On the Power Spectrum of Digital Frequency-Modulated Signals

M. K. Simon
Communications Systems and Research Section

Using a matrix method proposed a number of years ago by Prabhu and Rowe for computing the power spectrum of digital frequency-modulated carriers, we obtain explicit expressions for this power spectrum when the baseband signaling is a binary pulse-shaped random data stream with nonoverlapping pulses. Particular attention is paid to the conditions under which a discrete spectral component can exist.

I. Introduction

In a companion article [1], the power spectrum of angle-modulated phase-shift-keyed (PSK) signals was considered with emphasis on the effects of intersymbol interference (ISI). Using an approach taken by Prabhu and Rowe [2], it was shown that, depending on the shape and duration of the phase pulse, it was possible to have both continuous and discrete line spectrum components. In particular, even in the absence of ISI, pulse shaping alone can result in the presence of a discrete spectrum for the phase-modulated signal. In fact, for pulses that are time limited to a single transmission interval, the only pulse shape that does not produce a discrete line spectrum is a purely digital (± 1) one.

A formulation similar to that in [2] was developed one year later by Prabhu and Rowe to obtain the power spectrum of digital frequency-modulated signals [3]. As for the PSK case, it is possible to have both discrete and continuous spectrum components; however, the criterion for the existence of a discrete line spectrum is quite different for the frequency-modulation case. In this article, we illustrate a simple example of the application of this criterion for the case of binary full-response continuous-phase modulation (CPM). Special cases of this are continuous-phase frequency shift keying (CPFSK), where the frequency pulse is rectangular, and minimum shift keying (MSK), which itself is a special case of CPFSK corresponding to a modulation index equal to 0.5.

II. Mathematical Signal Model

Consider a digital angle-modulated carrier of the form

$$x(t) = \mathrm{Re}\left\{\exp\left[j2\pi f_c t + \phi(t) + \theta\right]\right\} \tag{1}$$

where f_c is the carrier frequency, θ is a random phase assumed to be uniformly distributed in $(-\pi, \pi)$, and $\phi(t)$ is a digital angle modulation. Then, the power spectrum of $x(t)$ can be expressed in terms of the power spectrum of the equivalent complex baseband modulation, $v(t) = \exp\left(j\phi(t)\right)$, by

$$S_{xx}(f) = \frac{1}{4} S_{vv}\left(f - f_c\right) + \frac{1}{4} S_{vv}\left(f + f_c\right) \tag{2}$$

Thus, it is sufficient to consider the power spectrum $S_{vv}(f)$.

Assume now that $\phi(t)$ is in the form of a continuous-phase binary FSK signal, namely,

$$\left.\begin{aligned} \phi(t) &= \int^{t} f_d(u)du \\ f_d(t) &= \sum_{n=-\infty}^{\infty} a_k h(t - kT) \end{aligned}\right\} \tag{3}$$

where a_k denotes the data bit in the kth signaling interval that takes on values ± 1 with equal probability and

$$h(t) = 2\pi h g(t) \tag{4}$$

with h the frequency modulation index and $g(t)$ the normalized frequency pulse having the property

$$\int_0^{KT} g(t)dt = \frac{1}{2} \tag{5}$$

Here K is the number of transmission intervals over which $g(t)$ is assumed to exist. Since, as stated in the introduction, we are interested here only in the case of full-response CPM, we shall specifically consider only the case of $K = 1$, frequency pulses time limited to a single transmission interval.

III. Conditions for the Existence of a Discrete Power Spectrum for Nonoverlapping Pulses—*K* = 1

Define the two equivalent complex baseband signals

$$\left.\begin{aligned} q_1(t) &= \exp\left(j2\pi h \int_0^t g(u)du\right) \\ q_2(t) &= \exp\left(-j2\pi h \int_0^t g(u)du\right) \end{aligned}\right\} \tag{6}$$

corresponding to the two possible data bit values transmitted in the zeroth transmission interval. Arrange these two signals in a column vector:

$$\mathbf{q}(t) \triangleq \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \tag{7}$$

Also, define the a priori probability vector corresponding to these two equiprobable signals, namely,

$$\mathbf{w} \triangleq \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \tag{8}$$

Then, it is shown in [3] that $S_{vv}(f)$ will contain a line (discrete) spectrum component when

$$\left|\mathbf{w}^T \mathbf{q}(T)\right| = w_1 q_1(T) + w_2 q_2(T) = 1 \tag{9}$$

Conversely, if $\left|\mathbf{w}^T \mathbf{q}(T)\right| < 1$, then a line spectrum component will not exist.

For the binary full-response CPM case under consideration, substituting Eq. (5) (with $K = 1$) into Eq. (6) gives

$$\left.\begin{aligned} q_1(T) &= \exp(j\pi h) \\ q_2(T) &= \exp(-j\pi h) \end{aligned}\right\} \tag{10}$$

Thus, from Eq. (9), the condition for the existence of a line spectrum is given by

$$\left|\mathbf{w}^T \mathbf{q}(T)\right| = \left|\frac{1}{2}\left(\exp(j\pi h) + \exp(-j\pi h)\right)\right| = \left|\cos \pi h\right| = 1 \tag{11}$$

Clearly, then for h noninteger (e.g., MSK), a line spectrum will not exist. On the other hand, for any integer h, a line spectrum *will* exist. The frequencies at which the spectral lines occur are best understood in terms of an alternate interpretation of the condition for the existence of the spectrum itself, as described below.

When Eq. (9) is satisfied, it is possible to express the equivalent phase arguments of $q_1(T)$ and $q_2(T)$ defined in Eq. (6) in the form

$$\left.\begin{aligned} 2\pi h \int_0^T g(t)dt &= 2\pi f_l + 2\pi m_1 \\ -2\pi h \int_0^T g(t)dt &= 2\pi f_l + 2\pi m_2, \quad -\frac{1}{2} \le f_l \le \frac{1}{2} \end{aligned}\right\} \tag{12}$$

where $2\pi f_l$ represents the total phase change (modulo 2π) in the carrier produced by both possible signals, and m_1 and m_2 are arbitrary integers. Stated another way, if a discrete spectrum is to exist, then both signals must produce the same total phase change (modulo 2π) in the carrier. Using Eq. (5) (with $K = 1$) in Eq. (12) gives

$$\left.\begin{aligned} \pi h &= 2\pi f_l + 2\pi m_1 \\ -\pi h &= 2\pi f_l + 2\pi m_2 \end{aligned}\right\} \tag{13}$$

for the existence of a line spectrum. Clearly, Eq. (13) can be satisfied only for h integer, as concluded previously. For example, if $h = 1$, then Eq. (13) is satisfied by the values $f_l = 1/2$, $m_1 = 0$, and $m_2 = -1$. The significance of determining f_l is that when a line spectrum exists, the frequencies at which the spectral lines of $S_{xx}(f)$ exist are given by

$$f = \pm\left(f_c + \frac{f_l + n}{T}\right); \quad n = \ldots, -1, 0, 1, \ldots \tag{14}$$

Thus, for example, if $h = 1$, we have a discrete spectrum at

$$f = \pm\left(f_c + \frac{n + (1/2)}{T}\right); \quad n = \ldots, -1, 0, 1, \ldots \tag{15}$$

The actual line spectrum itself can be obtained in an analogous manner to that given in [1,2] for the PSK wave. In particular, the discrete spectrum component of $S_{vv}(f)$, namely, $S_{vv_l}(f)$, is given by

$$S_{vv_l}(f) = \frac{1}{T^2}\left|\mathbf{w}^T\mathbf{R}(f)\right|^2 \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n + f_l}{T}\right) = \frac{1}{2T^2}\left|R_1(f) + R_2(f)\right|^2 \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n + f_l}{T}\right) \tag{16}$$

where

$$\left.\begin{aligned} \mathbf{R}(f) &= \begin{bmatrix} R_1(f) \\ R_2(f) \end{bmatrix} \\ R_i(f) &= \mathcal{F}\left\{r_i(t)\right\} \\ r_i(t) &= \exp\left(-\frac{j2\pi f_l t}{T}\right) q_i(t) \end{aligned}\right\} \tag{17}$$

Equivalently, if $Q_i(f)$ denotes the Fourier transform of $q_i(t)$, then

$$\begin{aligned} S_{vv_l}(f) &= \frac{1}{2T^2}\left|Q_1\left(f + \frac{f_l}{T}\right) + Q_2\left(f + \frac{f_l}{T}\right)\right|^2 \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n + f_l}{T}\right) \\ &= \frac{1}{2T^2}\sum_{n=-\infty}^{\infty}\left|Q_1\left(\frac{n + 2f_l}{T}\right) + Q_2\left(\frac{n + 2f_l}{T}\right)\right|^2 \delta\left(f - \frac{n + f_l}{T}\right) \end{aligned} \tag{18}$$

When $f_l = 0$, we have $r_i(t) = q_i(t)$, and the FSK spectra are identical to the previous PSK results. However, it is not necessarily true that in this case the wave is a PSK wave. To have a PSK wave, the stronger condition

$$\left.\begin{aligned} 2\pi h \int_0^T g(t)dt &= 0 \\ -2\pi h \int_0^T g(t)dt &= 0 \end{aligned}\right\} \tag{19}$$

must be met, i.e., m_1 and m_2 must both be equal to zero. Stated another way, a wave with $f_l = 0$ but with one or both of the signaling pulses producing a net phase change equal, for example, to $\pm 2\pi$ will have a spectrum given by the PSK formula, but will not be a PSK wave.

In a straightforward manner, the results of Section III can be extended to the partial-response CPM case where there is signal pulse overlap, i.e., $K > 1$. These results are well documented in [3].

IV. Conclusions

The criteria for the existence of discrete components in the power spectrum of digital frequency-modulated carriers are quite different from those for the analogous phase modulation case. For the case of binary full-response continuous-phase modulation (CPM), special cases of which are continuous-phase FSK (CPFSK) and minimum shift keying (MSK) (which itself is a special case of CPFSK), the condition for the existence of a discrete spectrum is independent of the pulse shape and merely dependent on the frequency modulation index. In particular, a discrete line spectrum will exist when the modulation index is integer and will not exist when the modulation index is noninteger. The frequencies at which the spectral lines occur can be obtained directly from the condition for the existence of the discrete spectrum.

References

[1] M. K. Simon, "On the Power Spectrum of Angle-Modulated PSK Signals Corrupted by Intersymbol Interference," *The Telecommunications and Data Acquisition Progress Report 42-131, July–September 1997*, Jet Propulsion Laboratory, Pasadena, California, pp. 1–10, November 15, 1997. http://tda.jpl.nasa.gov/tda/progress_report/42-131/131B.pdf

[2] V. K. Prabhu and H. E. Rowe, "Spectra of Digital Phase Modulation by Matrix Methods," *Bell System Technical Journal*, vol. 53, no. 5, pp. 899–934, May–June 1974.

[3] V. K. Prabhu and H. E. Rowe, "Power Spectrum of a Digital Frequency Modulation Signal," *Bell System Technical Journal*, vol. 54, no. 6, pp. 1095–1125, July–August 1975.