$.

In the carrier-arraying case, we are dealing with an arrayed loop (station 1) that has a single output, $\hat{\theta}_1$, and, in general, *multiple* inputs, i.e., $\theta_1, \theta_2, \cdots, \theta_N$. For such a system, the definition of loop bandwidth

for station 1's receiver is not uniquely defined since it indeed depends on the relation of the other receiver inputs, $\theta_2, \cdots, \theta_N$, to its own input, θ_1. In the simplest scenario with just two antennas, the received signals at antenna 1 and antenna 2 differ only in a constant delay, $\tau = \Delta\theta/\omega_c$, between them, where ω_c is the radian carrier frequency common to both. Thus, for this scenario, we would have $\theta_2 = \theta_1 + \Delta\theta$, i.e., θ_1 and θ_2 are totally correlated, in which case Eq. (5) becomes

$$\hat{\theta}_1 = H_1(p)\left\{\theta_1 + \frac{N_1}{\sqrt{P}} + \left(\frac{\beta_2}{\beta_1}\right)(1 - H_2(p))\left[\theta_1 + \Delta\theta - \hat{\theta}_1 + \frac{N_2}{\sqrt{P}}\right]\right\} \quad (10)$$

or equivalently,

$$\hat{\theta}_1 = \left[\frac{H_1(p)\left[1 + \left(\frac{\beta_2}{\beta_1}\right)(1 - H_2(p))\right]}{1 + \left(\frac{\beta_2}{\beta_1}\right)H_1(p)(1 - H_2(p))}\right]\theta_1 + \left[\frac{H_1(p)\left(\frac{\beta_2}{\beta_1}\right)(1 - H_2(p))}{1 + \left(\frac{\beta_2}{\beta_1}\right)H_1(p)(1 - H_2(p))}\right]\Delta\theta \quad (11)$$

Even for this assumption of totally correlated θ_1 and θ_2, the definition of loop bandwidth is not well defined since, in general, it depends on the value of $\Delta\theta$.

If one makes the assumption that $\Delta\theta = 0$ (corresponding to the case of perfect delay compensation), then in accordance with Eq. (7), the loop bandwidth would be given by

$$2B_{L1} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left|\frac{H_1(s)\left[1 + \left(\frac{\beta_2}{\beta_1}\right)(1 - H_2(s))\right]}{1 + \left(\frac{\beta_2}{\beta_1}\right)H_1(s)(1 - H_2(s))}\right|^2 ds \quad (12)$$

For first-order loops, Eq. (12) evaluates to

$$B_{L1} = \frac{K_1'\left[\beta_1^2 K_2' + K_1'\beta_1(\beta_1 + \beta_2)^2\right]}{4\left(\beta_1 K_2' + K_1'\beta_1(\beta_1 + \beta_2)\right)}$$

$$K_i' \triangleq K_i\sqrt{P} \quad (13)$$

which for $K_2' << K_1'\beta_1$, i.e., station 2's loop narrowband with respect to station 1's loop prior to arraying, simplifies to

$$B_{L1} = \frac{K_1\sqrt{P}\beta_1}{4}\left(1 + \frac{\beta_2}{\beta_1}\right) = (B_{L1})_0\left(1 + \frac{\beta_2}{\beta_1}\right) \quad (14a)$$

where $(B_{L1})_0 \triangleq K_1'\beta_1/4 = K_1\sqrt{P}\beta_1/4$ denotes the loop bandwidth of station 1's receiver prior to arraying (i.e., with $\beta_2 = 0$).

For second-order loops with imperfect integrating loop filters $F_i(s) = (1 + s\tau_{2i})/(1 + s\tau_{1i})$, $i = 1, 2$, and the same assumption regarding station 2's loop being narrowband with respect to that of station 1 prior to arraying, Eq. (12) evaluates to

$$B_{L1} = (B_{L1})_0 \left[\frac{1 + r_1 \left(1 + \frac{\beta_2}{\beta_1} \right)}{1 + r_1} \right]$$

(14b)

$$r_1 = \frac{\sqrt{P}K_1\beta_1\tau_{21}^2}{\tau_{11}}$$

Although not explicitly stated, the definition of loop bandwidth given in Eq. (14) corresponding to $\Delta\theta = 0$ is the case assumed in [1] (compare Eq. (14b) with Eq. (23) of [1]).

When $\Delta\theta \neq 0$ but constant, then one can argue that, under the assumption of station 2's loop being narrowband with respect to station 1's loop prior to arraying, the $1 - H_2(p)$ factor in the numerator of the second term in Eq. (11) will for all practical purposes filter out the dc component due to $\Delta\theta$. Hence, the dynamics of the output, $\hat{\theta}_1$, are dominated by the input, θ_1, and to a first approximation the loop bandwidth after arraying would be the same as in the previous case where $\Delta\theta = 0$ was assumed. In other situations, $\Delta\theta$ may not be constant, i.e., θ_1 and θ_2 may not be totally correlated and, hence, the loop bandwidth will depend on the statistical properties of $\Delta\theta$.

In summary then, the definition of loop bandwidth applied in a multiple input system such as a carrier-arrayed loop is dependent on the relation assumed among its inputs. In what follows, we suggest an alternate basis for comparing arrayed and unarrayed performances that has the advantage that it does not depend on $\Delta\theta$; hence, one does not have to make any assumption regarding its statistical properties. More generally, we suggest an alternate normalization of the problem wherein the relation between the other receiver inputs, $\theta_2, \theta_3, \cdots, \theta_N$, and that corresponding to the receiver being arrayed, i.e., θ_1, is inconsequential. Before presenting this alternate formulation, we briefly review the performance comparison in terms of rms phase jitter made in [1] using the criterion of equal loop bandwidths prior to and after arraying, which *strictly speaking* is only proper when $\Delta\theta = 0$. This is done so that it can later be contrasted to the analogous performance comparison using the suggested alternate basis.

IV. Evaluation of Phase Jitter Variance

Letting $\phi_1 \triangleq \theta_1 - \hat{\theta}_1$ denote the phase error in station 1's carrier-tracking loop, then it is straightforward to show that Eq. (11) can be written in the form

$$\phi_1 = \left[\frac{1 - H_1(p)}{1 + \left(\frac{\beta_2}{\beta_1} \right) H_1(p)(1 - H_2(p))} \right] \theta_1 - \left[\frac{H_1(p) \left(\frac{\beta_2}{\beta_1} \right) (1 - H_2(p))}{1 + \left(\frac{\beta_2}{\beta_1} \right) H_1(p)(1 - H_2(p))} \right] \Delta\theta$$

$$- \left[\frac{H_1(p)}{1 + \left(\frac{\beta_2}{\beta_1} \right) H_1(p)(1 - H_2(p))} \right] \left(\frac{N_1}{\sqrt{p}} \right) - \left[\frac{H_1(p) \left(\frac{\beta_2}{\beta_1} \right) (1 - H_2(p))}{1 + \left(\frac{\beta_2}{\beta_1} \right) H_1(p)(1 - H_2(p))} \right] \left(\frac{N_2}{\sqrt{P}} \right)$$

(15)

Denoting variances of the Gaussian noise random variables by $\sigma_{Ni}^2 = N_{0i}/2, i = 1, 2$, then the variance of the phase jitter in station 1's carrier-tracking loop in the presence of arraying would be

$$
\begin{aligned}
\sigma_{\phi 1}^2 &= \left(\frac{N_{01}}{2P}\right) \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left| \frac{H_1(s)}{1 + \left(\frac{\beta_2}{\beta_1}\right) H_1(s)(1 - H_2(s))} \right|^2 ds \\[2ex]
&+ \left(\frac{N_{02}}{2P}\right) \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left| \frac{H_1(s)\left(\frac{\beta_2}{\beta_1}\right)(1 - H_2(s))}{1 + \left(\frac{\beta_2}{\beta_1}\right) H_1(s)(1 - H_2(s))} \right|^2 ds
\end{aligned}
\tag{16}
$$

Evaluating Eq. (16) for first-order loops gives

$$
\sigma_{\phi 1}^2 = \left(\frac{N_{01}(B_{L1})_0}{P}\right) \left(\frac{\beta_1 + \frac{K_2'}{K_1'}}{\beta_1 + \beta_2 + \frac{K_2'}{K_1'}}\right) + \left(\frac{N_{02}(B_{L1})_0}{P}\right) \left[\left(\frac{\beta_2}{\beta_1}\right)^2 \left(\frac{\beta_1}{\beta_1 + \beta_2 + \frac{K_2'}{K_1'}}\right)\right]
\tag{17}
$$

which for $K_2' << K_1'\beta_1$ and $N_{01} = N_{02} = N_0$ simplifies to

$$
\sigma_{\phi 1}^2 = \left(\frac{N_0(B_{L1})_0}{P}\right) \left(\frac{1 + \left(\frac{\beta_2}{\beta_1}\right)^2}{1 + \frac{\beta_2}{\beta_1}}\right)
\tag{18}
$$

If we now write Eq. (18) in terms of the definition of loop bandwidth given in Eq. (14a), then we obtain

$$
\sigma_{\phi 1}^2 = \left(\frac{N_0 B_{L1}}{P}\right) \left(\frac{1 + \left(\frac{\beta_2}{\beta_1}\right)^2}{\left(1 + \frac{\beta_2}{\beta_1}\right)^2}\right)
\tag{19}
$$

If using this definition of loop bandwidth, we hold the loop bandwidth fixed prior to and after arraying, then since $\sigma_{\phi 1}^2 = N_0 B_{L1}/P$ represents the variance of the phase jitter prior to arraying, the arraying gain (loop SNR improvement factor) is given by

$$
G \triangleq \frac{\left(1 + \frac{\beta_2}{\beta_1}\right)^2}{1 + \left(\frac{\beta_2}{\beta_1}\right)^2}
\tag{20}
$$

which when optimized with respect to the combining weight ratio β_2/β_1 yields $(\beta_2/\beta_1)_{opt} = 1$ and, hence, $G_{opt} = 2$, i.e., *a 3-dB gain!* This result is what would be predicted by the results in [1] if two first-order loops and equal power-to-noise ratios at both antennas, i.e., $P_1/N_{01} = P_2/N_{02} \triangleq P/N_0$ were assumed. Also, in order to realize the gain predicted by Eq. (20), i.e., to hold the loop bandwidth fixed, station 1's total loop gain, K_1', must be renormalized (divided by) the factor $1 + \beta_2/\beta_1$, which, for the optimum design, implies division of K_1' by a factor of 2. This can be accomplished by including a 1/2 gain factor in series with the loop filter $F_1(s)$.

V. Computation of Arraying Gain Based on Fixed Steady-State Phase Error

It is well known that when a frequency offset exists in the received signal to a carrier-tracking loop, then a steady-state phase error, ϕ_{ss}, will exist. If the loop input phase is modeled as $\theta(t) = \theta + \Omega t$, then for a first-order loop, ϕ_{ss} is given by

$$\phi_{ss} = \frac{\Omega}{K\sqrt{p}} = \frac{\Omega}{4B_L} \tag{21}$$

whereas for a second-order loop with imperfect integrating loop filter $F(s) = (1 + s\tau_2)/(1 + s\tau_1)$, the result is

$$\phi_{ss} = \frac{\Omega}{r\tau_1/\tau_2^2} \cong \frac{\Omega}{r\tau_1/\tau_2}\left(\frac{r+1}{4B_L}\right) \tag{22}$$

where the approximation is in the same sense as that used in Eq. (9). Since from Eq. (21) or Eq. (22) we see that ϕ_{ss} is inversely proportional to B_L, then for the carrier-arraying application, keeping ϕ_{ss} fixed prior to and after arraying can be construed as maintaining a fixed-loop bandwidth. The importance of this normalization in terms of fixed ϕ_{ss} is that, as we shall soon see, the value of ϕ_{ss} in station 1's carrier-tracking loop is unaffected by the presence of the arraying, provided that the same frequency offset is present at the input of the other receivers, e.g., for the two-antenna case, assuming as before that the inputs to the two antennas only differ by a fixed delay $(\theta_2 - \theta_1 = \Delta\theta)$.

Setting N_1 and N_2 equal to zero in Eq. (15), we have

$$\phi_{ss_1} = \lim_{t\to\infty} \phi_1(t) = \lim_{s\to 0} s\Phi_1(s)$$

$$= \lim_{s\to 0}\left[\frac{s(1 - H_1(s))}{1 + \left(\dfrac{\beta_2}{\beta_1}\right)H_1(s)(1 - H_2(s))}\right]\left(\frac{\theta_1}{s} + \frac{\Omega_1}{s^2}\right) - \lim_{s\to 0}\left[\frac{sH_1(s)\left(\dfrac{\beta_2}{\beta_1}\right)(1 - H_2(s))}{1 + \left(\dfrac{\beta_2}{\beta_1}\right)H_1(s)(1 - H_2(s))}\right]\left(\frac{\Delta\theta}{s}\right)$$

$$\tag{23}$$

For first-order loops, Eq. (23) becomes

$$\phi_{ss_1} = \lim_{s \to 0} \left[\frac{s^2(s + K_2')}{s^2 + s\left[K_1'\beta_1\left(1 + \frac{\beta_2}{\beta_1}\right) + K_2'\right] + K_1'\beta_1 K_2'} \right] \left(\frac{\theta_1}{s} + \frac{\Omega_1}{s^2} \right)$$

$$- \lim_{s \to 0} \left[\frac{\left(\frac{\beta_2}{\beta_1}\right) s^2 K_1'\beta_1}{s^2 + s\left[K_1'\beta_1\left(1 + \frac{\beta_2}{\beta_1}\right) + K_2'\right] + K_1'\beta_1 K_2'} \right] \left(\frac{\Delta\theta}{s} \right) = \frac{\Omega}{K_1'\beta_1} = \frac{\Omega}{K_1\sqrt{P}\beta_1} = \frac{\Omega}{4(B_{L1})_0}$$

$$(24)$$

independent of $\Delta\theta$ and β_2. Thus, as previously predicted, the steady-state error in station 1's carrier-tracking loop is unchanged when carrier arraying is added to the receiver. A similar conclusion is reached for second-order loops, namely, the steady-state phase error in station 1's carrier-tracking loop, with or without arraying, is given by [see Eq. (22)]

$$\phi_{ss_1} = \frac{\Omega_1}{r_1\tau_{11}/\tau_{21}^2} \cong \frac{\Omega}{r_1\tau_{11}/\tau_{21}} \left(\frac{r_1 + 1}{4(B_{L1})_0} \right) \tag{25}$$

The upshot of the above is that in order to maintain a fixed steady-state phase error when carrier arraying is employed in a receiver, *the total loop gain should not be renormalized at all!* Hence, the phase jitter variance is given by Eq. (18) with no renormalization of the loop bandwidth and, thus, the arraying gain using fixed ϕ_{ss_1} as a basis of comparison with the unarrayed case is

$$G \triangleq \frac{\left(1 + \frac{\beta_2}{\beta_1}\right)}{1 + \left(\frac{\beta_2}{\beta_1}\right)^2} \tag{26}$$

which when optimized with respect to β_2/β_1 yields $(\beta_2/\beta_1)_{opt} = \sqrt{2} - 1 = 0.414$ and, hence, $G_{opt} = 1.207$, i.e., *only a 0.82-dB gain!* This result has been confirmed by computer simulation.

VI. Conclusions

What has been shown here is that, depending on the basis used for comparison, it is possible to predict different gains when applying carrier arraying to a conventional coherent receiver. Thus, it is concluded that one should exercise caution when attempting to compare the performance of a coherent receiver before and after carrier arraying in that one should have a complete knowledge of the relative phase and frequency conditions of the signals being received at the various antenna inputs.

Acknowledgments

References

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