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The Power Spectrum of Pulse-Position Modulation With Dead Time and Pulse Jitter

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Pulse-position modulation (PPM) is considered to be an efficient modulation format for transmitting information over optical communication channels, requiring only the delay of a single optical pulse in each word to convey the desired information. In this aricle, the power spectral density (PSD) of a random pulse train generated by a sequence of PPM symbols is derived. Variations of laser intensity, pulse jitter, and "dead time" are taken into account in the model. Results show that the PSD of the generalized PPM signal is composed of both discrete and continuous components.

I. Introduction

The power spectrum of a generalized pulse-position modulation (PPM) waveform is derived, motivated by the desire to understand the behavior of optical pulses generated by a Q-switched laser. When used as the transmitter in an optical communications system, the characteristics of the Q-switched laser pulses must be accounted for in the system design. Characteristics of Q-switched lasers relevant to the communications problem include a minimum delay between pulses due to the time it takes to reestablish a "population inversion" in the lasing material, and a small random delay—or jitter—between pulses due to uncertainty in the time it takes to build up the laser pulse after the quality factor, Q, of the cavity is restored.

In the context of optical PPM communications, these effects impact system design by requiring a "guard time" around each possible pulse location to account for pulse jitter and a predetermined "dead time" after each PPM word to allow the energy for the next laser pulse to build up. These concepts are illustrated in Fig. 1, which shows the format required to implement M-ary PPM with a commercially available Q-switched laser. Note that by eliminating guard time and dead time, the classical PPM waveform is obtained.

II. PPM Signal Model

Consider a PPM signal in which the rectangular signaling pulse $p(t)$ of width τ_p and average height n_s can, with equal probability $(1/M)$, occupy one of M slots of width $\tau \geq \tau_p$. The pulse is assumed to be symmetrically located within the slot and, thus, $\tau = \tau_p + 2\tau_d$, where τ_d is the slot time interval preceding and succeeding the pulse. The PPM frame consists of the M time slots followed by a dead time of length

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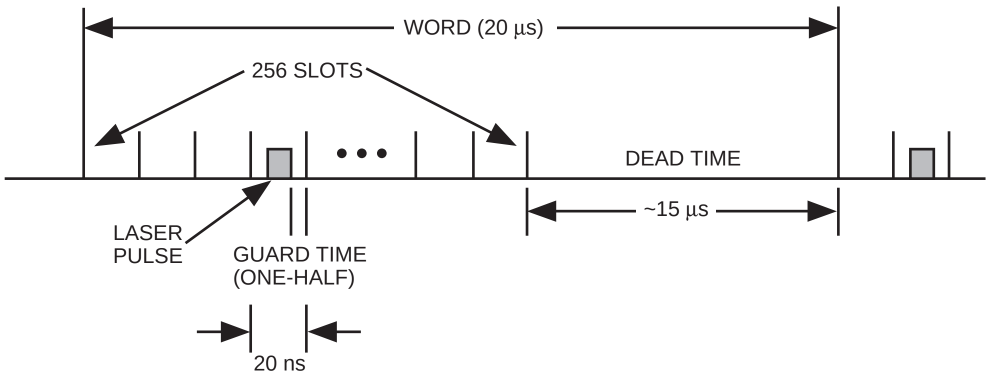

Fig. 1. Pulse-position modulation.

$K\tau$ (K integer). Thus, the frame length (in seconds) is $T_f = (M+K)\,\tau \triangleq N_f\tau$. Associated with each signaling pulse is a random timing jitter, ε, which causes the pulse to shift left or right in the τ_d intervals surrounding its nominal location. It is assumed that this random timing jitter is uniformly distributed in the interval $(-\tau_d, \tau_d)$ and is independent from transmission to transmission. As such, the transmitted PPM waveform $s(t)$ can be expressed mathematically as

$$s(t) = \sum_{i=-\infty}^{\infty} a_i p\left(t - \left(i - \frac{1}{2}\right) T_f - l_i\tau - \varepsilon_i\right) \tag{1}$$

where $\{a_i\}$ is an independent, identically (uniform) distributed (i.i.d.) sequence of random intensity variations in which a_i is a positive random variable with $E\{a_i\} = \overline{a_i} = 1$ and $E\{a_i^2\} = \overline{a_i^2}$; l_i is the PPM modulation corresponding to the ith transmission, which takes on values $0, 1, \cdots, M-1$ with equal probability $(1/M)$; and $\{\varepsilon_i\}$ is the i.i.d. timing jitter sequence. For simplicity, we have arbitrarily chosen the time origin to be at the midpoint of the PPM frame.

III. Evaluation of the Power Spectrum

The classical method for evaluating the power spectrum of a random process modeled as a periodic repetition of a random frame is to consider an observation of $s(t)$ over the symmetric interval $(-T, T)$ (for convenience chosen to be an integer number of frames), compute the power spectrum for this truncated observation (i.e., the statistical expectation of the magnitude squared of the Fourier transform of the observation normalized by the length of the interval $2T$), and then take the limit as the observation interval goes to infinity. Mathematically, the power spectrum of $s(t)$ is given by

$$S\left(\omega\right) = \lim_{T\to\infty} \frac{1}{2T} E\left\{\left|S_T\left(\omega\right)\right|^2\right\} \tag{2}$$

where $E\left\{\cdot\right\}$ is the expectation operator and $S_T\left(\omega\right)$ is the Fourier transform of the $2T$-second observation of $s(t)$, namely,

$$S_T\left(\omega\right) = \int_{-T}^{T} \sum_{i=-\infty}^{\infty} a_i p\left(t - \left[\left(i - \frac{1}{2}\right) N_f - l_i\right]\tau - \varepsilon_i\right) e^{-j\omega t} dt \tag{3}$$

Letting $2N+1$ denote the number of frames in the $(-T, T)$-second observation and $P\left(\omega\right)$ the Fourier transform of the PPM pulse $p\left(t\right)$, then it is straightforward to show that Eq. (3) becomes

$$S_T(\omega) = P(\omega)\, e^{j\omega N_f \tau/2} \sum_{i=-N}^{N} a_i e^{-j\omega(iN_f - l_i)\tau} e^{-j\omega\varepsilon_i} \tag{4}$$

where

$$P(\omega) = n_s \tau_p e^{-j\omega\tau_p/2} \frac{\sin(\omega\tau_p/2)}{\omega\tau_p/2} \tag{5}$$

Since we are dealing with an integer number of frames over the observation interval, we can rewrite Eq. (2) as

$$S(\omega) = \lim_{N\to\infty} \frac{1}{(2N+1)\,T_f} E\left\{|S_N(\omega)|^2\right\} \tag{6}$$

where we have replaced the T subscript on $S(\omega)$ in Eq. (4) with N to reflect this fact. Taking the squared magnitude of Eq. (4) and then performing the statistical expectation gives

$$E\left\{|S_N(\omega)|^2\right\} = |P(\omega)|^2 \left\{ (2N+1)\overline{a^2} + \sum_{\substack{i=-N \\ i\neq k}}^{N} \sum_{k=-N}^{N} \overline{a_i}\ \overline{a_k} e^{-j\omega(i-k)N_f\tau} E\left\{e^{j\omega(l_i-l_k)\tau}\right\} E\left\{e^{-j\omega(\varepsilon_i-\varepsilon_k)}\right\} \right\} \tag{7}$$

However, since the modulation sequence $\{l_i\}$ and $\{\varepsilon_i\}$ are i.i.d., then, since $i \neq k$, the expectations in Eq. (7) partition into products, i.e.,

$$E\left\{e^{j\omega(l_i-l_k)\tau}\right\} = E\left\{e^{j\omega l_i\tau}\right\} E\left\{e^{-j\omega l_k\tau}\right\} \tag{8}$$

and

$$E\left\{e^{-j\omega(\varepsilon_i-\varepsilon_k)}\right\} = E\left\{e^{-j\omega\varepsilon_i}\right\} E\left\{e^{j\omega\varepsilon_k}\right\} \tag{9}$$

where

$$E\left\{e^{j\omega l_i\tau}\right\} = \frac{1}{M}\sum_{i=0}^{M-1} e^{j\omega i\tau} = \frac{1}{M}\frac{\sin(\omega M\tau/2)}{\omega M\tau/2} e^{j\omega(M-1)\tau/2} \tag{10}$$

and

$$E\left\{e^{-j\omega\varepsilon_i}\right\} = \frac{1}{2\tau_d}\int_{-\tau_d}^{\tau_d} e^{-j\omega\varepsilon_i} d\varepsilon_i = \frac{\sin(\omega\tau_d)}{\omega\tau_d} \tag{11}$$

Substituting Eqs. (5), (8), and (9) together with Eqs. (10) and (11) in Eq. (7) and the fact that $\overline{a_i} = 1$ for all i gives, after simplification,

$$E\left\{\left|S_N\left(\omega\right)\right|^2\right\} =$$

$$\left(n_s\tau_p\right)^2 \frac{\sin^2\left(\omega\tau_p/2\right)}{\left(\omega\tau_p/2\right)^2} \left\{(2N+1)\overline{a^2} + \frac{1}{M^2} \frac{\sin^2\left(\omega M\tau/2\right)}{\left(\omega M\tau/2\right)^2} \frac{\sin^2\left(\omega\tau_d\right)}{\left(\omega\tau_d\right)^2} \sum_{\substack{i=-N \\ i\neq k}}^{N} \sum_{k=-N}^{N} e^{-j\omega(i-k)N_f\tau}\right\} \quad (12)$$

Dividing by $(2N+1)\,T_f$ and taking the limit as $N\to\infty$, we get

$$S\left(\omega\right) =$$

$$\frac{\left(n_s\tau_p\right)^2}{\left(M+K\right)\tau} \frac{\sin^2\left(\omega\tau_p/2\right)}{\left(\omega\tau_p/2\right)^2} \left\{\overline{a^2} + \frac{1}{M^2} \frac{\sin^2\left(\omega M\tau/2\right)}{\left(\omega M\tau/2\right)^2} \frac{\sin^2\left(\omega\tau_d\right)}{\left(\omega\tau_d\right)^2} \lim_{N\to\infty} \frac{1}{2N+1} \sum_{\substack{i=-N \\ i\neq k}}^{N} \sum_{k=-N}^{N} e^{-j\omega(i-k)N_f\tau}\right\} \quad (13)$$

The double sum in Eq. (13) can be evaluated as the single sum

$$\frac{1}{2N+1} \sum_{\substack{i=-N \\ i\neq k}}^{N} \sum_{k=-N}^{N} e^{-j\omega(i-k)N_f\tau} =$$

$$\sum_{m=1}^{2N} \left(1 - \frac{m}{2N+1}\right) \cos\, m\omega N_f\tau = 2\sum_{m=-2N}^{2N} \left(1 - \frac{m}{2N+1}\right) e^{-jm\omega N_f\tau} - 1 \quad (14)$$

Taking the limit of Eq. (14) as $N\to\infty$,

$$\lim_{N\to\infty} 2\sum_{m=-2N}^{2N} \left(1 - \frac{m}{2N+1}\right) e^{-jm\omega N_f\tau} - 1 =$$

$$2\sum_{m=-\infty}^{\infty} e^{-jm\omega N_f\tau} - 1 = 2\left(\frac{2\pi}{N_f\tau}\right) \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N_f\tau}\right) - 1 \quad (15)$$

Finally, substituting Eq. (14) together with Eq. (15) in Eq. (13) gives the desired result in the form of a continuous plus a discrete spectrum, namely,

$$S\left(\omega\right) = S_c\left(\omega\right) + S_d\left(\omega\right) \quad (16)$$

where

$$S_c\left(\omega\right) = \frac{\left(n_s\tau_p\right)^2}{\left(M+K\right)\tau} \frac{\sin^2\left(\omega\tau_p/2\right)}{\left(\omega\tau_p/2\right)^2} \left\{\overline{a^2} - \frac{1}{M^2} \frac{\sin^2\left(\omega M\tau/2\right)}{\left(\omega M\tau/2\right)^2} \frac{\sin^2\left(\omega\tau_d\right)}{\left(\omega\tau_d\right)^2}\right\} \quad (17)$$

and

$$S_d\left(\omega\right) = \frac{\left(n_s \tau_p\right)^2}{\left(M+K\right)\tau}\ \frac{1}{M^2}\left(\frac{4\pi}{N_f \tau}\right)\sum_{k=-\infty}^{\infty}\frac{\sin^2\left(\omega\tau_p/2\right)}{\left(\omega\tau_p/2\right)^2}\ \frac{\sin^2\left(\omega M\tau/2\right)}{\left(\omega M\tau/2\right)^2}\ \frac{\sin^2\left(\omega\tau_d\right)}{\left(\omega\tau_d\right)^2}\delta\left(\omega-\frac{2\pi k}{N_f \tau}\right) \tag{18}$$

The discrete spectrum has components at frequencies corresponding to integer multiples of the frame time.

IV. Conclusion

The power spectrum derived above forms the basis for determining the output spectra of optical detectors responding to light pulses generated by a PPM-modulated Q-switched laser. For the case of nonmultiplying optical detectors observing high-intensity pulses, the spectrum defined in Eqs. (16) through (18) can be used directly, while for avalanche photodiode detectors (APDs) and photomultiplier tube (PMT) detectors, it provides a spectral description of the stochastic intensity that governs the detector's output process.