On the Power Spectrum of Angle-Modulated Phase-Shift-Keyed Signals Corrupted by Intersymbol Interference

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Using a matrix method proposed a number of years ago by Prabhu and Rowe for computing the power spectrum of a sinusoidal carrier phase modulated by a random baseband pulse train, we obtain explicit expressions for this power spectrum when the signaling is M-ary *phase shift keying (M-PSK) in the presence of intersymbol interference (ISI) brought about by transmitter pulse shaping. Particular attention is paid to the conditions on the pulse shape (and hence the transmit filter) under which a discrete spectral component can exist.*

I. Introduction

In the study of digital phase modulation techniques, it is often of interest to compute the power spectrum of the resulting carrier-modulated waveform. In bandlimited systems, this computation must be made in the presence of intersymbol interference (ISI) applied to the baseband angle modulation. Many techniques exist in the literature for computing the power spectra [1–9] of a sinusoidal carrier phase modulated by a random baseband pulse train with symbols taken from an M-ary alphabet. Of these, the most convenient for the purpose at hand, namely, where the pulses in the train overlap in time, is a matrix method proposed by Prabhu and Rowe [9]. Although the formulation in [9] is quite general in that it allows for arbitrary pulse shapes and a priori probabilities for the M possible signals assigned to the M-ary alphabet, our purpose here is to abstract from this reference specific results for the special case of binary phase-shift-keyed (PSK) modulation transmitted through a bandlimited channel and document them in a form that is readily accessible to users of such a modulation scheme. As such, the signal model consists of passing a binary pulse train of rate $1/T$ with rectangular pulses through a filter of bandwidth B (which inherently introduces ISI) and then angle modulating this waveform onto a sinusoidal carrier with known frequency but random phase. As we shall see, the nonlinear nature of the modulation, namely, phase modulation of a carrier, results in an equivalent complex baseband process with, in general, both discrete and continuous spectrum components. In fact, for such a nonlinear phase modulation, the only instance[1] in which the discrete spectrum vanishes is when the received baseband pulses are equal to the transmitted rectangular (time limited to T s) ones, i.e., zero ISI with no pulse shaping. Even pulse shaping alone without ISI will result in the presence of a discrete spectrum for the phase-modulated waveform.

[1] Actually, any purely digital (± 1) pulse shape for $g(t)$ will result in a vanishing discrete spectrum and, thus, this statement is not restricted to only purely *rectangular* (single ± 1-level) pulses.

II. Mathematical Signal Model

Consider a digital angle-modulated carrier of the form

$$x(t) = \mathrm{Re}\left\{\exp\left[j2\pi f_c t + \phi(t) + \theta\right]\right\} \tag{1}$$

where f_c is the carrier frequency, θ is random phase assumed to be uniformly distributed in $(-\pi, \pi)$, and $\phi(t)$ is a digital angle modulation. Then, the power spectrum of $x(t)$ can be expressed in terms of the power spectrum of the equivalent complex baseband modulation, $v(t) = \exp(j\phi(t))$, by

$$S_{xx}(f) = \frac{1}{4} S_{vv}(f - f_c) + \frac{1}{4} S_{vv}(f + f_c) \tag{2}$$

Assume now that $\phi(t)$ is in the form of a binary PSK signal[2] (with rectangular T-s pulses) that has been passed through an ISI-producing filter, which results in a pulse shape, $g(t)$, that extends over K symbol (T-s) intervals. As such,

$$\phi(t) = \theta_m \sum_{k=-\infty}^{\infty} a_k g(t - kT) \tag{3}$$

where a_k denotes the data bit in the kth signaling interval that takes on values ± 1 with equal probability. Following the notation in [9], the domain of $g(t)$ is defined as follows:

$$g(t) = 0, \quad \begin{array}{lll} t \le -\dfrac{K-1}{2}T, & t > \dfrac{K+1}{2}T; & K \text{ odd} \\ t \le -\dfrac{K}{2}T, & t > \dfrac{K}{2}T; & K \text{ even} \end{array} \tag{4}$$

We shall begin by considering the power spectrum $S_{vv}(f)$ of the equivalent complex baseband modulation $v(t)$ for the case of time-limited pulses, i.e., $K = 1$.

III. Power Spectrum for Nonoverlapping Pulses—K = 1

Define the two equivalent complex baseband signals

$$\left.\begin{array}{l} r_1(t) = \begin{cases} \exp(j\theta_m g(t)); & 0 \le t \le T \\ 0; & \text{otherwise} \end{cases} \\ \\ r_2(t) = \begin{cases} \exp(-j\theta_m g(t)); & 0 \le t \le T \\ 0; & \text{otherwise} \end{cases} \end{array}\right\} \tag{5}$$

corresponding to the two possible data bit values transmitted in the zeroth transmission interval. Also, define the Fourier transforms of these complex signals by

[2] For generality, we assume an arbitrary modulation angle, θ_m, rather than the more conventional $\pi/2$ value.

$$\left.\begin{aligned} R_1(f) &= \mathcal{F}\{r_1(t)\} = \int_0^T \exp\left(j\theta_m g(t)\right) e^{-j2\pi f t} dt \\ R_2(f) &= \mathcal{F}\{r_2(t)\} = \int_0^T \exp\left(-j\theta_m g(t)\right) e^{-j2\pi f t} dt \end{aligned}\right\} \tag{6}$$

Then the line (discrete) component, $S_{vv_l}(f)$, and the continuous component, $S_{vv_c}(f)$, of the power spectrum of $v(t)$ are given by

$$\left.\begin{aligned} S_{vv_l}(f) &= \frac{1}{2T^2} \left|R_1(f) + R_2(f)\right|^2 \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \\ S_{vv_c}(f) &= \frac{1}{4T} \left|R_1(f) - R_2(f)\right|^2 \end{aligned}\right\} \tag{7}$$

where, from Eq. (6),

$$\left.\begin{aligned} \left|R_1(f) + R_2(f)\right|^2 &= \left(\int_0^T 2\cos\left(\theta_m g(t)\right) \cos 2\pi f t dt\right)^2 + \left(\int_0^T 2\cos\left(\theta_m g(t)\right) \sin 2\pi f t dt\right)^2 \\ \left|R_1(f) - R_2(f)\right|^2 &= \left(\int_0^T 2\sin\left(\theta_m g(t)\right) \cos 2\pi f t dt\right)^2 + \left(\int_0^T 2\sin\left(\theta_m g(t)\right) \sin 2\pi f t dt\right)^2 \end{aligned}\right\} \tag{8}$$

Note that if $g(t)$ is a unit rectangular pulse shape, i.e., no pulse shaping and zero ISI, then

$$\left.\begin{aligned} \left|R_1(f) + R_2(f)\right|^2 &= 4\cos^2\theta_m \left|\int_0^T e^{-j2\pi f t} dt\right|^2 = 4\cos^2\theta_m T^2 \left(\frac{\sin \pi f T}{\pi f T}\right)^2 \\ \left|R_1(f) - R_2(f)\right|^2 &= 4\sin^2\theta_m \left|\int_0^T e^{-j2\pi f t} dt\right|^2 = 4\sin^2\theta_m T^2 \left(\frac{\sin \pi f T}{\pi f T}\right)^2 \end{aligned}\right\} \tag{9}$$

and, from Eq. (7), the spectrum becomes

$$S_{vv}(f) = T\sin^2\theta_m \left(\frac{\sin \pi f T}{\pi f T}\right)^2 + 2\cos^2\theta_m \delta(f) \tag{10a}$$

For the special case of $\theta_m = \pi/2$, the discrete line spectrum vanishes and the continuous component of the spectrum becomes the well-known result

$$S_{vv}(f) = S_{vv_c}(f) = T\left(\frac{\sin \pi f T}{\pi f T}\right)^2 \tag{10b}$$

An intuitive explanation of why there exists a discrete spectrum in a carrier that is phase modulated by a pulse-shaped data stream is as follows: Consider the signal of Eq. (1), which, when combined with Eq. (3), is rewritten as (ignoring the random phase)

$$x(t) = \cos\left(2\pi f_c t + \frac{\pi}{2}\sum_{k=-\infty}^{\infty} a_k g\left(t - kT\right)\right) \tag{11}$$

Applying simple trigonometry to Eq. (11), we get

$$\begin{aligned} x(t) =& \cos\left(2\pi f_c t\right)\cos\left(\frac{\pi}{2}\sum_{k=-\infty}^{\infty} a_k g\left(t-kT\right)\right) - \sin\left(2\pi f_c t\right)\sin\left(\frac{\pi}{2}\sum_{k=-\infty}^{\infty} a_k g\left(t-kT\right)\right) \\ =& \cos\left(2\pi f_c t\right)\sum_{k=-\infty}^{\infty}\cos\left(\frac{\pi}{2}g\left(t-kT\right)\right) - \sin\left(2\pi f_c t\right)\sum_{k=-\infty}^{\infty} a_k \sin\left(\frac{\pi}{2}g\left(t-kT\right)\right) \end{aligned} \tag{12}$$

Defining the effective in-phase (I) and quadrature-phase (Q) pulse shapes by

$$\left.\begin{aligned} p_I(t) =& \cos\left(\frac{\pi}{2}g\left(t\right)\right) \\ p_Q(t) =& \sin\left(\frac{\pi}{2}g\left(t\right)\right) \end{aligned}\right\} \tag{13}$$

we can rewrite Eq. (12) in the form

$$x(t) = \cos\left(2\pi f_c t\right)\sum_{k=-\infty}^{\infty} p_I\left(t-kT\right) - \sin\left(2\pi f_c t\right)\sum_{k=-\infty}^{\infty} a_k p_Q\left(t-kT\right) \tag{14}$$

The first term in Eq. (14), which is data independent, represents a carrier modulated by a periodic (with period T) waveform and as such has a purely discrete spectrum. The second term in Eq. (14) is a PSK modulation with pulse shape $p_Q\left(t\right)$ and thus has a purely continuous spectrum. Hence, the first term of Eq. (14) accounts entirely for the discrete spectrum in Eq. (11), and the second term of Eq. (14) accounts entirely for the continuous spectrum in Eq. (12). Because of this partitioning, one could, if desired, remove the discrete spectrum from Eq. (11) by generating the first term in Eq. (14) and then subtracting this out from Eq. (11). However, since the subtraction occurs *after* the phase modulation, the resulting waveform no longer has a constant envelope. Also note that if $g(t)$ is a purely rectangular unit amplitude pulse, then from Eq. (13), $p_I(t) = 0$ and $p_Q(t) = g(t)$, in which case the first term (discrete spectrum) of Eq. (14) vanishes and the second term becomes a PSK modulation with the original pulse shape $g(t)$.

IV. Power Spectrum for Overlapping Pulses—*K* Arbitrary

Consider a data sequence whose corresponding pulses would contribute to the signal in the interval $0 \le t \le T$. In particular, let $\mathbf{a} \triangleq (a_{m_1}, a_{m_1+1}, \ldots, a_{m_2})$, where

$$\left.\begin{array}{lll} m_1 = \dfrac{-(K-1)}{2}, & m_2 = \dfrac{(K-1)}{2}; & K \text{ odd} \\ m_1 = -\left(\dfrac{K}{2}\right)+1, & m_2 = \dfrac{K}{2}; & K \text{ even} \end{array}\right\} \tag{15}$$

Also define the ith such sequence (corresponding to a particular set of $+1$'s and -1's for the a_i's) by $\mathbf{a}^{(i)} \triangleq \left(a_{m_1}^{(i)}, a_{m_1+1}^{(i)}, \ldots, a_{m_2}^{(i)}\right)$; $\quad i = 1, 2, \ldots, 2^K$. Analogous to Eq. (5), define the set of complex baseband signals

$$r_i(t) = \begin{cases} \exp\left(j\theta_m \sum_{m=m_1}^{m_2} a_m^{(i)} g\left(t - mT\right)\right); & 0 \le t \le T \\ 0; & \text{otherwise} \end{cases}, \quad i = 1, 2, \ldots, 2^K \tag{16}$$

with associated Fourier transforms

$$R_i(f) = \mathcal{F}\left\{r_i(t)\right\} = \int_0^T \exp\left(j\theta_m \sum_{m=m_1}^{m_2} a_m^{(i)} g\left(t - mT\right)\right) e^{-j2\pi f t} dt, \quad i = 1, 2, \ldots, 2^K \tag{17}$$

Then, the discrete line spectrum of $v(t)$ is, analogous to Eq. (6), given by

$$S_{vv_l}(f) = \frac{1}{2^K T^2} \left| \sum_{i=1}^{2^K} R_i(f) \right|^2 \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \tag{18}$$

Since for each sequence $\mathbf{a}^{(i)}$ there also exists the complement of this sequence in the complete set of 2^K sequences, then the Fourier transforms of Eq. (17) can be combined for each of these pairs of complementary sequences and, hence, Eq. (18) can be slightly simplified to

$$S_{vv_l}(f) = \frac{1}{2^K T^2} \left| \sum_{i=1}^{2^{K-1}} R_i'(f) \right|^2 \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \tag{19}$$

where

$$R_i'(f) = \int_0^T 2\cos\left[\theta_m \left(g(t) + \sum_{\substack{m=m_1 \\ m\neq 0}}^{m_2} a_m^{(i)} g\left(t - mT\right)\right)\right] e^{-j2\pi f t} dt \tag{20}$$

Note that even for $\theta_m = \pi/2$, a discrete spectrum component will, in general, exist.

In the case of overlapping pulses ($K > 1$), the computation of the continuous component of the spectrum of $v(t)$ is significantly more formidable. After considerable matrix simplification, the result can be put in the following form:

$$S_{vv_c}(f) = \frac{1}{2^K T} \sum_{i=1}^{2^K} |R_i(f)|^2 - \frac{1}{2^{2K} T} \left(\sum_{n=0}^{K-1} \varepsilon_n \cos 2\pi f n T \right) \sum_{i=1}^{2^K} \sum_{j=1}^{2^K} R_i(f) R_j^*(f)$$

$$+ \frac{1}{T} \sum_{n=1}^{K-1} \frac{1}{2^{K+n}} 2 \text{ Re} \left\{ e^{-j2\pi f n T} \sum_{k=1}^{2^{K-n}} \left(\sum_{l=1}^{2^n} R_{2^n \times (k-1)+l}(f) \right) \left(\sum_{i=0}^{2^n-1} R^*_{i \times 2^{K-n}+k}(f) \right) \right\} \quad (21)$$

where ε_n is the Neumann factor defined by

$$\varepsilon_n = \begin{cases} 1; & n = 0 \\ 2; & n \neq 0 \end{cases} \quad (22)$$

As an example, consider the case of ISI that results in a pulse duration of three bit intervals, i.e., $K = 3$. Then, the domain of definition for $g(t)$ is $-T < t \leq 2T$ and the set of signal Fourier transforms of Eq. (16) becomes

$$\left. \begin{aligned}
R_1(f) &= \int_0^T \exp\left(j\theta_m \left[g\left(t+T\right) + g(t) + g\left(t-T\right)\right]\right) e^{-j2\pi f t} dt \\
R_2(f) &= \int_0^T \exp\left(j\theta_m \left[g\left(t+T\right) + g(t) - g\left(t-T\right)\right]\right) e^{-j2\pi f t} dt \\
R_3(f) &= \int_0^T \exp\left(j\theta_m \left[g\left(t+T\right) - g(t) + g\left(t-T\right)\right]\right) e^{-j2\pi f t} dt \\
R_4(f) &= \int_0^T \exp\left(j\theta_m \left[g\left(t+T\right) - g(t) - g\left(t-T\right)\right]\right) e^{-j2\pi f t} dt \\
R_5(f) &= \int_0^T \exp\left(j\theta_m \left[-g\left(t+T\right) + g(t) + g\left(t-T\right)\right]\right) e^{-j2\pi f t} dt \\
R_6(f) &= \int_0^T \exp\left(j\theta_m \left[-g\left(t+T\right) + g(t) - g\left(t-T\right)\right]\right) e^{-j2\pi f t} dt \\
R_7(f) &= \int_0^T \exp\left(j\theta_m \left[-g\left(t+T\right) - g(t) + g\left(t-T\right)\right]\right) e^{-j2\pi f t} dt \\
R_8(f) &= \int_0^T \exp\left(j\theta_m \left[-g\left(t+T\right) - g(t) - g\left(t-T\right)\right]\right) e^{-j2\pi f t} dt
\end{aligned} \right\} \quad (23)$$

From Eq. (18), the discrete spectrum is given by

$$S_{vv_l}(f) = \frac{1}{8T^2} \left| \sum_{i=1}^{8} R_i(f) \right|^2 \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \quad (24)$$

and the continuous spectrum is from Eq. (21):

$$S_{vv_c}(f) = \frac{1}{8T}\left|\sum_{i=1}^{8} R_i(f)\right|^2 - \frac{1}{64T}\left(1 + 2\cos 2\pi fT + \cos 4\pi fT\right)\sum_{i=1}^{8}\sum_{j=1}^{8} R_i(f)R_j^*(f)$$

$$+ \frac{1}{8T}\,\mathrm{Re}\left\{e^{-j2\pi fT}\left[\left(R_1(f) + R_2(f)\right)\left(R_1^*(f) + R_5^*(f)\right)\right.\right.$$

$$+ \left(R_3(f) + R_4(f)\right)\left(R_2^*(f) + R_6^*(f)\right) + \left(R_5(f) + R_6(f)\right)\left(R_3^*(f) + R_7^*(f)\right)$$

$$\left.\left. + \left(R_7(f) + R_8(f)\right)\left(R_4^*(f) + R_8^*(f)\right)\right]\right\}$$

$$+ \frac{1}{16T}\,\mathrm{Re}\left\{e^{-j4\pi fT}\left[\left(R_1(f) + R_2(f) + R_3(f) + R_4(f)\right)\left(R_1^*(f) + R_3^*(f) + R_5^*(f) + R_7^*(f)\right)\right.\right.$$

$$\left.\left. + \left(R_5(f) + R_6(f) + R_7(f) + R_8(f)\right)\left(R_2^*(f) + R_4^*(f) + R_6^*(f) + R_8^*(f)\right)\right]\right\} \tag{25}$$

The intuitive explanation for the presence of a discrete spectrum in a carrier that is phase modulated by a data stream containing overlapping pulses follows along the same lines as that given at the end of Section III. In particular, if the duration of $g(t)$ extends K pulse intervals as specified in Eq. (4), then the phase-modulated carrier of Eq. (11) can be organized in the form

$$x(t) = \cos\left(2\pi f_c t\right)\sum_{k=-\infty}^{\infty} q_I\left(t - kT; \mathbf{a}_k\right) - \sin\left(2\pi f_c t\right)\sum_{k=-\infty}^{\infty} q_Q\left(t - kT; \mathbf{a}_k\right) \tag{26}$$

where $\mathbf{a}_k \triangleq \left(a_{m_1+k}, a_{m_1+k+1}, \ldots, a_{m_2+k}\right)$ is the sequence of data bits that contribute to the kth transmission interval, the integers m_1 and m_2 are defined in Eq. (15), and

$$\left.\begin{aligned} q_I\left(t - kT; \mathbf{a}_k\right) &= \begin{cases} \cos\left(\dfrac{\pi}{2}\displaystyle\sum_{m=m_1}^{m_2} a_{m+k}\, g\left(t - mT - kT\right)\right), & kT \le t \le (k+1)\,T \\ 0, & \text{otherwise} \end{cases} \\ q_Q\left(t - kT; \mathbf{a}_k\right) &= \begin{cases} \sin\left(\dfrac{\pi}{2}\displaystyle\sum_{m=m_1}^{m_2} a_{m+k}\, g\left(t - mT - kT\right)\right), & kT \le t \le (k+1)\,T \\ 0, & \text{otherwise} \end{cases} \end{aligned}\right\} \tag{27}$$

are the effective T-s duration pulse shapes for that same time interval that depend on the data sequence. Applying successive trigonometric expansions to the sine and cosine terms in Eq. (27), we finally arrive at the following form for Eq. (26):

$$x(t) = \cos\left(2\pi f_c t\right)\sum_{k=-\infty}^{\infty} p_I\left(t - kT\right) + \text{ terms that are data dependent} \tag{28}$$

where

$$p_I(t) = \begin{cases} \prod_{m=m_1}^{m_2} \cos\left(\frac{\pi}{2} g\left(t - mT\right)\right), & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{29}$$

is the equivalent T-s duration pulse shape that represents the envelope of the discrete carrier. Once again, since the first term in Eq. (28) is a periodic function with period T, it contributes a purely discrete spectrum, whereas the remaining data-dependent terms contribute the continuous component of the overall spectrum. Note that Eq. (29) suggests that if at least one of the K T-s segments of $g(t)$ is purely ± 1 digital, e.g., constant and equal to $+1$, then $p_I(t) = 0$ and the discrete spectrum vanishes. For example, phase modulating the carrier with a pulse stream having the $3T$-s duration trapezoid

$$g(t) = \begin{cases} \frac{t+T}{T}, & -T \le t \le 0 \\ 1, & 0 \le t \le T \\ -\frac{t-2T}{T}, & T \le t \le 2T \end{cases} \tag{30}$$

would have no discrete spectrum.

The form of Eq. (29) suggests that the general expression for the discrete spectrum given in Eq. (18) can be simplified. In fact, it is straightforward to show that continued pairwise combination of the terms in the sum of the $R_i(f)$'s results in the following expression:

$$S_{vv_l}(f) = \frac{1}{2^K T^2} \left|2^K P_I(f)\right|^2 \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \tag{31}$$

where $P_I(f)$ is the Fourier transform of $p_I(t)$ of Eq. (29), whose squared magnitude is given by

$$|P_I(f)|^2 = \left(\int_0^T \prod_{m=m_1}^{m_2} \cos\left(\frac{\pi}{2} g\left(t - mT\right)\right) \cos 2\pi f t dt\right)^2 + \left(\int_0^T \prod_{m=m_1}^{m_2} \cos\left(\frac{\pi}{2} g\left(t - mT\right)\right) \sin 2\pi f t dt\right)^2 \tag{32}$$

V. Extension to *M*-ary PSK

If now the phase modulation $\phi(t)$ corresponds to M-ary PSK, then Eq. (3) becomes

$$\phi(t) = \frac{\pi}{M} \sum_{k=-\infty}^{\infty} a_k g\left(t - kT\right) \tag{33}$$

where a_k takes on values $\pm 1, \pm 3, \ldots, \pm(M-1)$, each with probability $1/M$. Defining the set of M time-limited pulses by

$$r_i(t) = \begin{cases} \exp\left(j\dfrac{\pi}{M}\left(M-(2i-1)\right)g(t)\right); & 0 \le t \le T \\ 0; & \text{otherwise} \end{cases}, \quad i = 1, 2, \ldots, M \tag{34}$$

with corresponding Fourier transforms

$$R_i(f) = \mathcal{F}\left\{r_i(t)\right\} = \int_0^T \exp\left(j\frac{\pi}{M}\left(M-(2i-1)\right)g(t)\right)e^{-j2\pi ft}dt, \quad i = 1, 2, \ldots, M \tag{35}$$

then for nonoverlapping pulses ($K = 1$), we now get [analogous to Eq. (7)]

$$\left.\begin{aligned} S_{vv_l}(f) =& \frac{1}{MT^2}\left|\sum_{i=1}^{M} R_i(f)\right|^2 \sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T}\right) \\ S_{vv_c}(f) =& \frac{1}{2M^2T}\sum_{i=1}^{M}\sum_{j=1}^{M}\left|R_i(f)-R_j(f)\right|^2 \end{aligned}\right\} \tag{36}$$

For overlapping pulses with arbitrary K, we now get [analogous to Eqs. (18) and (21)]

$$S_{vv_l}(f) = \frac{1}{M^KT^2}\left|\sum_{i=1}^{M^K} R_i(f)\right|^2 \sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T}\right) \tag{37}$$

and

$$\begin{aligned} S_{vv_c}(f) =& \frac{1}{M^KT}\sum_{i=1}^{M^K}\left|R_i(f)\right|^2 - \frac{1}{M^{2K}T}\left(\sum_{n=0}^{K-1}\varepsilon_n \cos 2\pi fnT\right)\sum_{i=1}^{M^K}\sum_{j=1}^{M^K}R_i(f)R_j^*(f) \\ &+ \frac{1}{T}\sum_{n=1}^{K-1}\frac{1}{M^{K+n}}2\,\mathrm{Re}\left\{e^{-j2\pi fnT}\sum_{k=1}^{M^{K-n}}\left(\sum_{l=1}^{M^n}R_{M^n\times(k-1)+l}(f)\right)\left(\sum_{i=0}^{M^n-1}R^*_{i\times M^{K-n}+k}(f)\right)\right\} \end{aligned} \tag{38}$$

where, analogous to Eq. (17),

$$R_i(f) = \mathcal{F}\left\{r_i(t)\right\} = \int_0^T \exp\left(j\frac{\pi}{M}\sum_{m=m_1}^{m_2}a_m^{(i)}g\left(t-mT\right)\right)e^{-j2\pi ft}dt, \quad i = 1, 2, \ldots, M^K \tag{39}$$

and the a_i's in the sequence $\mathbf{a}^{(i)} \triangleq \left(a_{m_1}^{(i)}, a_{m_1+1}^{(i)}, \ldots, a_{m_2}^{(i)}\right);\ i = 1, 2, \ldots, M^K$ each range over the set of values $\pm1, \pm3, \ldots, \pm(M-1)$.

VI. Conclusions

For a sinusoidal carrier that is phase modulated by a random pulse train whose pulses are both shaped and overlapping, the discrete spectrum vanishes only when the transmitted pulse shape is purely digital (+1), e.g., NRZ or Manchester signaling, corresponding to zero ISI with no pulse shaping. Even pulse shaping alone without ISI will result in the presence of a discrete spectrum for the phase-modulated carrier.

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