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Power Spectral Density of Digital Pulse Streams in the Presence of Timing Jitter

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The spectral occupancy and composition of a chosen digital signaling technique when the data pulse stream is nonideal, due for instance to implementation imperfections, are important considerations in the design of a practical communication system. One source of imperfection is timing jitter where the rising and falling transitions do not occur at the nominal data transition time instants; nevertheless, the time instants are offset by random amounts relative to the nominal one. The amount of timing shift per transmission interval is random and typically is characterized by a discrete random process (independent of the data sequence) with known statistical properties. The purpose of this article is to characterize the power spectral density (PSD) of baseband signaling schemes in the presence of arbitrary timing jitter. Although general PSD results are first obtained for arbitrary timing jitter statistics, specific results are then given for the cases of practical interest, namely, uniform and Gaussian distributed jitter. Examples of an uncorrelated data pulse stream, an independent identically distributed data stream, and Markov sources are given. Interesting results emerge when the generating sequence $\{a_n\}$ is uncorrelated. For generating sequences $\{a_n\}$ that are nonzero mean, timing jitter has the effect of widening the main lobe of the spectrum and increasing the side lobes. When the generating sequence is zero mean and uncorrelated, a rather surprising result is that the timing jitter does not affect the PSD. Simulation results are also presented to verify the analysis.

I. Introduction

The spectral occupancy and composition of a chosen digital signaling technique are important considerations in the design of a communication system. These spectral properties can be derived from knowledge of the technique's power spectral density (PSD), which in the general case consists of both continuous and discrete components. Evaluation of the PSD for ideal synchronous data pulse streams (a baseband data waveform with a fixed transmission interval and fixed transmission epochs where the underlying data sequence has known statistical properties and the transmitted waveform is a single known pulse shape) is well documented in the literature (see [1, Chapter 2], for example). Most often the data sequence that generates the pulse stream is either wide sense stationary (WSS) or, more generally, wide sense cyclostationary (WSCS), both of which result in a pulse stream that is a WSCS process.

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Evaluation of the PSD when the data pulse stream is nonideal, due for instance to implementation imperfections, is also of interest. One such example is data symmetry where the rising and falling transitions do not occur at the nominal data transition time instants; nevertheless, the transition time instants are offset by fixed amounts relative to the nominal ones [2]. A complete PSD analysis for this form of nonideal condition is documented in [3], where it was shown that data asymmetry causes the presence of a line (discrete) spectrum (when none would exist in the ideal case) as well as distortion of the continuous component of the PSD. Another source of imperfection is timing jitter, which, like data asymmetry, causes a shift away from the nominal data transition instants. Here, however, the amount of timing shift per transmission interval is random and typically is characterized by a discrete random process (independent of the data sequence) with known statistical properties. The purpose of this article is to characterize the PSD of baseband signaling schemes of the type described above in the presence of such timing jitter. Although general PSD results are obtained first for arbitrary timing jitter statistics, specific results are then given for the cases of practical interest, namely, uniform and Gaussian distributed jitter. Simulation results also will be presented as verification of the analysis.

II. Signal Model and Preliminary Definitions

A digital pulse stream in the presence of timing jitter can be modeled as

$$m(t) = \sum_{n=-\infty}^{\infty} a_n p(t - nT - \epsilon_n) \tag{1}$$

where $\{a_n\}$ is an arbitrary digital sequence and $\{\epsilon_n\}$ represents the random timing jitter assumed to be independent of $\{a_n\}$. It is convenient to define a zero-mean process, $M(t)$, as

$$M(t) = m(t) - \overline{m}(t) \tag{2}$$

where $\overline{m}(t)$ is the mean of $m(t)$ given by

$$\overline{m}(t) \stackrel{\triangle}{=} \mathbb{E}\{m(t)\} = \sum_{n=-\infty}^{\infty} \mathbb{E}\{a_n\}\,\mathbb{E}\{p(t - nT - \epsilon_n)\} \tag{3}$$

The correlation function of $M(t)$ is derived in Appendix A as

$$\begin{aligned} R_M(t;\tau) &= \mathbb{E}\{M(t)M^*(t+\tau)\} \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_y\int_z K_{a,\epsilon}(n; m-n, -y, -z) \\ &\qquad \times P(y)P^*(z)e^{-j2\pi ynT}e^{+j2\pi zmT}e^{+j2\pi(y-z)t}e^{-j2\pi z\tau}dydz \end{aligned} \tag{4}$$

where $K_{a,\epsilon}(n; m-n, -y, -z)$ is defined to be

$$K_{a,\epsilon}(n; m-n, -y, -z) = \mathbb{E}\{a_n a_m^*\}\,\mathbb{E}\left\{e^{-j2\pi y\epsilon_n}e^{+j2\pi z\epsilon_m}\right\} - \mathbb{E}\{a_n\}\,\mathbb{E}\{a_m^*\}\,\mathbb{E}\left\{e^{-j2\pi y\epsilon_n}\right\}\mathbb{E}\left\{e^{+j2\pi z\epsilon_m}\right\} \tag{5}$$

III. The PSD of a Digital Pulse Stream With Arbitrary Jitter

In general, the PSD $S_m(f)$ of a digital pulse stream $m(t)$ consists of continuous as well as discrete components [1], namely, $S_m^c(f)$ and $S_m^d(f)$, respectively.

Irrespective of the properties of the generating sequence $\{a_n\}$, the pulse stream $m(t)$ is itself WSCS since $R_M(t;\tau)$ is, in addition to being a function of τ, a periodic function of t. Therefore, the continuous spectrum of $m(t)$ is

$$S_m^c(f) = \mathcal{F}_\tau \left\{ \langle R_M(t;\tau) \rangle_t \right\} \tag{6}$$

where $\langle \cdot \rangle_t$ denotes the time average. For a WSCS sequence $\{a_n\}$ with period N, $K_{a,\epsilon}(n, m-n, -y, -z)$ is periodic in n with period N. For such a sequence, $S_m^c(f)$ is calculated in Appendix B as

$$S_m^c(f) = S_p(f) S_{a,\epsilon}(f) \tag{7}$$

where

$$S_p(f) = \frac{1}{T} \left| P(f) \right|^2 \tag{8}$$

is the power spectral density of the pulse shape $p(t)$ and

$$S_{a,\epsilon}(f) = \sum_{l=-\infty}^{\infty} \left[\frac{1}{N} \sum_{n=1}^{N} K_{a,\epsilon}(n; l, f, f) \right] e^{-j2\pi f l T} \tag{9}$$

The function $S_{a,\epsilon}(f)$ can be interpreted conveniently as the continuous spectrum of the equivalent stationary data sequence; that is, the periodicity (with period N) of the covariance function caused by the WSCS behavior of the sequence $\{a_n\}$ is averaged out before taking the discrete Fourier transform.

The discrete power spectral density is found from

$$S_m^d(f) = \mathcal{F}_\tau \left\{ \left\langle \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathbb{E}\{a_n\} \mathbb{E}\{p(t-nT-\epsilon_n)\} \mathbb{E}\{a_m^*\} \mathbb{E}\{p^*(t+\tau-mT-\epsilon_m)\} \right\rangle \right\} \tag{10}$$

For a WSCS sequence $\{a_n\}$ with period N, $S_m^d(f)$ is calculated in Appendix C as

$$S_m^d(f) = \frac{1}{(NT)^2} \sum_{l=-\infty}^{\infty} \left| P\left(\frac{l}{NT}\right) \right|^2 \left| G_{a,\epsilon}\left(\frac{l}{NT}\right) \right|^2 \delta\left(f - \frac{l}{NT}\right) \tag{11}$$

where

$$G_{a,\epsilon}(f) = \sum_{n=1}^{N} \mathbb{E}\{a_n\} \mathbb{E}\left\{e^{+j2\pi f \epsilon_n}\right\} e^{+j2\pi f n T} \tag{12}$$

IV. The PSD for Specific Timing Jitter Statistics

The results obtained in Section III are general and apply to arbitrary timing jitter statistics. In this section, the expressions for PSD are evaluated for two specific cases of practical interest, namely, uniform distributed timing jitter (UDTJ) and Gaussian distributed timing jitter (GDTJ).

A. Uniform Distributed Timing Jitter

In this subsection, the PSD of a digital pulse train in the presence of UDTJ is considered. Specifically, $\{\epsilon_n\}$ is modeled as a sequence of independent identically distributed (i.i.d.) uniform random variables with probability density function

$$f_{\epsilon_n}(x) = \begin{cases} \frac{1}{\Delta} & \Delta_1 < x < \Delta_2 \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

where $\Delta = \Delta_2 - \Delta_1$. Note that, in general, ϵ_n has nonzero mean, and, thus, this model includes the asymmetry of the timing jitter.

It is easy to show that

$$\mathbb{E}\left\{e^{+j2\pi f\epsilon_n}\right\} = \left[\frac{\sin(\pi f\Delta)}{(\pi f\Delta)}\right] e^{+j2\pi f([\Delta_2+\Delta_1]/2)} \tag{14}$$

and

$$\mathbb{E}\left\{e^{+j2\pi f\epsilon_n} e^{-j2\pi f\epsilon_m}\right\} = \begin{cases} 1 & n = m \\ \left[\frac{\sin \pi f\Delta}{\pi f\Delta}\right]^2 & n \neq m \end{cases} \tag{15}$$

Therefore, from Eq. (5),

$$K_{a,\epsilon}(n;l,f,f) = \begin{cases} R_a(n;0) - |\overline{a}_n|^2 \left[\frac{\sin(\pi f\Delta)}{(\pi f\Delta)}\right]^2 & l = 0 \\ \left[R_a(n;l) - \overline{a}_n\overline{a}^*_{n+l}\right] \left[\frac{\sin(\pi f\Delta)}{(\pi f\Delta)}\right]^2 & l \neq 0 \end{cases} \tag{16}$$

Finally, substituting Eqs. (14) and (16) into Eqs. (7), (8), (9), (11), and (12) of Section III, the continuous and discrete PSD components of a WSCS sequence $\{a_n\}$ with correlation function $R_a(n;l)$ and mean $\overline{a}_n$ in the presence of i.i.d. UDTJ (asymmetric or symmetric) over the interval Δ becomes

$$S^c_m(f) = \frac{1}{T}|P(f)|^2 \left\{\frac{1}{N}\sum_{n=1}^{N}\left[R_a(n;0) - |\overline{a}_n|^2\left[\frac{\sin(\pi f\Delta)}{(\pi f\Delta)}\right]^2\right]\right\}$$

$$+ \frac{1}{T}|P(f)|^2 \left\{\sum_{l\neq 0}\left[\frac{1}{N}\sum_{n=1}^{N}\left[R_a(n;l) - \overline{a}_n\overline{a}^*_{n+l}\right]\right] e^{-j2\pi flT}\right\}\left[\frac{\sin(\pi f\Delta)}{(\pi f\Delta)}\right]^2 \tag{17}$$

and

$$S_m^d(f) = \frac{1}{(NT)^2} \sum_{l=-\infty}^{\infty} \left| P\left(\frac{l}{NT}\right)\right|^2 \left|\sum_{n=1}^{N} \overline{a}_n e^{+j2\pi(ln/N)}\right|^2 \left[\frac{\sin(\pi \frac{l}{NT}\Delta)}{(\pi \frac{l}{NT}\Delta)}\right]^2 \delta\left(f - \frac{l}{NT}\right) \tag{18}$$

respectively. Note that Eqs. (17) and (18) are independent of the jitter asymmetry.

When the sequence $\{a_n\}$ is WSS, the above results reduce to

$$S_m^c(f) = \frac{1}{T} |P(f)|^2 \left[R_a(0) - |\overline{a}|^2 \left[\frac{\sin(\pi f \Delta)}{(\pi f \Delta)}\right]^2\right]$$

$$+ \frac{1}{T} |P(f)|^2 \left\{ \sum_{l \neq 0} \left[R_a(l) - |\overline{a}|^2\right] e^{-j2\pi f l T} \right\} \left[\frac{\sin(\pi f \Delta)}{(\pi f \Delta)}\right]^2 \tag{19}$$

and

$$S_m^d(f) = \frac{1}{T^2} \sum_{l=-\infty}^{\infty} \left| P\left(\frac{l}{T}\right)\right|^2 \; |\overline{a}|^2 \left[\frac{\sin(\pi \frac{l}{T}\Delta)}{(\pi \frac{l}{T}\Delta)}\right]^2 \delta\left(f - \frac{l}{T}\right) \tag{20}$$

respectively.

B. Gaussian Distributed Timing Jitter

In this subsection, the PSD of a digital pulse train in the presence of GDTJ is considered. Specifically, $\{\epsilon_n\}$ is modeled as a sequence of i.i.d. Gaussian random variables with probability density function

$$f_{\epsilon_n}(x) = \frac{1}{\sqrt{2\pi}\Delta} e^{-(x-\theta)^2/2\Delta^2} \tag{21}$$

In general, $\theta \neq 0$, and it is important to point out again that this model includes the *asymmetry* of the timing jitter. It is easy to verify that

$$\mathbb{E}\left\{e^{+j\pi f \epsilon_n}\right\} = e^{+(1/2)\left\{j(2\pi f \theta) - (2\pi f \Delta)^2\right\}} \tag{22}$$

and

$$\mathbb{E}\left\{e^{+j2\pi f \epsilon_n} e^{-j2\pi f \epsilon_m}\right\} = \begin{cases} 1 & n = m \\ e^{-(2\pi f \Delta)^2} & n \neq m \end{cases} \tag{23}$$

Therefore,

$$K_{a,\epsilon}(n; l, f, f) = \begin{cases} R_a(n; 0) - |\overline{a}_n|^2 e^{-(2\pi f \Delta)^2} & l = 0 \\ \left[R_a(n; l) - \overline{a}_n \overline{a}_{n+l}^*\right] e^{-(2\pi f \Delta)^2} & l \neq 0 \end{cases} \tag{24}$$

Finally, substituting Eqs. (22) and (24) into Eqs. (7), (8), (9), (11), and (12) of Section III gives the desired results,

$$S_m^c(f) = \frac{1}{T}|P(f)|^2 \left\{ \frac{1}{N}\sum_{n=1}^{N}\left[R_a(n;0) - |\overline{a}_n|^2 e^{-(2\pi f\Delta)^2}\right]\right\}$$

$$+ \frac{1}{T}|P(f)|^2 \left\{ \sum_{l\neq 0}\left[\frac{1}{N}\sum_{n=1}^{N}\left[R_a(n;l) - \overline{a}_n\overline{a}_{n+l}^*\right]\right] e^{-j2\pi flT}\right\} e^{-(2\pi f\Delta)^2} \tag{25}$$

and

$$S_m^d(f) = \frac{1}{(NT)^2}\sum_{l=-\infty}^{\infty}\left|P\left(\frac{l}{NT}\right)\right|^2 \left|\sum_{n=1}^{N}\overline{a}_n e^{+j2\pi(ln/N)}\right|^2 e^{-(2\pi[l/NT]\Delta)^2}\delta\left(f - \frac{l}{NT}\right) \tag{26}$$

respectively. The above two equations are the continuous and discrete PSD components of a WSCS sequence, $\{a_n\}$, in the presence of i.i.d. GDTJ with mean θ and standard deviation Δ. Note that, as for the UDTJ case, Eqs. (25) and (26) are independent of the jitter asymmetry.

When the sequence $\{a_n\}$ is WSS, the above results reduce to

$$S_m^c(f) = \frac{1}{T}|P(f)|^2\left\{R_a(0) - |\overline{a}|^2 e^{-(2\pi f\Delta)^2}\right\} + \frac{1}{T}|P(f)|^2\left\{\sum_{l\neq 0}\left[R_a(l) - |\overline{a}|^2\right] e^{-j2\pi flT}\right\} e^{-(2\pi f\Delta)^2} \tag{27}$$

and

$$S_m^d(f) = \frac{1}{T^2}\sum_{l=-\infty}^{\infty}\left|P\left(\frac{l}{T}\right)\right|^2 |\overline{a}|^2 e^{-(2\pi[l/T]\Delta)^2}\delta\left(f - \frac{l}{T}\right) \tag{28}$$

C. Limiting Cases

The limiting case of $\Delta_1 \to \Delta_2$ or $\Delta \to 0$ for both UDTJ and GDTJ implies constant clock delay with no timing jitter. In this case, Eqs. (17) and (18) (for the UDTJ) and Eqs. (25) and (26) (for the GDTJ) reduce to the well-known results for the PSD of an ideal synchronous data pulse stream given by Eqs. (2.32) and (2.39) of [1, pp. 60–61].

Similarly, the results obtained for the WSS sequence, given by Eqs. (19) and (20) (for the UDTJ) and Eqs. (27) and (28) (for the GDTJ), reduce to the well-known results for the PSD of an ideal synchronous data pulse stream with a nonzero-mean WSS sequence, given by Eqs. (2.17) and (2.40) of [1, pp. 59–61].

V. Classical Examples

A. Uncorrelated Data Pulse Stream

Consider an uncorrelated data sequence, that is, one with correlation function

$$R(l) = \begin{cases} \mathbb{E}\{a_n a_n^*\} & l = 0 \\ |\overline{a}|^2 & l \neq 0 \end{cases} \tag{29}$$

Then the PSD of this pulse stream in the presence of UDTJ has components

$$S_m^c(f) = \frac{1}{T} |P(f)|^2 \left\{ R_a(0) - |\overline{a}|^2 \left[\frac{\sin(\pi f \Delta)}{(\pi f \Delta)} \right]^2 \right\} \tag{30}$$

and

$$S_m^d(f) = \frac{|\overline{a}|^2}{T^2} \sum_{l=-\infty}^{\infty} \left| P\left(\frac{l}{T}\right) \right|^2 \left[\frac{\sin(\pi \frac{l}{T} \Delta)}{(\pi \frac{l}{T} \Delta)} \right]^2 \delta\left(f - \frac{l}{T}\right) \tag{31}$$

In the case of GDTJ, the PSD components become

$$S_m^c(f) = \frac{1}{T} |P(f)|^2 \left\{ R_a(0) - |\overline{a}|^2 e^{-(2\pi f \Delta)^2} \right\} \tag{32}$$

and

$$S_m^d(f) = \frac{|\overline{a}|^2}{T^2} \sum_{l=-\infty}^{\infty} \left| P\left(\frac{l}{T}\right) \right|^2 e^{-(2\pi [l/T] \Delta)^2} \delta\left(f - \frac{l}{T}\right) \tag{33}$$

B. Independent Identically Distributed Binary Antipodal Data Pulse Stream

Consider next an unbalanced binary i.i.d. sequence with

$$\Pr\{a_i\} = \begin{cases} p & a_i = 1 \\ 1 - p & a_i = -1 \end{cases} \tag{34}$$

and its associated correlation function

$$R(l) = \begin{cases} 1 & l = 0 \\ (2p-1)^2 & l \neq 0 \end{cases} \tag{35}$$

with mean $\overline{a} = \mathbb{E}\{a_i\} = 2p - 1$.

In the case of uniform timing jitter, simple substitution shows that

$$S_m^c(f) = \frac{1}{T}|P(f)|^2 \left\{1 - (2p-1)^2 \left[\frac{\sin(\pi f \Delta)}{(\pi f \Delta)}\right]^2\right\} \tag{36}$$

and

$$S_m^d(f) = \frac{(2p-1)^2}{T^2} \sum_{l=-\infty}^{\infty} \left|P\left(\frac{l}{T}\right)\right|^2 \left[\frac{\sin(\pi \frac{l}{T}\Delta)}{(\pi \frac{l}{T}\Delta)}\right]^2 \delta(f - \frac{l}{T}) \tag{37}$$

When the timing jitter is normally distributed, then the PSD of this data pulse stream has components

$$S_m^c(f) = \frac{1}{T}|P(f)|^2 \left\{1 - (2p-1)^2 e^{-(2\pi f \Delta)^2}\right\} \tag{38}$$

and

$$S_m^d(f) = \frac{(2p-1)^2}{T^2} \sum_{l=-\infty}^{\infty} \left|P\left(\frac{l}{T}\right)\right|^2 e^{-(2\pi[l/T]\Delta)^2} \delta\left(f - \frac{l}{T}\right) \tag{39}$$

C. Zero-Mean Markov Source Data Stream

Consider a binary, zero-mean Markov source characterized by

$$\left.\begin{aligned} \Pr\{a_{n+1} \neq a_n\} &= p_t \\ \Pr\{a_{n+1} = a_n\} &= 1 - p_t \end{aligned}\right\} \tag{40}$$

It can be shown that the correlation function of such a source is given by

$$R(l) = (1 - 2p_t)^{|l|} \tag{41}$$

Since its generating sequence is a zero-mean sequence, one can conclude immediately that the discrete spectrum is identically zero. Substituting Eq. (41) into Eq. (19), the continuous PSD in the presence of UDTJ becomes

$$S_m^c(f) = \frac{1}{T}|P(f)|^2 \left\{1 + \left[\sum_{l \neq 0} (1-2p_t)^{|l|} e^{-j2\pi f l T}\right] \left[\frac{\sin(\pi f \Delta)}{(\pi f \Delta)}\right]^2\right\} \tag{42}$$

Note that

$$\sum_{l \neq 0} (1-2p_t)^{|l|} e^{-j2\pi f l T} = 2\sum_{l=1}^{\infty} (1-2p_t)^{|l|} \cos(j2\pi f l T) \tag{43}$$

Using the identity

$$\sum_{k=1}^{\infty} b^k \cos(k\beta) = \frac{b\cos\beta - b^2}{1 - 2b\cos\beta + b^2} \tag{44}$$

Eq. (42) reduces to

$$S_m^c(f) = \frac{1}{T}|P(f)|^2 \left\{1 + \left\{\frac{2(1-2p_t)\cos(j2\pi fT) - 2(1-2p_t)^2}{2(1-2p_t)\left[1-\cos(j2\pi fT)\right] + 4p_t^2}\right\}\left[\frac{\sin(\pi f\Delta)}{(\pi f\Delta)}\right]^2\right\} \tag{45}$$

Similarly, the continuous PSD in the presence of GDTJ can be shown to be

$$S_m^c(f) = \frac{1}{T}|P(f)|^2 \left\{1 + \left\{\frac{2(1-2p_t)\cos(j2\pi fT) - 2(1-2p_t)^2}{2(1-2p_t)\left[1-\cos(j2\pi fT)\right] + 4p_t^2}\right\} e^{-(2\pi[l/T]\Delta)^2}\right\} \tag{46}$$

D. Interleaving *N* Independent First-Order Markov Sources

Consider the sequence formed by interleaving N independent first-order zero-mean Markov sources with respective transition probabilities $p_{t_n}; n = 1, 2, \ldots, N$. Then the resulting sequence is WSCS with correlation function

$$R(n;l) = \begin{cases} (1-2p_{t_n})^{|l|/N} & l = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{all other integers} \end{cases} \tag{47}$$

Again, one can conclude immediately that the discrete spectrum is identically zero, since its generating sequence is a zero-mean sequence. Substituting Eq. (47) into Eq. (17), the continuous PSD in the presence of UDTJ becomes

$$S_m^c(f) = \frac{1}{T}|P(f)|^2 \left\{1 + \left[\frac{1}{N}\sum_{n=1}^{N}\sum_{k\neq 0}(1-2p_{t_n})^{|k|} e^{-j2\pi fkNT}\right]\left[\frac{\sin(\pi f\Delta)}{(\pi f\Delta)}\right]^2\right\} \tag{48}$$

Using Eqs. (43) and (44), Eq. (48) reduces to

$$S_m^c(f) = \frac{1}{T}|P(f)|^2 \frac{1}{N}\sum_{n=1}^{N}\left\{1 + \left\{\frac{2(1-2p_{t_n})\cos(j2\pi fNT) - 2(1-2p_{t_n})^2}{2(1-2p_{t_n})\left[1-\cos(j2\pi fNT)\right] + 4p_{t_n}^2}\right\}\left[\frac{\sin(\pi f\Delta)}{(\pi f\Delta)}\right]^2\right\} \tag{49}$$

Similarly, the continuous PSD in the presence of GDTJ can be shown to be

$$S_m^c(f) = \frac{1}{T}|P(f)|^2 \frac{1}{N}\sum_{n=1}^{N}\left\{1 + \left\{\frac{2(1-2p_{t_n})\cos(j2\pi fNT) - 2(1-2p_{t_n})^2}{2(1-2p_{t_n})\left[1-\cos(j2\pi fNT)\right] + 4p_{t_n}^2}\right\} e^{-(2\pi[l/T]\Delta)^2}\right\} \tag{50}$$

VI. Numerical Evaluations and Simulation Results

Illustrated in Figs. 1 through 3 are evaluations of the analytical PSD results in the presence of uniform jitter obtained from Eqs. (36) and (37) for intervals $\Delta = T$, $\Delta = T/2$, and $\Delta = T/16$, respectively. For all cases, the pulse shape is rectangular with transition probabilities of $p = 0.5$ and 0.9. Also shown are computer simulation results obtained for the same uniform jitter values. For the simulations, a sampling rate of 256 Hz and a bit rate of 1 Hz are maintained so that there are 256 samples per bit in each bit duration, and a fast Fourier transform (FFT) size of 32,768 points is used. Hence, the resolution bandwidth for the continuous PSD is given as 256/32,768. The continuous PSD for the analytical results is given at a resolution bandwidth of 1 Hz. In order to compare the analytical and simulation continuous PSDs at the same resolution, the analytical PSDs were adjusted in Figs. 1 through 3 so that they have the same resolution bandwidth as the simulation. For all simulation cases, 100 FFTs were averaged in order to decrease the noise due to the randomness of the data sequence. As can be seen, the analytical and simulation results are in excellent agreement.

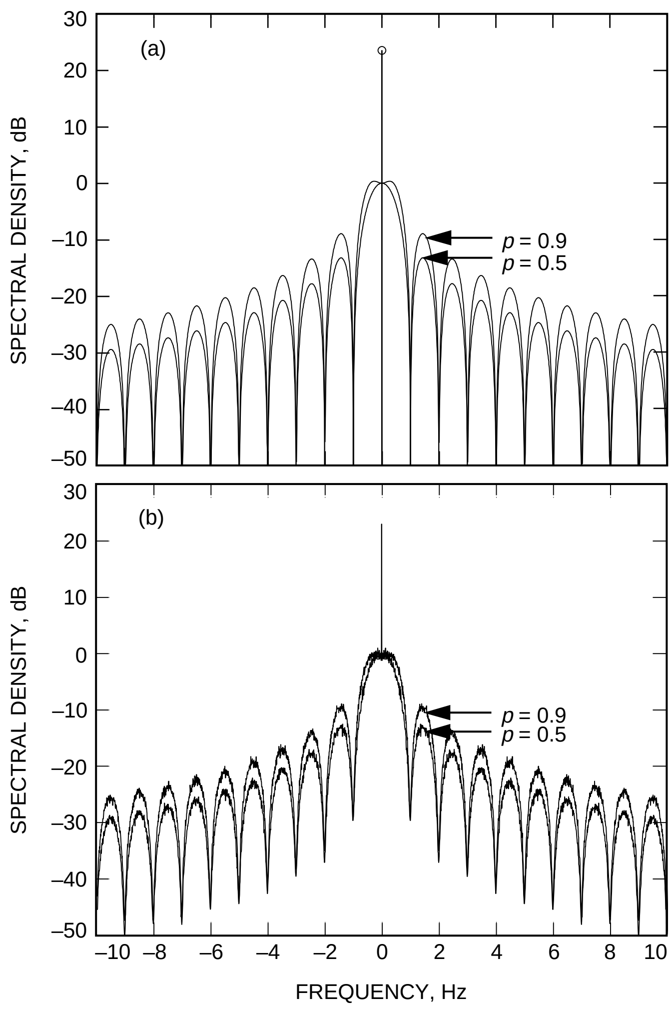

Fig. 1. Results for Δ = *T* with uniform jitter: (a) analytical PSD and (b) simulation FFT.

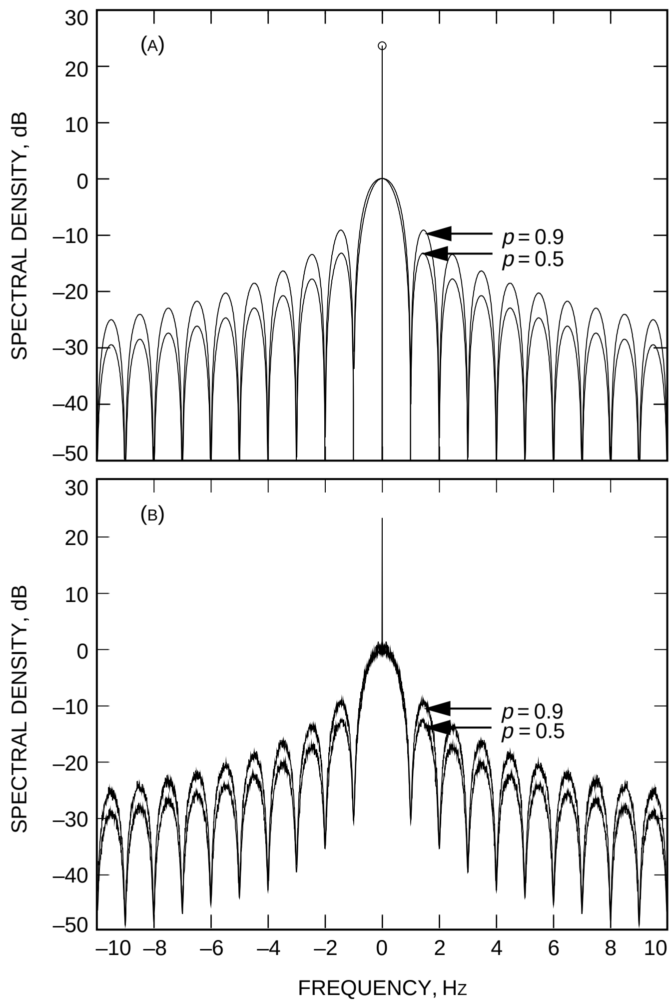

Fig. 2. Results for Δ = *T*/2 with uniform jitter: (a) analytical PSD and (b) simulation FFT.

VII. Observations and Caveats

Results obtained for the discrete PSD component are in the same form as those given in [1] for the case of an ideal data pulse stream (i.e., no timing jitter), except that

$$\left.\begin{aligned} P\left(\frac{l}{NT}\right) &\longrightarrow P\left(\frac{l}{NT}\right)\left[\frac{\sin\left(\pi\frac{l}{NT}\Delta\right)}{\left(\pi\frac{l}{NT}\Delta\right)}\right] \\ P\left(\frac{l}{NT}\right) &\longrightarrow P\left(\frac{l}{NT}\right)e^{-(1/2)(2\pi[l/NT]\Delta)^2} \end{aligned}\right\} \tag{51}$$

for UDTJ and GDTJ, respectively.

Results obtained for the continuous PSD component are *almost* in the same form as those given in [1] for the ideal data pulse stream, but the sum $\sum_{l=-\infty}^{\infty}$ must be separated into $l = 0$ and $l \neq 0$ terms. For $l \neq 0$,

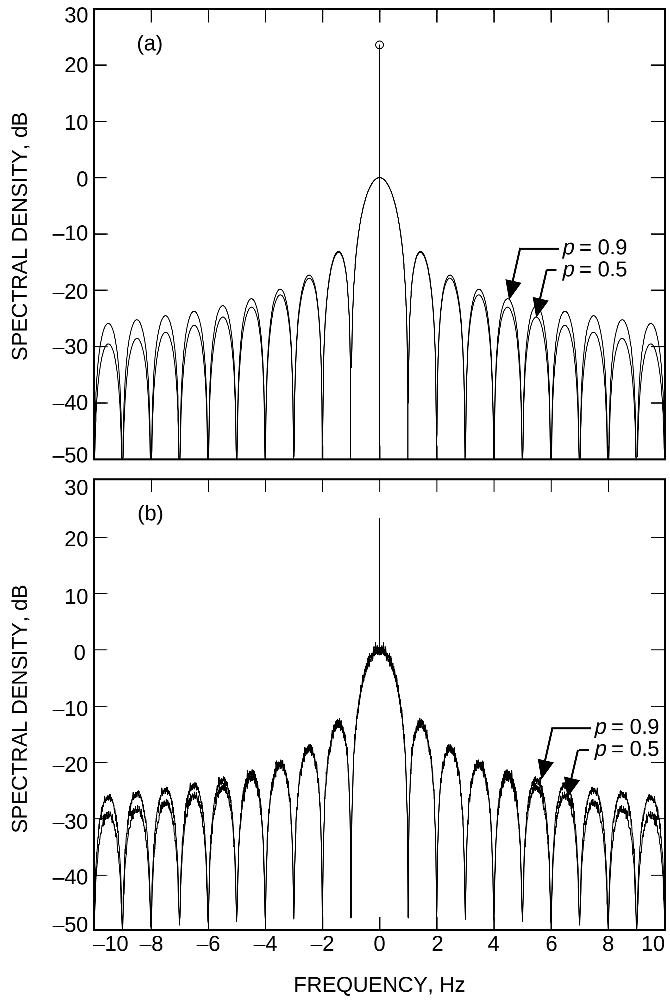

Fig. 3. Results for Δ = *T*/16 with uniform jitter: (a) analytical PSD and (b) simulation FFT.

$$\left.\begin{aligned} P(f) &\longrightarrow P(f)\left[\frac{\sin(\pi f\Delta)}{(\pi f\Delta)}\right] \\ P(f) &\longrightarrow P(f)\, e^{-(1/2)(2\pi f\Delta)^2} \end{aligned}\right\} \tag{52}$$

for UDTJ and GDTJ, respectively.

The discrete PSD of a digital pulse stream generated by a WSS sequence and an uncorrelated sequence are identical, since the discrete PSD depends only on the mean of the sequence, and the fact that a given sequence is WSS or uncorrelated affects only the correlation function and has no effect on the mean of the sequence. As previously pointed out, the *asymmetry* of the timing jitter does *not* affect the PSD at all. This is reasonable and agrees with one's intuition, since adding a constant delay to an ideal synchronous data pulse stream should not affect its PSD.

Interesting results emerge when the generating sequence $\{a_n\}$ is uncorrelated. In this case, the $l \neq 0$ terms of the continuous PSD in Eqs. (17) and (19) for UDTJ and Eqs. (25) and (27) for GDTJ are zero, and only the $l = 0$ term survives. This also can be noticed in Eqs. (30) and (32) of Section V.A. For generating sequences $\{a_n\}$ that are nonzero mean, the simulation results of Section VI show that timing jitter has the effect of widening the main lobe of the spectrum and increasing the side lobes. This can be explained mathematically by rewriting Eqs. (30) and (32) as

$$S_m^c(f) = \frac{1}{T}\left|P(f)\right|^2 \left\{ \left[R_a(0) - |\overline{a}|^2\right] + \underbrace{|\overline{a}|^2 \left\{1 - \left[\frac{\sin(\pi f \Delta)}{(\pi f \Delta)}\right]^2\right\}}_{>\ 0} \right\} \tag{53}$$

and

$$S_m^c(f) = \frac{1}{T}\left|P(f)\right|^2 \left\{ \left[R_a(0) - |\overline{a}|^2\right] + \underbrace{|\overline{a}|^2 \left\{1 - e^{-(2\pi f \Delta)^2}\right\}}_{>\ 0} \right\} \tag{54}$$

respectively.

When the generating sequence is zero mean and uncorrelated, a rather surprising result is that the timing jitter does *not* affect the PSD. This fact was also observed in Section VI.

VIII. Conclusion

General expressions for the PSD of digital pulse streams generated by WSCS sequences in the presence of arbitrary timing jitter are derived using methods similar to those given in [1, Chapter 2]. The results for a WSS sequence are immediately obtained from the results of the WSCS case by setting the period of the WSCS sequence equal to one. Specific results are derived for the cases of practical interest, namely, uniform and Gaussian distributed timing jitter. The results show that the *asymmetry* of the timing jitter does *not* affect the PSD. Examples of an uncorrelated data pulse stream, an i.i.d. data stream, and Markov sources are given. Interesting results emerge when the generating sequence $\{a_n\}$ is uncorrelated. For generating sequences $\{a_n\}$ that are nonzero mean, timing jitter has the effect of widening the main lobe of the spectrum and increasing the side lobes. When the generating sequence is *zero mean* and *uncorrelated*, a rather surprising result is that the timing jitter does *not* affect the PSD. Simulation results are also presented and are in excellent agreement with the analysis. The well-known results for the PSD of an ideal synchronous data pulse stream in [1] are shown to be a special case of the results obtained in this article and can be obtained as the limiting case of timing jitter going to zero (i.e., $\Delta_1 \to \Delta_2$ or $\Delta \to 0$).

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Appendix A

Correlation Function of the Zero-Mean Process *M*(*t*)

The correlation function of the zero-mean process $M(t)$ is given by

$$\begin{aligned} R_M(t;\tau) &= \mathbb{E}\left\{M(t)M^*(t+\tau)\right\} \\ &= \mathbb{E}\left\{\sum_{n=-\infty}^{\infty}\left[a_n p(t-nT-\epsilon_n) - \mathbb{E}\left\{a_n\right\}\mathbb{E}\left\{p(t-nT-\epsilon_n)\right\}\right]\right. \\ &\qquad \left. \times \sum_{m=-\infty}^{\infty}\left[a_m p(t+\tau-mT-\epsilon_m) - \mathbb{E}\left\{a_m\right\}\mathbb{E}\left\{p(t+\tau-mT-\epsilon_m)\right\}\right]^*\right\} \end{aligned}$$

This can be rewritten as

$$\begin{aligned} R_M(t,\tau) = &\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\mathbb{E}\left\{a_n a_m^* p(t-nT-\epsilon_n)p^*(t+\tau-mT-\epsilon_m)\right\} \\ &\times \mathbb{E}\left\{a_n\right\}\mathbb{E}\left\{a_m^*\right\}\mathbb{E}\left\{p(t-nT-\epsilon_n)\right\}\mathbb{E}\left\{p^*(t+\tau-mT-\epsilon_m)\right\} \end{aligned} \tag{A-1}$$

Since

$$p(t) = \int_{-\infty}^{\infty} P(f) e^{+j2\pi f t} dt \tag{A-2}$$

where $P(f)$ is the Fourier transform of $p(t)$, substituting Eq. (A-2) into Eq. (A-1) gives

$$R_m(t,\tau) = \sum_{n=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_y \int_z \Big[R_a(n; m-n) \mathbb{E}\left\{ e^{-j2\pi y \epsilon_n} e^{+j2\pi z \epsilon_m} \right\}$$

$$-\mathbb{E}\{a_n\} \mathbb{E}\{a_m^*\} \mathbb{E}\left\{e^{-j2\pi y \epsilon_n}\right\} \mathbb{E}\left\{e^{+j2\pi z \epsilon_m}\right\}\Big]$$

$$\times P(y) P^*(z) e^{-j2\pi y n T} e^{+j2\pi z m T} e^{+j2\pi (y-z) t} e^{-j2\pi z \tau} dydz$$

It is convenient to define

$$K_{a,\epsilon}(n; m-n, -y, -z) = \mathbb{E}\{a_n a_m^*\} \mathbb{E}\left\{e^{-j2\pi y \epsilon_n} e^{+j2\pi z \epsilon_m}\right\}$$

$$- \mathbb{E}\{a_n\} \mathbb{E}\{a_m^*\} \mathbb{E}\left\{e^{-j2\pi y \epsilon_n}\right\} \mathbb{E}\left\{e^{+j2\pi z \epsilon_m}\right\} \tag{A-3}$$

In terms of the above definition, $R_M(t;\tau)$ becomes

$$R_M(t;\tau) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int_y \int_z K_{a,\epsilon}(n; m-n, -y, -z)$$

$$\times P(y) P^*(z) e^{-j2\pi y n T} e^{+j2\pi z m T} e^{+j2\pi (y-z) t} e^{-j2\pi z \tau} dydz \tag{A-4}$$

Appendix B
Derivation of the Continuous PSD

The continuous PSD of $m(t)$ is given by

$$
\begin{aligned}
S_m^c(F) &= \mathcal{F}_\tau\{\langle R_M(t;\tau)\rangle_t\} \qquad \text{(B-1)}\\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_y\int_z K_{a,\epsilon}(n; m-n, -y, -z)\\
&\quad \times P(y)P^*(z)e^{-j2\pi ynT}e^{+j2\pi zmT}\left\langle e^{+j2\pi(y-z)t}\right\rangle \underbrace{\mathcal{F}\left\{e^{-j2\pi z\tau}\right\}}_{\delta(f+z)} dydz\\
&= \int_y \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} K_{a,\epsilon}(n; m-n, -y, f)\\
&\quad \times P(y)P^*(-f)e^{-j2\pi ynT}e^{-j2\pi fmT}\left\langle e^{+j2\pi(y+f)t}\right\rangle dy\\
&= \int_y \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} K_{a,\epsilon}(n; m-n, -y, f)\\
&\quad \times P(y)P^*(-f)e^{-j2\pi f(m-n)T}e^{-j2\pi(y+f)nT}\left\langle e^{+j2\pi(y+f)t}\right\rangle dy \qquad \text{(B-2)}
\end{aligned}
$$

Letting $l = m - n$ and, with $K_{a,\epsilon}(n; l, -y, f)$ being periodic in n with period N for a WSCS sequence, it is easy to show that

$$
\sum_{n=-\infty}^{\infty} K_{a,\epsilon}(n; l, -y, f)e^{-j2\pi(y+f)nT} = \sum_{n=1}^{N} K_{a,\epsilon}(n; l, -y, f)e^{-j2\pi(y+f)nT}\sum_{i=-\infty}^{\infty} e^{-j2\pi(y+f)iNT} \qquad \text{(B-3)}
$$

Using the Poisson sum formula

$$
\sum_{n=-\infty}^{\infty} e^{-j2\pi xnT} = \frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(x - \frac{k}{T}\right) \qquad \text{(B-4)}
$$

and performing the integration over y in Eq. (B-2) gives

$$S_m^c(f) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{1}{NT} \sum_{n=1}^{N} K_{a,\epsilon}\left(n; l, f - \frac{k}{NT}, f\right)$$

$$\times P\left(-f + \frac{k}{NT}\right) P^*(-f) e^{-j2\pi(k/N)n} e^{-j2\pi flT} \left\langle e^{j2\pi(k/NT)t} \right\rangle \tag{B-5}$$

Note that

$$\left\langle e^{j2\pi(k/NT)t} \right\rangle = \frac{1}{NT} \int_{-(NT/2)}^{NT/2} e^{j2\pi(k/NT)t} dt = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \tag{B-6}$$

Therefore,

$$S_m^c(f) = \frac{1}{T} |P(f)|^2 \sum_{l=-\infty}^{\infty} \left[\frac{1}{N} \sum_{n=1}^{N} K_{a,\epsilon}(n; l, f, f) \right] e^{-j2\pi flT} \tag{B-7}$$

Appendix C
Derivation of the Discrete PSD

The discrete PSD of $m(t)$ is given by

$$S_m^d(f) = \mathcal{F}_\tau \left\{ \left\langle \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathbb{E}\{a_n\} \mathbb{E}\{p(t - nT - \epsilon_n)\} \mathbb{E}\{a_m^*\} \mathbb{E}\{p^*(t + \tau - mT - \epsilon_m)\} \right\rangle \right\}$$

$$= \mathcal{F}_\tau \left\{ \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathbb{E}\{a_n\} \mathbb{E}\{a_m^*\} \right.$$

$$\times \left\langle \mathbb{E}\left\{ \int_y P(y) e^{-j2\pi ynT} e^{-j2\pi y\epsilon_n} e^{+j2\pi yt} dy \right\} \right.$$

$$\left. \left. \times \mathbb{E}\left\{ \int_z P^*(z) e^{-j2\pi z\tau} e^{+j2\pi zmT} e^{+j2\pi z\epsilon_m} e^{-j2\pi zt} dz \right\} \right\rangle \right\} \tag{C-1}$$

$$= \int_y \int_z \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathbb{E}\left\{a_n\right\} \mathbb{E}\left\{a_m^*\right\} \mathbb{E}\left\{e^{-j2\pi y\epsilon_n}\right\} \mathbb{E}\left\{e^{+j2\pi z\epsilon_m}\right\}$$

$$\times P(y)P^*(z)e^{-j2\pi ynT}e^{+j2\pi zmT}\left\langle e^{+j2\pi(y-z)t}\right\rangle \underbrace{\mathcal{F}_\tau\left\{e^{-j2\pi z\tau}\right\}}_{\delta(f+z)} dydz \tag{C-2}$$

Integrating over z and rearranging terms gives

$$S_m^d(f) = \int_y \sum_{n=-\infty}^{\infty} \mathbb{E}\left\{a_n\right\} \mathbb{E}\left\{e^{-j\pi y\epsilon_n}\right\} e^{-j2\pi ynT}$$

$$\times \sum_{m=-\infty}^{\infty} \mathbb{E}\left\{a_m^*\right\} \mathbb{E}\left\{e^{-j2\pi f\epsilon_m}\right\} e^{-j2\pi fmT} P(y)P^*(-f)\left\langle e^{+j2\pi(y+f)t}\right\rangle dy$$

Since, for a WSCS sequence, $\mathbb{E}\left\{a_n\right\} \mathbb{E}\left\{e^{-j2\pi y\epsilon_n}\right\}$ is periodic in n with period N, then it is easy to show that

$$\sum_{n=-\infty}^{\infty} \mathbb{E}\left\{a_n\right\} \mathbb{E}\left\{e^{-j2\pi y\epsilon_n}\right\} e^{-j2\pi ynT} = \sum_{n=1}^{N} \mathbb{E}\left\{a_n\right\} \mathbb{E}\left\{e^{-j2\pi y\epsilon_n}\right\} e^{-j2\pi ynT} \sum_{i=-\infty}^{\infty} e^{-j2\pi yiNT} \tag{C-3}$$

Using this together with the Poisson sum formula given in Eq. (B-4) and integrating over y yields, after simplifications,

$$S_m^d(f) = \frac{1}{NT} \sum_{k=-\infty}^{\infty} \sum_{n=1}^{N} \mathbb{E}\left\{a_n\right\} \mathbb{E}\left\{e^{-j2\pi(k/NT)\epsilon_n}\right\} e^{-j2\pi(kn/N)}$$

$$\times \frac{1}{NT} \sum_{l=-\infty}^{\infty} \sum_{m=1}^{N} \mathbb{E}\left\{a_m^*\right\} \mathbb{E}\left\{e^{-j2\pi(l/NT)\epsilon_m}\right\} e^{-j2\pi(lm/N)} \delta\left(f - \frac{l}{NT}\right)$$

$$\times P\left(\frac{k}{NT}\right) P^*\left(-\frac{l}{NT}\right) \left\langle e^{+j2\pi([k+l]/NT)t}\right\rangle \tag{C-4}$$

Using Eq. (B-6), the above expression reduces to

$$S_m^d(f) = \frac{1}{(NT)^2} \sum_{l=-\infty}^{\infty} \left| P\left(\frac{l}{NT}\right)\right|^2 \left| \sum_{n=1}^{N} \mathbb{E}\left\{a_n\right\} \mathbb{E}\left\{e^{+j2\pi(l/NT)\epsilon_n}\right\} e^{+j2\pi(ln/N)}\right|^2 \delta\left(f - \frac{l}{NT}\right) \tag{C-5}$$