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False Lock Performance of I-Q Costas Loops for Pulse-Shaped Binary Phase Shift Keying

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The false lock performance and false lock margin of I-Q Costas loops with matched-filter/sample-and-hold (S&H) arm filters are derived for the case of a random data input with symbols of arbitrary pulse shape.

I. Introduction

In the late 1970s, considerable analytical work was performed and reported on in the literature[1] [1–5] dealing with the theory of false lock in Costas loop receivers associated with acquisition of suppressed-carrier signals with frequency uncertainty greater than half the data symbol rate (assuming random data). Indeed, the phenomenon had been observed and reported on prior to that time [6], but it was not until 1976 that an adequate theoretical explanation of it that was able to confirm the measured results became available.[2] Much of the work performed at that time focused on Costas loops with single-pole passive arm filters. A smaller amount of attention was paid to active arm filters such as integrate-and-dump (I&D) circuits, but only for nonreturn-to-zero (NRZ) data and, even in that case, the effect on the signal-by-signal (S $\times$ S) component of the error signal was all that was accounted for [4,5].

In this article, we derive the false lock performance and assess the false lock margin of in-phase–quadrature-phase (I-Q) Costas loops with matched-filter/sample and hold (S&H) arm filters for a random data input with symbols having an arbitrary pulse shape. We shall show that, for this type of Costas loop, the false lock performances of NRZ and Manchester data are *identical* and, furthermore, the false lock margins for both are entirely predicted by the effect on the S $\times$ S components of the error signals. These results are in contrast to those for Costas loops with passive arm filters, wherein the false lock performance and margins for NRZ and Manchester coded data are quite different from one another.

II. Characterization of the Loop Error Signal Under False Lock Conditions

The input $x(t)$ to the I-Q Costas loop of Fig. 1 is the sum of pulse-shaped phase shift keying (PSK) and a bandpass white Gaussian noise process, i.e.,

$$x(t) = s(t,\theta) + n(t) \tag{1}$$

[1] M. K. Simon, "False Lock Behavior of Costas Receivers," Appendix J, *Integrated Source and Channel Encoded Digital Communication System Design Study*, Final Report, Axiomatix Report R7607-3, Marina del Rey, California, July 31, 1976.

[2] Ibid.

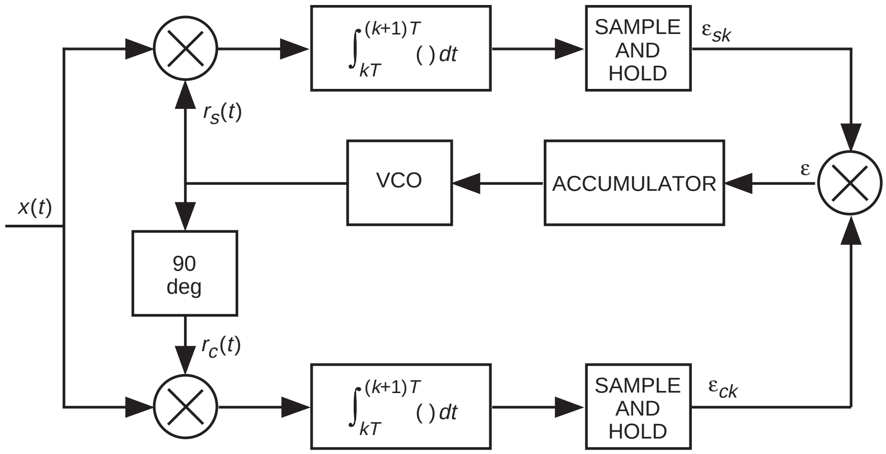

Fig. 1. Block diagram of an I-Q Costas loop for pulse-shaped BPSK.

where

$$s(t, \theta) = \sqrt{2S} m(t) \sin(\omega_c t + \theta) \tag{2}$$

with S the signal power, ω_c the carrier radian frequency, θ the unknown carrier phase to be tracked, and

$$m(t) = \sum_{n=-\infty}^{\infty} d_n p(t - nT) \tag{3}$$

the baseband modulation with ± 1 independent identically distributed (i.i.d.) data sequence $\{d_n\}$ and arbitrary unit power pulse shape $p(t)$ time-limited to the interval $0 \le t \le T$. The noise $n(t)$ has power spectral density N_0 W/Hz.

Under false lock conditions, the in-phase (I) and quadrature-phase (Q) demodulation reference signals are expressable as

$$\left.\begin{aligned} r_s(t) &= \sqrt{2} \sin\left((\omega_c - \omega_f) t + \hat{\theta}\right) \\ r_c(t) &= \sqrt{2} \cos\left((\omega_c - \omega_f) t + \hat{\theta}\right) \end{aligned}\right\} \tag{4}$$

where $\hat{\theta}$ is the loop's estimate of the input carrier phase and ω_f is the false lock radian frequency (as yet to be determined) that represents the difference between the true radian frequency of the received signal and the nominal radian frequency of the voltage-controlled oscillator (VCO). For the analysis that follows, it is convenient to expand the additive bandpass noise around the VCO frequency rather than the more usual expansion around the input carrier frequency. Thus, expressing $n(t)$ as

$$n(t) = \sqrt{2} N_c(t) \cos((\omega_c - \omega_f) t + \theta) - \sqrt{2} N_s(t) \sin((\omega_c - \omega_f) t + \theta) \tag{5}$$

and letting $\phi \triangleq \theta - \hat{\theta}$ denote the loop phase error, then, ignoring second harmonic terms of the carrier, the I and Q sample-and-hold outputs are piecewise constant waveforms, which in the $k + 1$st T-s interval are given by

$$
\left.\begin{aligned}
\varepsilon_{sk} &= d_k\sqrt{S}\int_{kT}^{(k+1)T} p^2\left(t-kT\right)\cos\left(\omega_f t+\phi\right)dt \\
&\quad - N_1\sin\phi - N_2\cos\phi \\
\varepsilon_{ck} &= d_k\sqrt{S}\int_{kT}^{(k+1)T} p^2\left(t-kT\right)\sin\left(\omega_f t+\phi\right)dt \\
&\quad + N_1\cos\phi - N_2\sin\phi
\end{aligned}\quad , \quad (k+1)\,T \le t \le (k+2)\,T \right\} \tag{6}
$$

where

$$
\left.\begin{aligned}
N_1 &\triangleq \int_{kT}^{(k+1)T} N_c(t)p\left(t-kT\right)dt \\
N_2 &\triangleq \int_{kT}^{(k+1)T} N_s(t)p\left(t-kT\right)dt
\end{aligned}\right\} \tag{7}
$$

are independent zero-mean Gaussian random variables with variance

$$
\sigma_{N_1}^2 = \sigma_{N_2}^2 = \frac{N_0}{2}\int_{kT}^{(k+1)t} p^2(t-kT)dt = \frac{N_0}{2}\int_0^T p^2(t)dt = \frac{N_0T}{2} \triangleq \sigma_N^2 \tag{8}
$$

which is independent of k.

The signal terms in Eq. (6) can be rewritten as follows:

$$
\left.\begin{aligned}
\int_{kT}^{(k+1)T} p^2\left(t-kT\right)\cos\left(\omega_f t+\phi\right)dt &= \int_0^T p^2\left(t\right)\cos\left(\omega_f\left(t+kT\right)+\phi\right)dt \\
&= A_{ck}\cos\phi - A_{sk}\sin\phi = \mathrm{Re}\left\{A_k e^{j(\phi+\alpha_k)}\right\} \\
\int_{kT}^{(k+1)T} p^2\left(t-kT\right)\sin\left(\omega_f t+\phi\right)dt &= \int_0^T p^2\left(t\right)\sin\left(\omega_f\left(t+kT\right)+\phi\right)dt \\
&= A_{ck}\sin\phi + A_{sk}\cos\phi = \mathrm{Im}\left\{A_k e^{j(\phi+\alpha_k)}\right\}
\end{aligned}\right\} \tag{9}
$$

where

$$\left.\begin{aligned} A_{ck} &\triangleq \int_0^T p^2(t)\cos\left(\omega_f\left(t+kT\right)\right)dt \\ A_{sk} &\triangleq \int_0^T p^2(t)\sin\left(\omega_f\left(t+kT\right)\right)dt \\ A_k &\triangleq A_{ck}+jA_{sk} \triangleq \int_0^T p^2(t)\,e^{j(\omega_f(t+kT))}dt \\ \alpha_k &\triangleq \tan^{-1}\frac{A_{sk}}{A_{ck}} \end{aligned}\right\} \tag{10}$$

Substituting Eq. (9) together with Eq. (10) in Eq. (6) and multiplying the results gives the loop error signal, namely,

$$\begin{aligned} \varepsilon_k \triangleq \varepsilon_{sk}\varepsilon_{ck} &= \left(d_k\sqrt{S}\left(\mathrm{Re}\left\{A_k e^{j(\phi+\alpha_k)}\right\}\right) - N_1\sin\phi - N_2\cos\phi\right) \\ &\quad\times\left(d_k\sqrt{S}\left(\mathrm{Im}\left\{A_k e^{j(\phi+\alpha_k)}\right\}\right) + N_1\cos\phi - N_2\sin\phi\right) \\ &= S\left|A_k\right|^2\cos\left(\phi+\alpha_k\right)\sin\left(\phi+\alpha_k\right) + N_e\left(t,\phi\right) \\ &= \frac{ST^2}{2}\left|V\right|^2\sin\left(2\left(\phi+\alpha_k\right)\right) + N_{e_k} \end{aligned} \tag{11}$$

where

$$\left|V\right|^2 \triangleq \left|\frac{1}{T}A_k\right|^2 = \left|\frac{1}{T}\int_0^T p^2(t)\,e^{j(\omega_f(t+kT))}dt\right|^2 = \left|\frac{1}{T}\int_0^T p^2(t)\,e^{j\omega_f t}dt\right|^2 \tag{12}$$

which is independent of k, and $N_e(t,\phi)$ is an equivalent noise process, which includes both signal-times-noise (S × N) and noise-times-noise (N × N) terms, that is piecewise constant over T-s intervals (e.g., the $k+1$st) with value N_{e_k}. The statistics of $N_e(t,\phi)$ will be investigated shortly. We recognize from Eq. (12) that V represents a normalized (by $1/T$) version of the Fourier transform of $p^2(t)$ evaluated at $\omega = -\omega_f$. Note that for true lock, i.e., $\omega_f = 0$, we have $|V| = 1$. Using simple trigonometry, the phase shift α_k defined in Eq. (10) can be expressed as

$$\alpha_k = \tan^{-1}\frac{A_{s0}\cos k\omega_f T + A_{c0}\sin k\omega_f T}{A_{c0}\cos k\omega_f T - A_{s0}\sin k\omega_f T} = \tan^{-1}\frac{A_0\sin\left(k\omega_f T+\alpha_0\right)}{A_0\cos\left(k\omega_f T+\alpha_0\right)} = k\omega_f T + \alpha_0 \tag{13}$$

which is linear in k. Thus, the signal component of Eq. (11) can be rewritten as

$$\varepsilon_k\left|_{signal}\right. = \frac{ST^2}{2}\left|V\right|^2\sin\left(2\phi+2k\omega_f T+2\alpha_0\right), \quad (k+1)\,T \le t \le (k+2)\,T \tag{14}$$

III. Characterization of the Equivalent Noise Process

As previously mentioned, the equivalent noise process $N_e(t,\phi)$ in the error signal is piecewise constant over T-s intervals. As long as the loop bandwidth is much less than the data bandwidth (which is always the case of interest), then, as for the true lock situation, $N_e(t,\phi)$ can be modeled as a delta-correlated process with the correlation function given by[3]

$$R_{N_e}(\tau) = E\left\{N_e(t)N_e(t+\tau)\right\} = \begin{cases} \sigma_{N_e}^2\left[1-\dfrac{|\tau|}{T}\right], & |\tau| \le T \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

where $\sigma_{N_e}^2$ is the variance of N_{e_k}, which from Eq. (11) can be evaluated as[4]

$$\begin{aligned} \sigma_{N_e}^2 &= E\left\{\left(A_{ck}d_k\sqrt{S}N_1 - A_{sk}d_k\sqrt{S}N_2 - N_1N_2\right)^2\right\} \\ &= SA_k^2\sigma_N^2 + \sigma_N^4 = \sigma_N^2\left[ST^2|V|^2 + \sigma_N^2\right] \end{aligned} \tag{16}$$

and again is independent of k. Finally, the equivalent noise spectral density is given by

$$N_0' = 2\int_{-\infty}^{\infty} R_{N_e}(\tau)\,d\tau = 2\sigma_{N_e}^2 T = N_0T^2\left[ST^2|V|^2 + \frac{N_0T}{2}\right] \tag{17}$$

IV. Establishing the False Lock Frequencies

In order for the loop to track, the error signal must have a nonzero dc component. This component is represented by the time average (i.e., the average over the index k) of the signal in Eq. (14). It is clear from this equation that for a nonzero dc component to exist, we must have $2\omega_f T$ equal to a multiple of 2π, i.e.,

$$2\omega_f T = 2\pi n \quad \Rightarrow \quad f_f = \frac{\omega_f}{2\pi} = \frac{n}{2T} \tag{18}$$

At these frequencies, the signal component of the error (i.e., the so-called loop S-curve) becomes

$$\eta(\phi) \triangleq \langle \varepsilon_k|_{\text{signal}}\rangle = \frac{ST^2}{2}|V|^2\sin(2\phi + 2\alpha_0) \tag{19}$$

where $\langle\bullet\rangle$ denotes the time average. Thus, under false lock conditions, the Costas loop will track perfectly but with a phase offset given by $2\alpha_0$ and a reduced strength (slope of the S-curve) relative to true lock given by $|V|^2$.

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[3] Also, for large loop signal-to-noise ratios (SNRs), it is sufficient to examine this process only at $\phi = 0$, as is customary for true lock operation. Hence, we herein shorten our notation to $N_e(t)$, which denotes $N_e(t,\phi)$ evaluated at $\phi = 0$.

[4] Note that N_{e_k} is zero mean since d_k, N_1, and N_2 are all zero mean.

V. Tracking Performance of the Loop Under False Lock Conditions

Assuming a loop signal-to-noise ratio (SNR) sufficiently large for the loop to be operating in the linear region, and denoting the slope of the S-curve at its lock point by

$$K_\eta \triangleq \frac{d\eta(\phi)}{d\phi}\,|_{\phi=-\alpha_0} = ST^2 \left|V\right|^2 \tag{20}$$

then the phase error variance is given by

$$\sigma_\phi^2 = \frac{N_0' B_L}{K_\eta^2} \triangleq \frac{1}{\rho S_L} \tag{21}$$

where $\rho \triangleq S/N_0 B_L$ is the linear phase-locked loop (PLL) SNR with B_L denoting the single-sided loop noise bandwidth, and S_L is the squaring loss which, from Eqs. (17), (20), and (21), is given by

$$S_L = \frac{K_\eta^2/S}{N_0'/N_0} = \frac{2R_d \left|V\right|^4}{1 + 2R_d \left|V\right|^2} \tag{22}$$

Since, as previously mentioned, $|V| = 1$ for true lock, then for this condition, Eq. (22) reduces to the well-known result for I-Q Costas loops:

$$S_L = \frac{2R_d}{1 + 2R_d} \tag{23}$$

where $R_d \triangleq ST/N_0$ is the detection SNR.

VI. Evaluation of $|V|^2$ for NRZ and Manchester Pulse Shapes

The evaluation of $|V|^2$ for NRZ pulses was previously performed in [4,5] and is easily obtainable from Eq. (12), namely,

$$\left|V\right|^2 \Big|_{f_f = n/2T} = \left|\frac{1}{T}\int_0^T e^{j\pi n t/T} dt\right|^2 = \left(\frac{\sin\dfrac{n\pi}{2}}{\dfrac{n\pi}{2}}\right)^2 = \begin{cases} \left(\dfrac{2}{n\pi}\right)^2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \tag{24}$$

In [5, p. 1707], it was stated that "similar results can be derived for Manchester data." However, from the definition of $|V|^2$ given here in Eq. (12), we see that it depends only on the square of the pulse shape; hence, we immediately see that Eq. (24) also applies to a Manchester data waveform. In fact, Eq. (24) would apply to any digital (± 1) waveform for the pulse shape $p(t)$. Thus, *for I-Q Costas loops with matched-filter/S&H arm filters, the loop will have identical false lock behavior at frequency offsets equal to odd multiples of half the data rate both for NRZ and Manchester data.* This is an interesting result when contrasted with the equivalent results for single-pole arm filters. In the case of the latter, it was shown in the previously cited references that NRZ and Manchester data had quite different false lock strength behaviors as a function of false lock frequency. In particular, it was shown that for NRZ data, the strongest false lock occurred at $f_f = 1/2T$, whereas for Manchester data, the strongest false lock occurred at $f_f = 1/T$. Here the false lock strengths are identical for both at all false lock frequencies and, hence, the strongest false lock for both occurs at $f_f = 1/2T$.

VII. Evaluation of $|V|^2$ for Root Raised Cosine Pulse Shapes

The evaluation of $|V|^2$ for a root-raised cosine transmitted pulse shape [i.e., an overall (transmitter plus matched receiver) raised cosine pulse shape] can be similarly performed. In particular, for

$$p(t) = \sqrt{2}\sin\frac{\pi}{T}t, \quad 0 \le t \le T \tag{25}$$

it is straightforward to show from Eq. (12) that

$$|V|^2\Big|_{f_f=n/2T} = \begin{cases} \left(\dfrac{2}{n\pi}\right)^2 \dfrac{1}{\left[1-\left(\dfrac{n}{2}\right)^2\right]^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \tag{26}$$

Thus, while the first false lock at $f_f = 1/2T(n=1)$ is a factor of 16/9 stronger than that of NRZ or Manchester, the remaining false locks at $f_f = n/2T,\ \ n \ge 3$ (n odd) are all weaker. This result is a direct consequence of the fact that the Fourier transform of the raised cosine squared pulse has a wider main lobe than that of the rectangular pulse, but has smaller side lobes.

VIII. Evaluation of False Lock Margin

The tendency of a Costas loop to lock at one of the potential false lock frequencies as compared to its likelihood of staying in lock at the true carrier frequency can be measured by examining the relative loop SNRs under true and false lock conditions. A measure of this tendency can be expressed in terms of the false lock margin [4,5],[5] which is defined as the amount (in dB) by which the input S/N_0 must increase to produce a false lock loop SNR equal to that obtained under true lock conditions (equivalently, produce equal rms phase errors under the two conditions). In mathematical terms, for false lock at the nth false lock frequency $f_f = n/2T$, the margin is defined by

$$M_n \triangleq 10\log_{10}\frac{S'/N_0}{S/N_0} \tag{27}$$

where S'/N_0 denotes the SNR under false lock conditions and S/N_0 denotes the SNR under true lock conditions. Since we desire equal loop SNRs under the two conditions, then the following relation must be met:

$$\frac{S}{N_0 B_L}S_L = \frac{S'}{N_0 B_L}S'_L \tag{28}$$

where S_L is given by Eq. (23) for true lock, itself a function of S/N_0, and S'_L is given by Eq. (22) for false lock, itself a function of S'/N_0. Substituting Eqs. (22) and (23) and solving for the SNR required in Eq. (27), it is straightforward to show that the false lock margin is simply given by

$$M_n \triangleq -10\log_{10}|V|^2\Big|_{f_f=n/2T} \tag{29}$$

[5] M. K. Simon, *Improved Calculation of False Lock Margin in Costas Loop Receivers*, Axiomatix Report R7803-5, Marina del Rey, California, March 31, 1978.

which is easily evaluated for NRZ and Manchester data using Eq. (24). Here again, the result in Eq. (29) should be contrasted with the previously reported results for Costas loops with passive single-pole arm filters, where the false lock margins for NRZ and Manchester data signals were quite different.

IX. Further Observations

Although not explicitly stated previously, several of the cited references developed the theory of false lock for Costas loops with arbitrary arm-filter transfer function $H(j\omega)$ as well as arbitrary pulse-shape Fourier transform $P(j\omega)$. At first glance, it is tempting simply to substitute the transfer function of a matched filter, namely, $H(j\omega) = e^{-j\omega T}P(-j\omega)$, in the expressions contained in these references in an effort to obtain the specific results for the I-Q Costas loop. Unfortunately, this procedure does not lead to the results presented in this article for the following reason: The output of a matched filter is a continuous function of time, whereas the output of a matched filter coupled with a sample-and-hold circuit (as considered here) is a piecewise constant function of time. While one could include the transfer function of the sample-and-hold circuit in $H(j\omega)$ along with that of the matched filter, it is simpler to proceed in the direct manner considered in this article.

X. Conclusion

For an I-Q Costas loop with matched-filter/sample-and-hold (S&H) arm filters, it has been shown that the false lock performances for NRZ and Manchester data inputs are *identical* and, furthermore, the false lock margin for both is entirely predicted by the effect on the S $\times$ S component of the error signal. These results are in contrast to those for Costas loops with passive single-pole arm filters, wherein the false lock performances and margins for NRZ and Manchester coded data are quite different from one another.

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