Eigen Theory for Optimal Signal Combining: A Unified Approach

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In this article, we establish a general theoretical framework for optimal signal combining using the eigen theory approach. We also describe how the eigen approach can be applied to current and future arraying implementation.

I. Introduction

In the past few years, there has been much interest at JPL in the investigation and development of efficient signal arraying techniques. As JPL missions begin to adopt Ka-band (32-GHz) for downlink communications, and the mechanical control and structural integrity of large antennas approach their limits, using advanced signal processing to array smaller antennas becomes a cost effective and flexible option for the Deep Space Network (DSN) to meet its current and future needs. Arraying can be applied to the feed horn at the front end of the antenna to increase the efficiency of the front end, to perform atmospheric noise cancellation, and to assist antenna pointing [1–4]. It can also be applied to intra- and inter-complex antenna arraying to increase the effective aperture, for example, in full spectrum combining, complex symbol combining, and symbol combining [5,6]. The effects of colored noise on full spectrum combining and complex symbol combining have also been analyzed [7].

Arraying has been used in many missions since the 1970s [8]. The more recent ones are the Voyager encounters with Uranus and Neptune that used symbol-stream combining; the ongoing Galileo S-band (2.3-GHz) mission, which uses both full-spectrum combining and complex-symbol combining; and the upcoming 34-m arraying task that uses full-spectrum combining to array several intracomplex 34-m antennas to mimic the functionality of a single 70-m antenna. Currently, the DSN uses different combiner architectures to combine signals for different combining schemes.

The key signal processing step in signal arraying is to find the set of combining weight coefficients that maximizes the combined signal-to-noise ratio (SNR). The current full-spectrum combiner and complex-symbol combiner assume uncorrelated noise[1] and weights the signal streams proportionally to the estimated SNRs. More sophisticated schemes that take into account the noise correlation were proposed by Victor Vilnrotter, Eugene Rodemich, Sam Dolinar, Harry Tan, and others [1–3,10,11]. Their approach for finding optimal weights is to apply the Cauchy–Schwartz inequality to the SNR expression of the combined output samples and to evaluate the weight vector that achieves the maximum SNR. This approach

[1] The full-spectrum combiner has added an ad hoc suboptimal scheme that subtracts the correlated noise from the received signals where it is present (D. Rogstad and R. Kahn, personal communication, Jet Propulsion Laboratory, Pasadena, California, April 16, 1996).

for finding the optimal combining weights is in fact a well-known technique in the adaptive arraying area [9]. In this formulation, the optimal weight vector is expressed as the conjugate of the product of the inverse of the noise correlation matrix, Θ_Z, and the signal vector, $\underline{s}$. This approach requires estimating both the noise statistics and the signal statistics from the received observables.

In this article, we propose a unified approach to performing optimal signal combining using eigen theory. Some special features of this approach are as follows:

(1) This scheme requires one to estimate only the pairwise noise correlations, and not the signal statistics. The other statistics used are derived from the directly observed samples. We will show in later sections that, by bypassing the signal statistics estimation, this approach allows new ways to do signal arraying.

(2) The eigen approach is a generalization of the optimal signal combining techniques previously developed in [1] and [11].

(3) Tan [11] proposed an optimal combining scheme for uncorrelated and correlated noise, but for residual carrier signals only. Rogstad [6] developed a full-spectrum combining scheme for residual and suppressed carrier signals, but for uncorrelated noise only.[2] This scheme works on both residual and suppressed carriers, and for both correlated and uncorrelated noise.

The rest of this article is organized as follows: Section II lays the theoretical framework of optimal arraying. We first discuss the basic ideas of signal arraying and define the terminologies. We show that we can cast the problem of finding the optimal weights into an eigen theory problem. We derive the main theoretical results, which show that the optimal weights can be evaluated without using the signal statistics. We prove that this scheme achieves the same optimal weights as derived from the Cauchy–Schwartz inequality when specific assumptions required by the previous approaches are imposed (see Section II.C). Section III ties our results to an information theoretic result described in [12] and shows that the linear combining scheme proposed in this article achieves array channel capacity for a broadband source in the presence of Gaussian noise. Section IV describes how the eigen approach for signal combining can be applied to the current array feed combining, complex-symbol combining, and full-spectrum combining implementations. Section V discusses future work and gives concluding remarks.

II. Arraying Theory

A. Basic Definitions

We use the same arraying model as described in [12]. We consider transmitting a single source through n channels, as illustrated in the array system in Fig. 1. Unless otherwise stated, all quantities are assumed to be complex. Symbols in capital letters are used to denote random variables or random vectors, and symbols in lower-case letters are used to represent deterministic quantities.[3] Let $\underline{X}^T = (X_1, X_2, \cdots, X_n)$ be the vector of transmitted signals, $\underline{Z}^T = (Z_1, Z_2, \cdots, Z_n)$ be the vector of noise components, and $\underline{Y}^T = (Y_1, Y_2, \cdots, Y_n)$ be the vector of channel outputs. Let $\underline{w}^T = (w_1, w_2, \cdots, w_n)$ be a complex weight vector. Let Θ_X be the correlation matrix of $\underline{X}$,[4] Θ_Z be the correlation matrix of $\underline{Z}$, and Θ_Y

[2] Ibid.

[3] An exception is the notation for correlation matrices, Θ, which we represent with capital Greek letters.

[4] See Section II.C on the discussion of Θ_X for signals that are assumed to be deterministically related within the estimation window and for signals that are stochastically related within the estimation window.

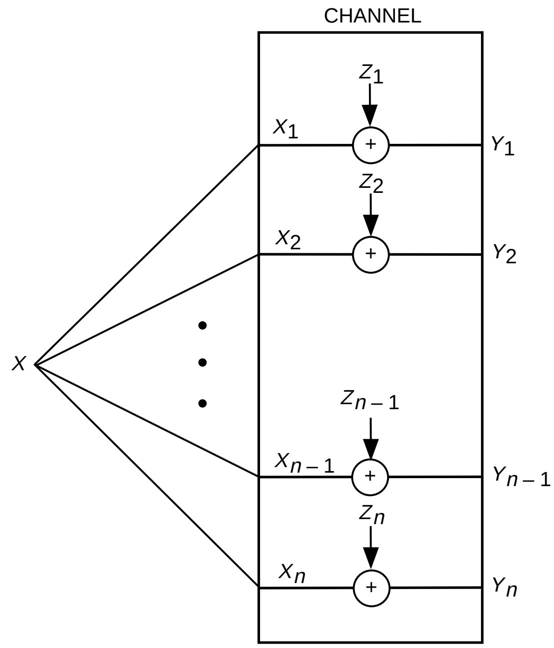

Fig. 1. The array channel model.

be the correlation matrix of $\underline{Y}$. The channel outputs are combined as shown in Fig. 2, and the SNR of the scalar output $V = \underline{w}^T \underline{Y}$ is defined by Definition 1:

$$
\begin{aligned}
SNR &= \frac{E(|\underline{w}^T \underline{X}|^2)}{E(|\underline{w}^T \underline{Z}|^2)} \\
&= \frac{\underline{w}^T \Theta_X \underline{w}^*}{\underline{w}^T \Theta_Z \underline{w}^*} \qquad (1)
\end{aligned}
$$

B. Optimal Weights for Arraying

With the above problem formulation, we now state and derive our first theorem in arraying.

Theorem 1. *Based on the definition of SNR as given in Definition 1, the maximum SNR of an array system is the largest eigenvalue of* $\Theta_Z^{-1}\Theta_X$, *and the weight vector that maximizes the SNR is the conjugate of the corresponding eigenvector.*

Proof: Using the standard vector gradient method, we differentiate Eq. (1) with respect to $\underline{w}$ and set the resulting expression to $\underline{0}$. We obtain

$$
\Theta_Z^{-1}\Theta_X \underline{w}^* - \frac{\underline{w}^T \Theta_X \underline{w}^*}{\underline{w}^T \Theta_Z \underline{w}^*} \underline{w}^* = \underline{0}
$$

This shows that the maximum SNR is the largest eigenvalue of $\Theta_Z^{-1}\Theta_X$, and the optimal weight vector is the conjugate of the corresponding eigenvector. ❐

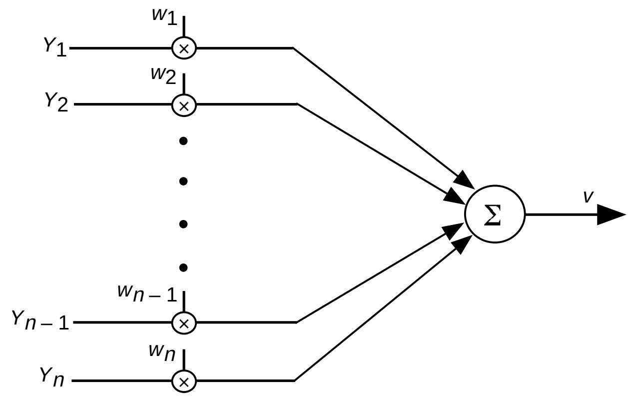

Fig. 2. Linear combining of signals.

We have a few comments on Theorem 1:

(1) The general result of the ratio of two quadratic forms $(x^T Ax)/(x^T Bx)$ (where B is nonsingular) achieving stationary values at the roots of $B^{-1}A$ is not new; in fact, it can be found in the *CRC Standard Mathematical Tables* [13]. Here we provide a simple proof to this result, and we believe that the application of this result in the context of feed-forward linear combining is new.

(2) The result in Theorem 1 is directly derived from Definition 1. In this case, we still need to estimate the signal statistics to find the optimal weights. For additive data-independent noise, which accounts for most of our noise environment, $\Theta_Y = \Theta_X + \Theta_Z$. We rewrite the SNR definition as Definition 2:

$$SNR = \frac{E(|\underline{w}^T \underline{Y}|^2)}{E(|\underline{w}^T \underline{Z}|^2)} - 1$$

$$= \frac{\underline{w}^T \Theta_Y \underline{w}^*}{\underline{w}^T \Theta_Z \underline{w}^*} - 1 \tag{2}$$

We now state the next theorem:

Theorem 2. *Using Definition 2, the maximum SNR of an array system is the largest eigenvalue of* $\Theta_Z^{-1}\Theta_Y$ *minus one, and the weight vector that maximizes the SNR is the conjugate of the corresponding eigenvector.*

Proof: Using the same gradient technique as in the proof of Theorem 1, we show that the maximum of the ratio $(\underline{w}^T \Theta_Y \underline{w}^*)/(\underline{w}^T \Theta_Z \underline{w}^*)$ (and thus $[(\underline{w}^T \Theta_Y \underline{w}^*)/(\underline{w}^T \Theta_Z \underline{w}^*)] - 1$) is the largest eigenvalue of $\Theta_Z^{-1}\Theta_Y$, and the optimal weight vector is the conjugate of the corresponding eigenvector. This completes the proof. ❒

The important consequence of Theorem 2 is that it only requires estimation of the noise correlations. The statistics of $\underline{Y}$ are derived from the directly observed samples. This avoids making additional assumptions on the signal statistics and opens up new ways to perform faster and more accurate estimation of combining weights. We will show in the following sections that this theoretical formulation offers new practical ways to perform signal combining.

C. Comparisons With Previous Results

In this section, we show that Theorems 1 and 2 are consistent and generalize the previous results in optimal signal combining. We discuss the results in [1] (uncorrelated noise) and [11] (correlated noise), which evaluate the maximum SNR and find the optimal weights in terms of the noise and signal statistics.

(1) Uncorrelated noise [1]: In Vilnrotter's article, the noise correlation matrix Θ_Z is a diagonal matrix of (uncorrelated noise assumption) with elements of noise variance $2\sigma_{Z,1}^2$, $2\sigma_{Z,2}^2$, $\cdots$, $2\sigma_{Z,n}^2$. The signal correlation matrix is of the form $\Theta_X = \underline{s}\underline{s}^\dagger$, where $\dagger$ is defined as the conjugate transpose of a vector ($\underline{s}^\dagger = \underline{s}^{*T}$). The optimal weight vector is given by Eq. (24) of [1]: $\underline{w}^T = (s_1^*/2\sigma_1^2, s_2^*/2\sigma_2^2, \cdots, s_n^*/2\sigma_n^2)$. It can also be written as $\underline{w} = (\Theta_Z^{-1}\underline{s})^*$. The maximum achievable SNR is given by Eq. (23) of [1]: $SNR_{max} = \sum_{i=1}^n (|s_i|^2)/(2\sigma_i^2)$, and can be written as $SNR_{max} = \underline{s}^\dagger \Theta_Z^{-1}\underline{s}$. We multiply $\Theta_Z^{-1}\Theta_X$ with $\underline{w}^*$ and we get

$$
\begin{aligned}
\Theta_Z^{-1}\Theta_X \underline{w}^* &= \Theta_Z^{-1}\underline{s}\underline{s}^\dagger \Theta_Z^{-1}\underline{s} \\
&= SNR_{max}\Theta_Z^{-1}\underline{s} \\
&= SNR_{max}\underline{w}^*
\end{aligned}
$$

This shows that SNR_{max} is an eigenvalue of $\Theta_Z^{-1}\Theta_X$, and $\underline{w}^* = \Theta_Z^{-1}\underline{s}$ is the corresponding eigenvector (Theorem 1).

(2) Correlated noise [11]: In Tan's article, the optimal weight is given by Eq. (12) of [11]: $\underline{w} = (\Theta_Z^{-1}\underline{s})^*$; and the maximum SNR is given by Eq. (13): $SNR_{max} = \underline{s}^\dagger \Theta_Z^{-1}\underline{s}$. Following the same argument as in the uncorrelated noise case, we show that SNR_{max} is an eigenvalue of $\Theta_Z^{-1}\Theta_X$, and $\underline{w}^* = \Theta_Z^{-1}\underline{s}$ is the corresponding eigenvector (Theorem 1).

We conclude this section with a discussion of Θ_X. Both [1] and [11] discuss the arraying problem in the context of array feed combining of residual carrier signals; [1] and [11] measure the relative magnitudes and phases between signals by estimating the individual residual carrier component s_i from the observable Y_i. This requires averaging out the data and noise (assuming both the data and noise have zero means) within an estimation window. The underlying assumption in this approach is that the residual carrier components s_i's (which contain the magnitude and phase information of signals) remain constant in the span of estimation. This assumption is valid for array feed, since all feed horns see the same signal at the same time and the different phases are due solely to the fixed geometry of the feed arrangement. This assumption is probably true for DSN ground antenna arraying, where the factors (e.g., wind gusts, antenna mechanical movements, etc.) affecting signal magnitudes and phases are usually slow changing compared to the estimation span. Thus, Θ_X can be written as $\underline{s}\underline{s}^\dagger$, and $\underline{w} = (\Theta_Z^{-1}\underline{s})^*$, given in [11], gives the optimal weight vector in this case.

In a general dynamic situation when the relative magnitudes and phases of signals can vary within the estimation window, Θ_X is not of the form $\underline{s}\underline{s}^\dagger$, and the optimal weights can be found using Theorem 2. Thus, this approach might be particularly useful for radio frequency interference cancellations, mobile communications with multipaths, and military communications with jamming noise.

III. Arraying Broadband Signals in the Presence of Gaussian Noise

In [12], we formulated the framework for evaluating the channel capacity of an array system. We derived a closed-form general formula for the channel capacity when arraying a broadband signal (with Gaussian distribution) in n Gaussian noise channels:

$$C_{array} = \frac{1}{2}\log_2(1 + \underline{s}^{\dagger}\Theta_Z^{-1}\underline{s}) \quad \text{bits/channel use} \tag{3}$$

This formula applies to correlated and uncorrelated noise, as long as the Gaussian assumption holds and the second-order statistics (correlation) of the signal and noise sources can be characterized. As discussed in Section IV of [12], this formula depicts an arraying receiver structure that achieves channel capacity as given in Eq. (3). This is done by linear combining of the received signals with the optimal weight, $\underline{w}_{opt}$, given by

$$\underline{w}_{opt} = a(\Theta_Z^{-1}\underline{S})^* \tag{4}$$

where a is an arbitrary complex constant. As discussed in Section II.C, Eq. (4) is a special case of Theorems 1 and 2. Thus, the eigen approach for optimal linear combining achieves channel capacity for a broadband source in the presence of Gaussian noise.

A more interesting problem is to find out whether the eigen approach can achieve channel capacity for a narrowband M-ary source in the presence of Gaussian noise. We conjecture that the eigen approach can also achieve channel capacity in this case.

IV. Eigen Approach for Current Arraying Implementations

The theoretical treatment of arraying described in Section II provides a general framework for performing optimal arraying using the pairwise correlations of the directly observed samples, $\underline{Y}$, and the estimated pairwise correlations of noise, $\underline{Z}$.[5] In this section, we discuss how this approach can be applied to enhance the current implementations.

A. Galileo's Full-Spectrum Combining Task and 34-m Antenna-Arraying Task

Both implementations use full-spectrum combining, with the 34-m antenna-arraying task using more hardware to improve the throughput rate.[6] Both systems perform combining before carrier, subcarrier, and symbol synchronizations,[7] and are essentially designed to combine signals with uncorrelated noise only.[8] Both implementations apply to the square-wave subcarrier modulated signals: they first open-loop downconvert the IF signals to baseband, bandpass the 1st, 3rd, 5th, and 7th harmonics (positive and negative) of the subcarrier signals, and translate each of the 8 harmonics to baseband. The signals of each harmonic are first phase aligned before they are combined according to their respective estimated SNR. The noise statistics can be obtained from any even harmonics where there are no spacecraft signals. In Galileo's implementation, the 10th harmonic is used internally as a noise reference channel. The eigen approach is ideal for this setup. The 10th harmonics of the ith and jth subcarrier signals can be used to compute the pairwise correlation $EZ_iZ_j^*$ (thus the noise correlation matrix Θ_Z). Similarly,

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[5] This is guaranteed by Theorem 2.

[6] Galileo's maximum throughput requirement is 640 symbols/second, and the 34-m-antenna arraying task's maximum throughput requirement is 6.6 megasymbols/second.

[7] In actual implementations, a coarse symbol synchronization is performed on each receiving signal stream to parse out the samples of each symbol in order to boost up the SNR and to facilitate weight estimations.

[8] See Footnote 1.

the received signals from different antennas of the kth harmonic can be used to construct the signal correlation matrix $\Theta_{Y,k}$. The optimal weight vector of the kth harmonic is the eigenvector corresponding to the largest eigenvalue of $\Theta_Z^{-1}\Theta_{Y,k}$. Thus, this approach achieves optimal weights for both uncorrelated and correlated noises and for both residual and suppressed carrier modulations.

B. Galileo's Symbol Combiner Demodulator Task

Galileo's symbol combiner demodulator performs complex symbol combining after subcarrier and symbol synchronizations. Like the full-spectrum combining scheme, this scheme phase rotates to align the received signals before combining, thus compromising its ability to optimally combine signals in the presence of correlated noises. This approach of computing the eigen vector (the optimal weights) of $\Theta_Z^{-1}\Theta_Y$ orients the received signals to maximize the combined SNR.

C. Advanced Systems Array Feed Task

The current real-time seven-channel array feed prototype computes signal correlations and noise correlations between the peripheral horns and the central horn only.[9] This setup works on broadband signals only and requires that the antenna go on-source and off-source to collect signal and noise statistics. As discussed in Section II, for a narrowband telemetry source, Tan [11] proposed to combine the signals with the weight vector $\underline{w} = (\Theta_Z^{-1}\underline{s})^*$, which is a special case of the eigen approach as discussed in this article. The key difference is that Tan's approach requires the estimation of $\underline{s}$, and this demands the existence of a carrier tone. The eigen approach requires the second-order statistics of the direct observables $\underline{Y}$ instead; thus, it can apply to both residual and suppressed carrier modulations.

V. Conclusion

In this article, we described an optimal approach for performing signal combining using eigen theory. This scheme applies to array feed combining, full-spectrum combining, complex symbol combining, and symbol combining. This unified, generalized, yet optimized, approach to arraying implementation can translate into a common implementation framework that would result in lower costs for production, maintenance, and upgrading for current needs and to anticipate future needs. We believe that this scheme is also suitable for radio frequency interference cancellations, mobile communications, and military communications, but we do not discuss these issues here. We will perform extensive simulations to verify and quantify the performance of this scheme. Future research includes the following:

(1) Investigate efficient techniques for matrix inversion, matrix multiplication, and eigenvalue and eigenvector computations.

(2) Investigate techniques to estimate noise statistics for different kinds of noise (e.g., additive white Gaussian noise, colored noise, jamming noise, etc.).

(3) Quantify and bound the number of samples required to obtain the reliable statistics for different noise distributions.

(4) Investigate the update rules for Θ_Y and Θ_Z.

(5) Investigate applying this technique to mitigate radio frequency interference effects.

(6) Investigate applying this arraying scheme in the more dynamic military environment, which is usually coupled with highly correlated noise (jamming) and fast changing signal and noise statistics.

[9] All pairwise correlations of signal and noise can be used in an off-line mode to combine prerecorded data and to deliver theoretically optimal combining performance.

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