The Power Spectrum of Unbalanced NRZ and Biphase Signals in the Presence of Data Asymmetry

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The impact of data unbalance and asymmetry on the power spectral density of nonreturn-to-zero (NRZ) and biphase baseband modulations is presented. Previously reported results for this problem assumed an incorrect model for the shape and number of elementary pulse shapes that characterize an arbitrary random data stream and, thus, led to erroneous computations of these power spectra. The purpose of this article is to provide the correct analytical model and then use it to obtain theoretical power spectrum results that are in full agreement with those obtained from computer simulation.

I. Introduction

A study of the average bit-error probability performance of pulse code modulation/phase modulation/nonreturn-to-zero (PCM/PM/NRZ) and PCM/PM biphase[1] receivers in nonideal channels has recently been presented in several places in the literature [1–5].[2] Included in the analyses and simulations were results obtained in the presence of two separate effects that degrade the performance of the receiver, namely, unbalanced data (the unbalance between the $+1$'s and -1's in the data stream) and data asymmetry (the unequal rise and fall times of the logic gating circuits producing data transitions at other than the nominal time instants). In [1–3], the bit-error probability performance of the receivers was evaluated taking into account the effect of the data unbalance and asymmetry on the carrier phase demodulation process. Aside from this consideration, the power spectral density (PSD) of these same modulations is of interest. The impact of data unbalance and asymmetry on the PSD of NRZ and biphase baseband modulations was previously studied in [4,5]. It was concluded there that the presence of these two effects in the transmitted data stream produces a line (discrete) spectrum in its PSD as well as distortion of the continuous component of the PSD. Recently, when attempting to compare the theoretical results for the discrete and continuous components of the spectra obtained in [4] and Footnote 2 with computer simulation results, a lack of agreement was discovered in both. Upon further investigation of the theoretical model used in [4,5] to characterize the elementary pulse shapes (to be defined shortly) and their probabilities that exist in unbalanced, asymmetric NRZ and biphase modulations, it was found that an error was committed and, hence, the spectrum results obtained therein were incorrect. The specifics of

[1] Biphase modulation is often referred to in the literature as Manchester code.

[2] L. Lam and S. Million, "Performance of PCM/PM Receivers With Non-Ideal Data," to appear in a future issue of *The Telecommunications and Data Acquisition Progress Report*, Jet Propulsion Laboratory, Pasadena, California.

the modeling errors committed in [4,5] will be discussed later on in the article. The purpose of this article is to provide the correct analytical model for analyzing the power spectra of unbalanced, asymmetric NRZ and biphase modulations and to provide analytical and simulation results for these spectra that are in agreement.

II. Unbalanced, Asymmetric NRZ Modulation

In [6, Chapter 2], a formulation is presented for calculating the power spectral density of a generalized M-ary Markov data source that is characterized by one of M signals (referred to as elementary signals) transmitted in each T-second interval with given a priori probabilities (called stationary probabilities) and given transition probabilities, i.e., the probability that a particular elementary signal is transmitted after the occurrence of another elementary signal. In general, such a source produces a random pulse train that contains both discrete and continuous power spectrum components. Since unbalanced, asymmetric binary baseband modulations such as NRZ and biphase can be modeled as above, the approach taken in [6, Chapter 2] is well suited to computing their PSD. While this approach was indeed used in [4,5] to evaluate these spectra, an error (to be described in more detail shortly) in the underlying model used to characterize the elementary signals that appear in a typical data stream and the stationary (steady-state) probabilities of these signals led to incorrect discrete and continuous spectrum results. In this section, we present an M-ary Markov data model for asymmetric, unbalanced NRZ modulation and then proceed to compute its PSD using the formulation in [6, Chapter 2]. This formulation was originally described in [7]. In the next section, we use a similar analytical model and the same formulation to compute the PSD of asymmetric, unbalanced biphase modulation.

In this section, we used, as did [4,5], the data asymmetry model proposed in [8], which assumes that $+1$ bits are elongated by $\Delta T/2$ (relative to their nominal value of T seconds) when a negative-going data transition occurs and -1 bits are shortened by the same amount when a positive-going transition occurs. The parameter Δ is referred to as the relative fractional asymmetry. When no data transitions occur, the bits maintain their T-second value. The NRZ waveform corresponding to a typical data sequence is illustrated in Fig. 1. The generalized M-ary source model discussed in [6, Chapter 2] requires that one find the set of so-called elementary signals, which represent the possible waveforms that can occur between adjacent integer multitudes of the bit time T in a waveform generated by an infinitely long data sequence. As such, each of these elementary signals is characterized by a T-second duration. This is where the signal model proposed in [4,5] is in error, namely, the unbalanced, asymmetric NRZ source is characterized there in terms of four elementary signals, two of which have durations unequal to the nominal bit time duration, T. From Fig. 1, we see that the correct source model is given in terms of three elementary signals, each of T-second duration, as described below:[3]

$$\left.\begin{aligned}
s_1(t) &= A, \quad 0 \le t \le T \\[2ex]
s_2(t) &= \begin{cases} A & 0 \le t \le \eta T \\ -A & \eta T < t \le T \end{cases} \\[2ex]
s_3(t) &= -A \quad 0 \le t \le T \\[1ex]
&= -s_1(t)
\end{aligned}\right\} \tag{1}$$

[3] Note that the elementary waveforms are described here in the interval $0 \le t \le T$, whereas in [6], they are described symmetrically around the origin. Since the PSD is invariant to a time shift, either characterization when correctly done will yield the same result.

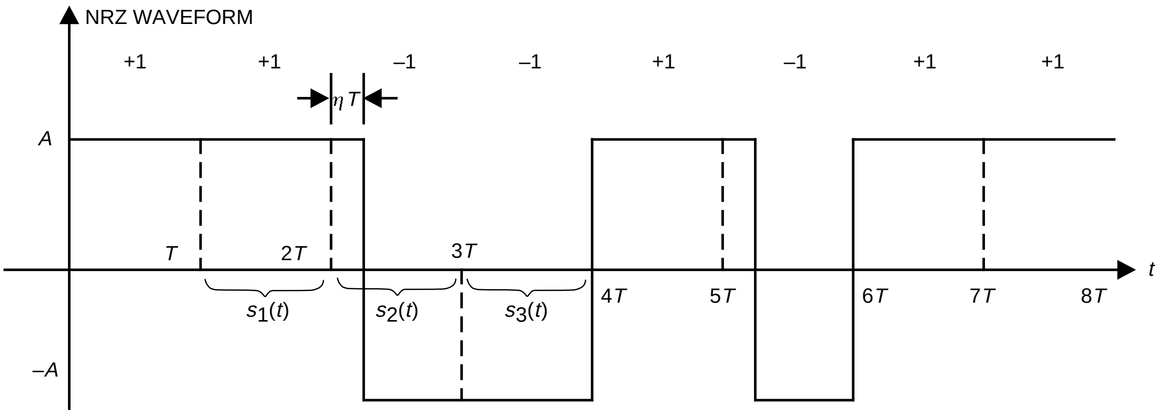

Fig. 1. An unbalanced, asymmetric NRZ waveform for a typical data sequence.

where A is the signal amplitude that can be related to the signal power and, following the notation in [5], we let $\eta \triangleq \Delta/2$. Note that $s_1(t)$ occurs whenever the data sequence (1,1) or $(-1,1)$ occurs. Similarly, $s_2(t)$ occurs whenever a $(1,-1)$ sequence occurs and, finally, $s_3(t)$ occurs whenever a $(-1,-1)$ occurs.

The Fourier transforms of each of the elementary signals in Eq. (1) are given by

$$\left.\begin{aligned} S_1(f) &= ATe^{-j\pi fT}\left(\frac{\sin \pi fT}{\pi fT}\right) \\ S_2(f) &= 2A\eta Te^{-j\pi f\eta T}\left(\frac{\sin \pi f\eta T}{\pi f\eta T}\right) - S_1(f) \\ S_3(f) &= -ATe^{-j\pi fT}\left(\frac{\sin \pi fT}{\pi fT}\right) = -S_1(f) \end{aligned}\right\} \tag{2}$$

In order to compute the discrete spectrum component, we must compute the stationary probabilities of the three elementary signals, i.e., their probabilities of occurrence in a waveform generated by an infinitely long data sequence. Here the model assumed for the data source, i.e., the manner in which the +1's and −1's are generated, is important. If we assume a purely random data source, where +1's and −1's are independently generated with probabilities p and $q = 1-p$, respectively, then based on the sequence pairs that generate the three elementary signals as discussed above, the stationary probabilities of these three signals are

$$\left.\begin{aligned} p_1 &= p^2 + pq = p^2 + p(1-p) \\ p_2 &= pq = p(1-p) \\ p_3 &= q^2 = (1-p)^2 \end{aligned}\right\} \tag{3}$$

In [4,5], the stationary probabilities were calculated using a mixture of a random p, q data source (as assumed here) and a first-order Markov source with transition probability p_t. For example, the probability of a (1,1) sequence, which generates the elementary signal $s_1(t)$, was computed as $p(1-p_t)$ (the probability of a +1 and no transition in the data) rather than as p^2, which is the joint probability of two successive independent +1's.

From [6, Eq. (2.53)], the discrete component of the PSD for an arbitrary M-ary source (here $M = 3$) is given by

$$S_d(f) = \frac{1}{T^2} \sum_{n=-\infty}^{\infty} \left| \sum_{i=1}^{M} p_i S_i\left(\frac{n}{T}\right) \right|^2 \delta\left(f - \frac{n}{T}\right) \tag{4}$$

which is a line spectrum with components at integer multiples of the data rate. Using Eqs. (2) and (3) in Eq. (4) results after simplification in

$$S_d(f) = A^2(-1 + 2p + 2p(1-p)\eta)^2 \delta(f) + A^2(2p(1-p))^2 \eta^2 \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(\frac{\sin n\pi\eta}{n\pi\eta}\right)^2 \delta\left(f - \frac{n}{T}\right) \tag{5}$$

or, in terms of the signal energy $E = A^2 T$ and the transition probability of a random p, q sequence $p_t \triangleq 2p(1-p)$, we have the normalized form

$$\frac{S_d(f)}{E} = \frac{1}{T}(-1 + 2p + p_t\eta)\delta(f) + \frac{1}{T} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} p_t^2 \eta^2 \left(\frac{\sin n\pi\eta}{n\pi\eta}\right)^2 \delta\left(f - \frac{n}{T}\right) \tag{6}$$

The continuous component of the PSD is a bit more tedious to compute. Here we need to first compute the set of equivalent elementary signals $s_i'(t), i = 1, 2, \cdots, M$ generated by subtracting from each elementary signal the statistical mean of the set, i.e.,

$$s_i'(t) \triangleq s_i(t) - \sum_{k=1}^{M} p_k s_k(t), \quad i = 1, 2, \cdots, M \tag{7}$$

From Eqs. (2) and (3), the Fourier transforms of these equivalent elementary signals can be expressed in the form

$$S_i'(f) = \alpha_{i1} F_1(f) + \alpha_{i2} F_2(f), \quad i = 1, 2, \cdots, M \tag{8}$$

where

$$\left.\begin{aligned} F_1(f) &= ATe^{-j\pi fT}\left(\frac{\sin \pi fT}{\pi fT}\right) \\ F_2(f) &= A\eta Te^{-j\pi f\eta T}\left(\frac{\sin \pi f\eta T}{\pi f\eta T}\right) \end{aligned}\right\} \tag{9}$$

and

$$\left.\begin{aligned} \alpha_{11} &= 2(1-p) \quad \alpha_{12} = -p_t \\ \alpha_{21} &= \;-2p, \quad \alpha_{22} = 2 - p_t \\ \alpha_{31} &= \;-2p, \quad \alpha_{32} = -p_t \end{aligned}\right\} \tag{10}$$

We also need to specify the transition probability matrix $\mathbf{P} = \{p_{ik}\}$ where p_{ik} denotes the probability that elementary signal $s_k(t)$ is transmitted in a given transmission interval after the occurrence of elementary signal $s_i(t)$ in the previous transmission interval. For the problem at hand, the matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{pmatrix} p & q & 0 \\ p & 0 & q \\ p & 0 & q \end{pmatrix} \tag{11}$$

Note the zeros in the matrix of Eq. (11), which implies that certain elementary signals cannot occur just after the occurrence of certain others. For example, because of the assumed asymmetry model, $s_3(t)$ cannot occur after $s_1(t)$.

To evaluate the continuous PSD component, we need to compute the elements of $\mathbf{P}$ raised to integer powers. Multiplying $\mathbf{P}$ by itself gives

$$\mathbf{P}^2 = \begin{pmatrix} p & pq & q^2 \\ p & pq & q^2 \\ p & pq & q^2 \end{pmatrix} \triangleq \left\{p_{ik}^{(2)}\right\} \tag{12}$$

which is a matrix with three identical rows. From this, it is straightforward to show that

$$\mathbf{P}^n \mathbf{P}^2 = \mathbf{P}^2 \tag{13}$$

for all integer n. As in [6], defining

$$p_{ik}(z) = \sum_{n=1}^{\infty} p_{ik}^{(n)} z^n \tag{14}$$

where z is a parameter and $p_{ik}^{(n)}$ is the ikth element of $\mathbf{P}^n$, and then using Eqs. (11) through (13), we arrive at

$$p_{ik}(z) = p_{ik} z + p_{ik}^{(2)} \sum_{n=2}^{\infty} z^n = \left(p_{ik} - p_{ik}^{(2)}\right) z + \frac{p_{ik}^{(2)}}{1-z} \tag{15}$$

where we have also recognized that $p_{ik}^{(1)} = p_{ik}$. Evaluating Eq. (15)and arranging the results in the form of a matrix gives

$$P_{ik}(z) \triangleq \{p_{ik}(z)\} = \begin{pmatrix} \dfrac{p}{1-z} & q^2 z + \dfrac{pq}{1-z} & -q^2 z + \dfrac{q^2}{1-z} \\ \dfrac{p}{1-z} & -pqz + \dfrac{pq}{1-z} & pqz + \dfrac{q^2}{1-z} \\ \dfrac{p}{1-z} & -pqz + \dfrac{pq}{1-z} & pqz + \dfrac{q^2}{1-z} \end{pmatrix} \tag{16}$$

Finally, from [6, Eq. (2.53)], the continuous component of the PSD in normalized form is given by

$$\frac{S_c(f)}{E} = \frac{1}{T^2}\sum_{i=1}^{3} p_i |S_i'(f)|^2 + \frac{2}{T^2}\mathbf{Re}\left\{\sum_{i=1}^{3}\sum_{k=1}^{3} p_i S_i'^{*}(f) S_k'(f) p_{ik}\left(e^{-j2\pi fT}\right)\right\} \tag{17}$$

where

$$p_{ik}\left(e^{-j2\pi fT}\right) \triangleq p_{ik}(z)\big|_{z=e^{-j2\pi fT}}, \quad i,k = 1,2,3 \tag{18}$$

III. Unbalanced, Asymmetric Biphase Modulation

In this section, we again follow [4,5] by using the data asymmetry model originally proposed in [8], which assumes that, for a $+1$ bit, the first half is elongated by $\Delta T/4$ (relative to its nominal value of $T/2$ seconds) and, for a -1 bit, the first half is shortened by the same amount. When a data transition follows, the second half of the bit retains its nominal $T/2$-second value. When no data transition occurs, the second half of the bit adjusts itself so that the total bit maintains its nominal T-second value. A biphase waveform for a typical data sequence is illustrated in Fig. 2. Based on this waveform, we see that once again there are three elementary signals each of T-second duration, as described below:

$$\begin{aligned} s_1(t) &= \begin{cases} A, & 0 \le t \le (2\eta+1)\dfrac{T}{2} \\ -A, & (2\eta+1)\dfrac{T}{2} < t \le T \end{cases} \\ s_2(t) &= \begin{cases} -A, & 0 \le t \le \dfrac{T}{2} \\ A, & \dfrac{T}{2} < t \le T \end{cases} \\ s_3(t) &= \begin{cases} A, & 0 \le t \le \eta T \\ -A, & \eta T < t \le \dfrac{T}{2} \\ A, & \dfrac{T}{2} < t \le T \end{cases} \end{aligned} \tag{19}$$

where A again is the signal amplitude that can be related to the signal power and, following the notation in [5], we let $\eta \triangleq \Delta/4$. Note that, as in the NRZ case, $s_1(t)$ occurs whenever the data sequence (1,1) or $(-1,1)$ occurs. Similarly, $s_2(t)$ occurs whenever a $(1,-1)$ sequence occurs and, finally, $s_3(t)$ occurs whenever a $(-1,-1)$ occurs.

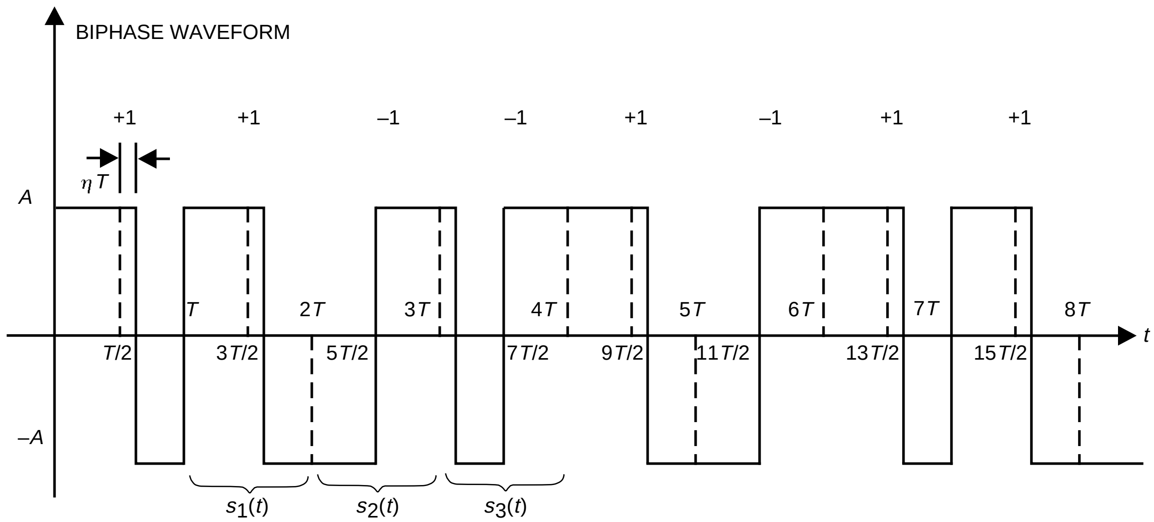

Fig. 2. An unbalanced, asymmetric biphase waveform for a typical data sequence.

The Fourier transforms of each of the elementary signals in Eq. (19) are given by

$$\left.\begin{aligned} S_1(f) &= 2A\eta T e^{-j\pi f(1+\eta)T}\left(\frac{\sin \pi f\eta T}{\pi f \eta T}\right) - S_2(f) \\ S_2(f) &= -jATe^{-j\pi fT}\left(\frac{\sin^2 \pi f\dfrac{T}{2}}{\pi f\dfrac{T}{2}}\right) \\ S_3(f) &= 2A\eta T e^{-j\pi f\eta T}\left(\frac{\sin \pi f\eta T}{\pi f\eta T}\right) + S_2(f) \end{aligned}\right\} \tag{20}$$

Since the data sequence pairs that generate the stationary signal of Eq. (19) are identical to those that generate the corresponding NRZ stationary signals, then the stationary probabilities are once again given by Eq. (3). Thus, using Eqs. (3) and (20) in Eq. (4), the discrete component of the PSD is, after considerable simplification, given by

$$\begin{aligned} S_d(f) &= 4A^2(1-p+p^2)^2\eta^2\delta(f) + 4A^2(1-p+p^2)^2\eta^2 \sum_{\substack{n=-\infty \\ n\neq 0 \\ n \text{ even}}}^{\infty} \left(\frac{\sin n\pi\eta}{n\pi\eta}\right)^2 \delta\left(f-\frac{n}{T}\right) \\ &\quad + A^2 \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \left[4(1-3p+p^2)^2\eta^2\left(\frac{\sin n\pi\eta}{n\pi\eta}\right)^2 - 4(1-3p+p^2)\eta\left(\frac{2}{n\pi}\right)(1-2p)\left(\frac{\sin^2 n\pi\eta}{n\pi\eta}\right)\right. \\ &\quad \left. + \left(\frac{2}{n\pi}\right)^2(1-2p)^2\right]\delta\left(f-\frac{n}{T}\right) \end{aligned} \tag{21}$$

or, in terms of the signal energy $E = A^2T$,

$$\frac{S_d(f)}{E} = \frac{1}{T}4(1-p+p^2)^2\eta^2\delta(f) + \frac{1}{T}4(1-p+p^2)^2\eta^2 \sum_{\substack{n=-\infty \\ n\neq 0 \\ n \text{ even}}}^{\infty} \left(\frac{\sin n\pi\eta}{n\pi\eta}\right)^2 \delta\left(f - \frac{n}{T}\right)$$

$$+ \frac{1}{T} \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \left[4(1-3p+p^2)^2\eta^2 \left(\frac{\sin n\pi\eta}{n\pi\eta}\right)^2 - 4(1-3p+p^2)\eta \left(\frac{2}{n\pi}\right)(1-2p)\left(\frac{\sin^2 n\pi\eta}{n\pi\eta}\right)\right.$$

$$\left. + \left(\frac{2}{n\pi}\right)^2 (1-2p)^2\right] \delta\left(f - \frac{n}{T}\right) \tag{22}$$

For no data symmetry, i.e., $\eta = 0$, Eq. (22) reduces to

$$\frac{S_d(f)}{E} = \frac{1}{T}(1-2p)^2 \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \left(\frac{2}{n\pi}\right)^2 \delta\left(f - \frac{n}{T}\right) \tag{23}$$

which agrees with [6, Eq. (2.67)].

It is also straightforward to show that the Fourier transforms of the equivalent elementary signals defined in Eq. (7) are also given in a form similar to Eq. (8), namely,

$$S_i'(f) = \alpha_{i1}F_1(f) + \alpha_{i2}F_2(f) + \alpha_{i3}F_3(f), \quad i = 1, 2, \cdots, M \tag{24}$$

where

$$\left.\begin{aligned}
F_1(f) &= jATe^{-j\pi fT}\left(\frac{\sin^2 \pi f\frac{T}{2}}{\pi f\frac{T}{2}}\right) \\
F_2(f) &= A\eta Te^{-j\pi f(1+\eta)T}\left(\frac{\sin \pi f\eta T}{\pi f\eta T}\right) \\
F_3(f) &= A\eta Te^{-j\pi f\eta T}\left(\frac{\sin \pi f\eta T}{\pi f\eta T}\right)
\end{aligned}\right\} \tag{25}$$

and

$$\left.\begin{aligned}
&\alpha_{11} = 2(1-p), \quad &&\alpha_{12} = 2(1-p), \quad &&\alpha_{13} = -2(1-p)^2 \\
&\alpha_{21} = -2p, &&\alpha_{22} = -2p, &&\alpha_{23} = -2(1-p)^2 \\
&\alpha_{31} = -2p, &&\alpha_{32} = -2p, &&\alpha_{33} = 2p(2-p)
\end{aligned}\right\} \tag{26}$$

The transition probability matrix $\mathbf{P} = \{p_{ik}\}$ is identical to Eq. (11), and thus the formula for the continuous spectrum as given by Eq. (17) still applies.

IV. Numerical Evaluations and Simulation Results

Illustrated in Figs. 3 and 4 are evaluations of the analytical PSD results for NRZ and biphase modulations as given by the combinations of Eqs. (5) and (17) and Eqs. (22) and (17), respectively. Also shown are computer simulation results obtained for the same unbalance and asymmetry parameter values. In particular, for the NRZ simulations, a sampling rate of 64 Hz and a bit rate of 1 Hz are maintained so that there are 64 samples per bit in each bit duration. Moreover, the fast Fourier transform (FFT) size, p, and η were set to 8192 points, 0.55, and 4.7 percent, respectively. For the biphase simulations, on the other hand, all the parameter values were kept the same as the NRZ case except that the sampling rate and FFT size were increased to 128 Hz (providing 64 samples per half-bit duration) and 16,384 points, respectively. Hence, the resolution bandwidth for the continuous NRZ and biphase PSD are 64/8192 and 128/16,384, respectively. The continuous PSD for the analytical results are given at a resolution bandwidth of 1 Hz. In order to compare the analytical and simulation continuous PSD at the same resolution, the analytical PSDs were adjusted in Figs. 3 and 4 so that they have the same resolution bandwidth as the simulation. Note that the discrete components are independent of the resolution bandwidth. For both NRZ and biphase simulations, 100 FFTs were averaged in order to decrease the noise due to the data sequence. As can be seen, the analytical and simulation results are in excellent agreement.

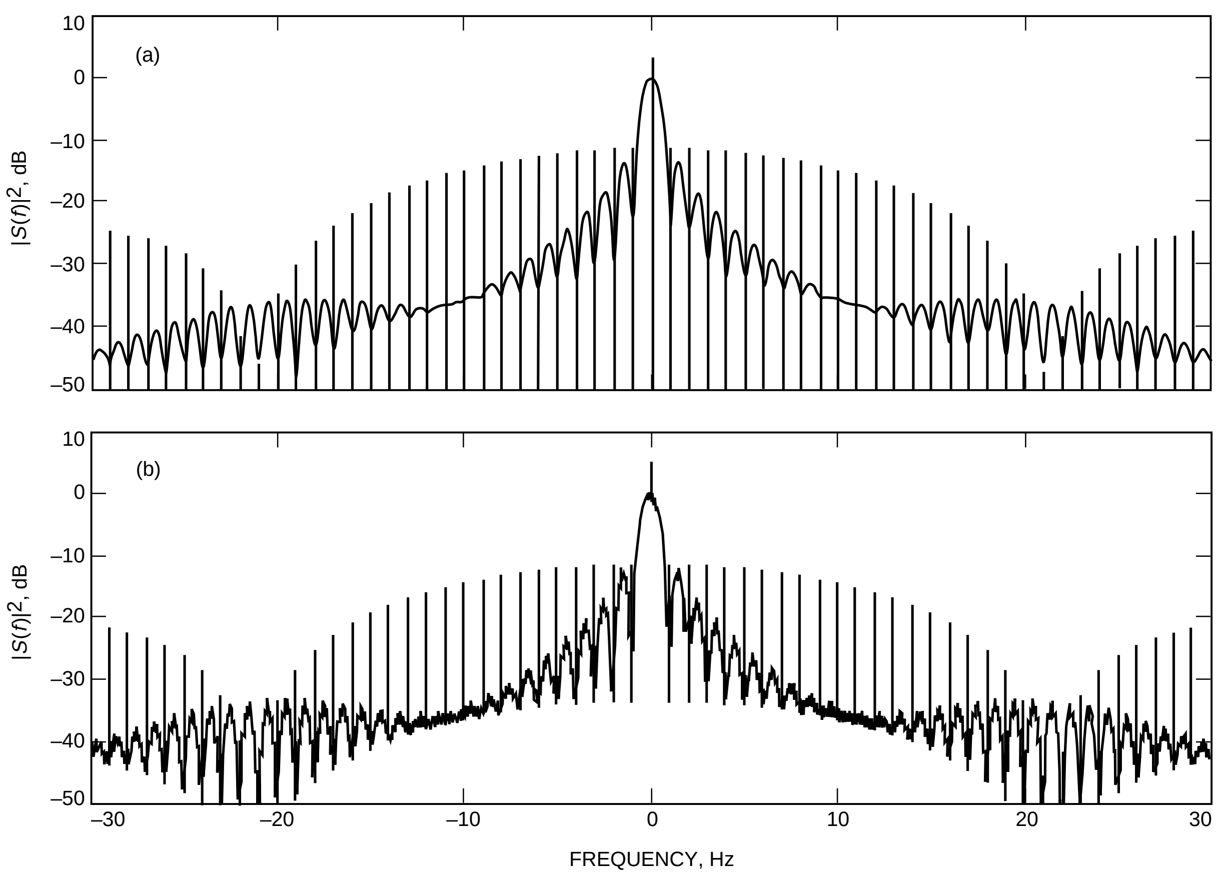

Fig. 3. PSD for NRZ data with η = 4.7 percent and p = 0.55: (a) analytical and (b) simulation.

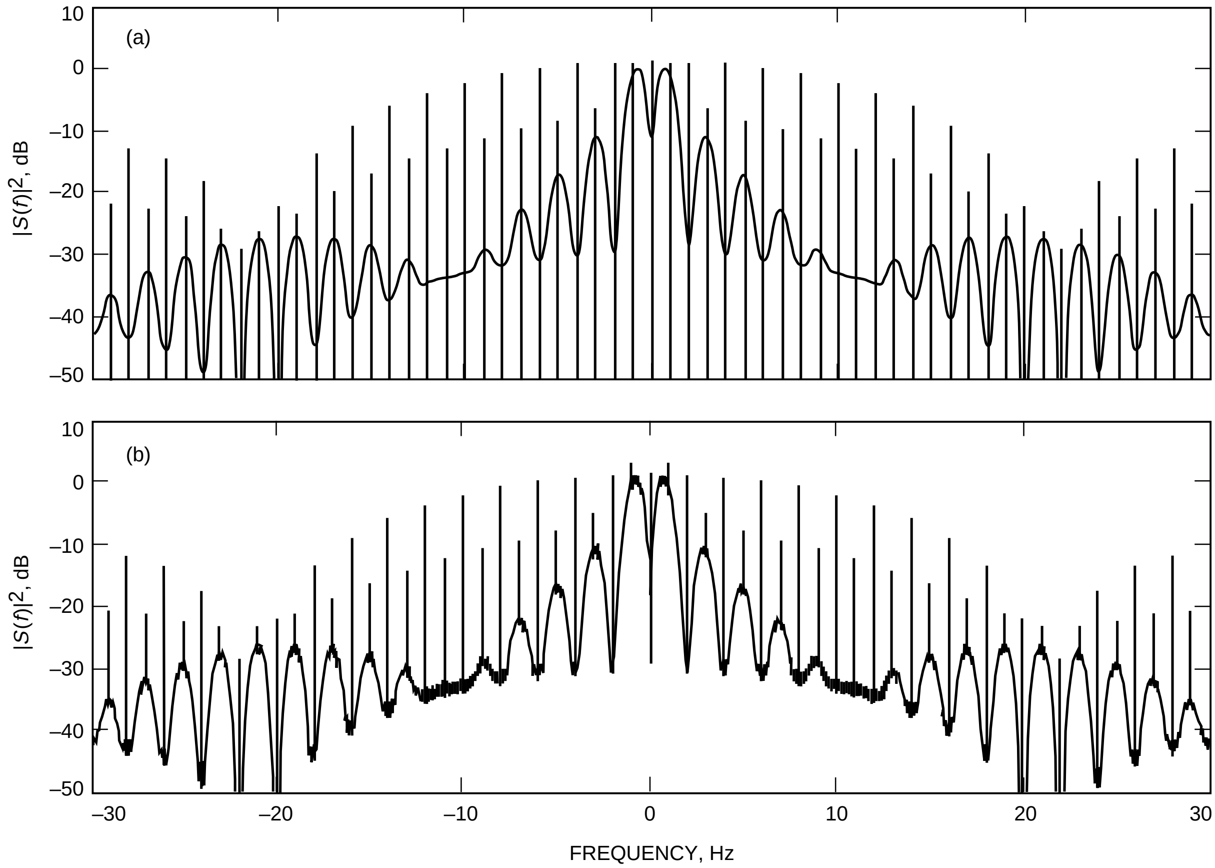

Fig. 4. PSD for biphase data with η = 4.7 percent and p = 0.55: (a) analytical and (b) simulation.

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