Performance of the Optimum Receiver for Pulse-Position Modulation Signals With Avalanche Photodiode Statistics

M. Srinivasan[1] and V. Vilnrotter[1]

Statistical modeling of the avalanche photodiode electron output is discussed. The exact McIntyre–Conradi density function for the output electron count is reviewed, along with the Webb and Gaussian approximations. In addition, the maximum-likelihood decision statistic for detection of pulse-position-modulation (PPM) signals is derived using the more accurate Webb density rather than the Poisson or Gaussian approximations that have been used previously. It is shown that for Webb-distributed output electrons, the maximum-likelihood rule is to choose the PPM word corresponding to the slot with the maximum electron count. Error probabilities for 256-ary PPM signals are calculated using the maximum-electron-count decision rule in the absence of thermal noise.

I. Introduction

The detection of weak optical signals is hampered by the presence of additive thermal noise, generated by resistors in the electrical circuit, in the measurement. This problem can be overcome by the use of photomultiplier tubes (PMTs) or avalanche photodiodes (APDs), both of which amplify the electrical current generated by absorbed photons. PMTs have very high gain and, hence, can be used to detect individual photons. However, the spectral response of these devices is limited to visible wavelengths. APDs provide less gain but remain useful over a wider range of wavelengths for overcoming thermal noise.

The output statistics of APDs in response to absorbed photons have been derived in two key articles by McIntyre and Conradi [1,2]. Although exact, APD density is difficult to compute and use due to its complex functional form. Therefore, a simpler approximate density that yields good results in communications and related applications was derived by Webb et al. [3]. Gaussian approximations to APD statistics also have been attempted for computing error probabilities but have not produced good results except in the case of high background noise levels [4].

This article is divided into two main sections. We first discuss the distribution of APD output electrons and approximations to the density function. We then examine the application of these statistical models to the problem of pulse-position modulation (PPM) detection. In particular, the maximum-likelihood decision rule for PPM signaling using the Webb approximation for the APD electron output is derived.

[1] Communications Systems and Research Section.

PPM word-error probabilities then are calculated for 256-ary PPM signaling using the Webb model in the absence of additive thermal noise. Error probabilities also are calculated using the exact McIntyre–Conradi model and the very simple Gaussian model. These results show that the Webb function provides an accurate model of APD output electrons that is relatively easy to use in error-probability computations.

II. APD Output Statistics

The average number of photons absorbed by an APD illuminated with optical intensity $P(t)$ in T seconds can be expressed as

$$\lambda = \frac{\eta}{h\nu} \int_0^T P(t) dt \tag{1}$$

where h is Planck's constant, ν is the optical frequency, and η is the detector's quantum efficiency, which is defined as the ratio of absorbed photons to incident photons. Here we assume that each *absorbed* photon gives rise to one photoelectron in the detector, i.e., λ may be interpreted as the mean number of primary photoelectrons released in the detector. The actual number of photons absorbed, n, is a Poisson distributed random variable with probability

$$p(n|\lambda) = \frac{\lambda^n}{n!} e^{-\lambda} \tag{2}$$

The conditional probability that an APD generates m electrons in response to exactly n absorbed photons, $m \geq n$, $n > 0$, has been shown by McIntyre to be [1]

$$p(m|n) = \frac{n\Gamma\left(\dfrac{m}{1-k} + 1\right)}{m(m-n)!\Gamma\left(\dfrac{km}{1-k} + n + 1\right)} \left[\frac{1 + k(G-1)}{G}\right]^{n+km/(1-k)} \left[\frac{(1-k)(G-1)}{G}\right]^{m-n} \tag{3}$$

where G is the average APD gain; k, the ionization ratio, is a property of the semiconductor material; and $0 < k < 1$.

If no photons are absorbed, then no electrons can be generated according to this model. Conversely, no electrons are generated only if no photons have been absorbed. Hence, from Eq. (2) we have

$$p(m = 0|\lambda) = p(n = 0|\lambda) = e^{-\lambda} \tag{4}$$

The conditioning on n can be removed by averaging, as in [2], which yields

$$p(m|\lambda) = \sum_{n=1}^{m} p(m|n) \frac{\lambda^n}{n!} e^{-\lambda}, \qquad m \geq 1 \tag{5}$$

Here, the summation limit is m instead of infinity because, according to the model, the number of absorbed photons, n, never can exceed the number of output electrons, m. The conditioning on λ in Eq. (5) allows modeling of the APD's response to stochastic optical fields.

An approximation to Eq. (5) that provides a much simpler expression for the density of m in response to the integrated optical intensity λ has been derived by Webb et al. in 1974 [3]:

$$p(m|\lambda) = \frac{1}{\sqrt{2\pi\lambda G^2 F}\Big(1 + \dfrac{m - G\lambda}{\lambda G F/(F-1)}\Big)^{3/2}} \exp\left(- \frac{(m - G\lambda)^2}{2\lambda G^2 F \Big(1 + \dfrac{m - G\lambda}{\lambda G F/(F-1)}\Big)} \right) \tag{6}$$

where $F = kG + (2 - 1/G)(1 - k)$.

The density in Eq. (6) is a continuous function that is defined for $m > -\lambda G/(F-1)$ and integrates to one in that range. The discrete probabilities in Eq. (5) are approximated by evaluating Eq. (6) at integer values of m. Note that Eq. (6) is defined for negative values of m, provided $m > -\lambda G/(F-1)$, even though negative m has no physical meaning. However, if the probability density is summed over the non-negative integers, a value less than one is obtained. This apparent contradiction can be reconciled by mapping probabilities corresponding to the negative integers in the interval $(-\lambda G/(F-1), 0]$ onto the origin, or by extending the domain of the function out to the negative values of m, an option that does not correspond to physical reality. Also note that the discrete version of the Webb density, consisting of Eq. (6) sampled at integer points, is not a true probability mass function since it does not sum to one and is only an approximation to Eq. (5). A true probability mass function could be obtained by assigning to each integer value of m the integral of Eq. (6) in the interval $(m - 1/2, m + 1/2)$. This is a more computationally intensive alternative, however, and therefore has not been investigated.

An even simpler Gaussian approximation for the APD output density has been suggested in the literature [4], but its range of applicability is limited to the case of high background noise levels. This approximation replaces the envelope of the Webb density with a Gaussian envelope with matched second-order statistics, i.e., mean λG and variance $\lambda G^2 F$.

Using a value of $k = 0.028$, as in [2], the exact density values from Eq. (5) and the approximate density values from Eq. (6) were calculated using average APD gain values of 30, 50, and 100, and assuming an average absorbed photon count of $\lambda = 20$. The results are shown in Fig. 1. The agreement between the Webb approximation and the exact distribution in the region near the peak is evident in this figure. Figures 2 and 3 correspond to low photon absorption cases, namely $\lambda = 0.1$ and $\lambda = 1.0$, respectively, in which case the peak is reached at the minimum value of m shown in the figures, $m = 1$. Figure 2(a) shows the divergence between the exact expressions and the Webb approximations for both small and large m, whereas Fig. 2(b) expands the scales for the high-probability, low electron-count region. For the range of APD gains considered, the intermediate-count intensity region of $\lambda = 1$ seems to provide the best agreement between the exact and approximate expressions for the low electron-count region, as can be seen in Figs. 3(a) and 3(b). However, for the case of $m > 100$, the probabilities assigned to electron counts begin to diverge once again, as can be seen in Fig. 3(a). Although not shown in these figures, the probability at $m = 0$ is given by Eq. (4) for the McIntyre–Conradi density, which is defined only for non-negative m, whereas the Webb approximation extends to negative values of m as well.

III. Applications to PPM Detection

The detection of information-bearing laser pulses by means of APD devices is a topic of interest in the design of optical communications systems. Although several key articles have been published that derive the output statistics of APDs, provide accurate approximations to these statistics, and discuss applications to problems in optical detection (e.g., [4]), the authors have not found a derivation of the optimal receiver structure for pulse-position modulation (PPM) detection with valid APD statistics. Most studies assume an overly simplistic Poisson or Gaussian model in the derivation of optimal receiver

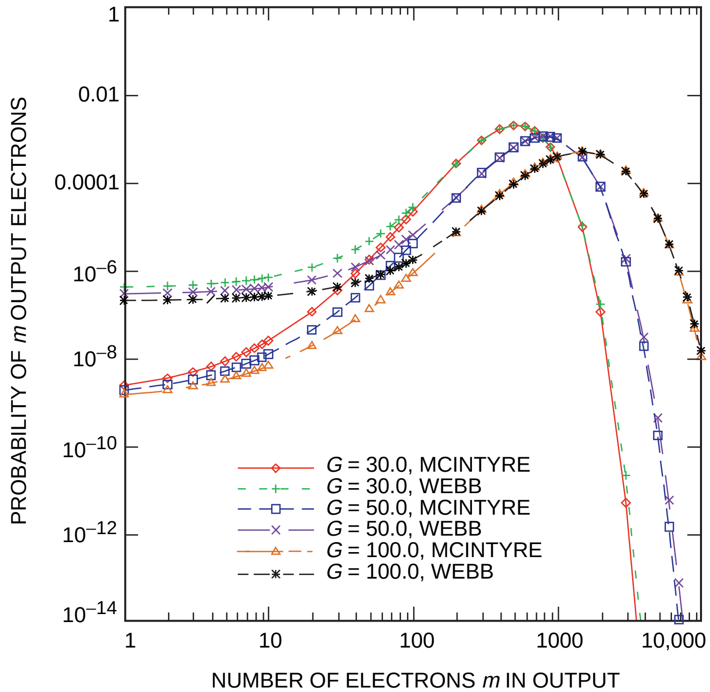

Fig. 1. Exact (McIntyre) and approximate (Webb) APD electron output probability density functions for λ = 20.0 and k = 0.028.

structure. For these simple models, one easily can show that the maximum-likelihood detection rule is to pick the PPM word that corresponds to the slot with the maximum electron count. Here we provide a derivation of the structure of the optimum (maximum-likelihood or maximum a posteriori) receiver for PPM pulses using the Webb density to model avalanche electrons generated by an APD in response to absorbed signal and background photons. This modeling and derivation does not take into account additive thermal noise generated by the follow-on electronics.

A. The Log-Likelihood Function

In the optical communication system that we are considering, the transmitter sends one of Q PPM symbols in the form of a laser pulse located in one of Q time slots, where each slot is T_s seconds long. The output of the APD detector is integrated synchronously within each slot to form a vector of Q observables, $\mathbf{m}$, on which to base the PPM decision. Only one of the slots contains the signal, and it is assumed that all slots are equally likely to contain the signal. Since the slots are disjointed in time, the components of the observable vector are independent. The probability density of each component depends upon the presence of background and signal photon intensities. Let the average number of absorbed photons over time T_s due to the background be denoted by λ_b, and let the average number of absorbed photons over time T_s due to presence of the signal be denoted by λ_s. Let m be the number of output APD electrons in a particular time slot, and let $x = m - \lambda_b G$, where G is the gain of the APD device. We shall now deal with the vector of observables, $\mathbf{x}$. According to the Webb approximation, the conditional density of x given the presence of background photons only is

$$p(x|\lambda_b) = \frac{1}{\sqrt{2\pi\sigma_b^2}(1+x/\beta_b)^{3/2}} \exp\left(-\frac{x^2}{2\sigma_b^2(1+x/\beta_b)}\right) \tag{7}$$

for $x \geq -\lambda_b G$, and the conditional density of x given the presence of signal and background photons is

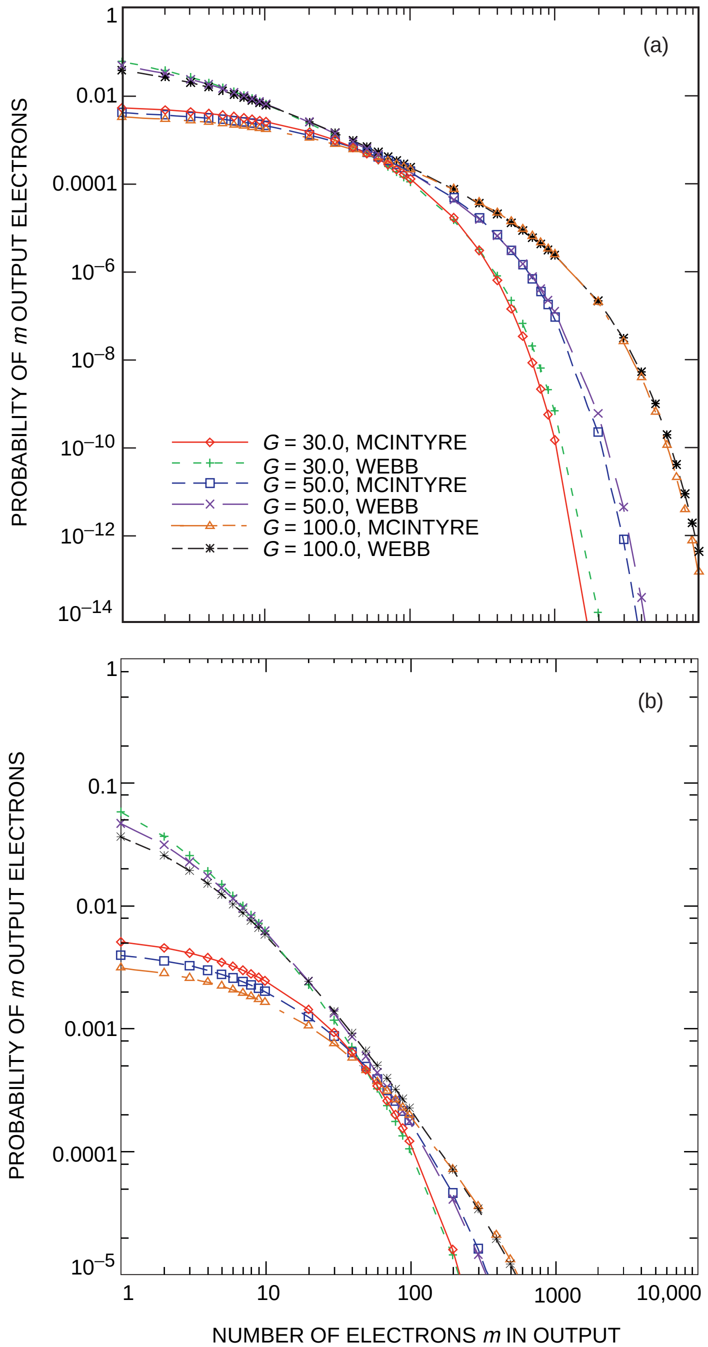

Fig. 2. Exact (McIntyre) and approximate (Webb) APD electron output probability density functions for: (a) λ = 0.1 and k = 0.028 and (b) a close-up of Fig. 2(a) for a small number of electrons.

$$p(x|\lambda_b + \lambda_s) = \frac{1}{\sqrt{2\pi\sigma_{sb}^2}(1 + (x - \lambda_s G)/\beta_{sb})^{3/2}} \exp\left(-\frac{(x - \lambda_s G)^2}{2\sigma_{sb}^2(1 + (x - \lambda_s G)/\beta_{sb})}\right) \tag{8}$$

for $x \geq -\lambda_b G$, where $\sigma_b = \sqrt{\lambda_b G^2 F}$, $\sigma_{sb} = \sqrt{(\lambda_b + \lambda_s)G^2 F}$, $\beta_b = (\lambda_b GF)/(F-1)$, and $\beta_{sb} = [(\lambda_b + \lambda_s)GF]/(F-1)$.

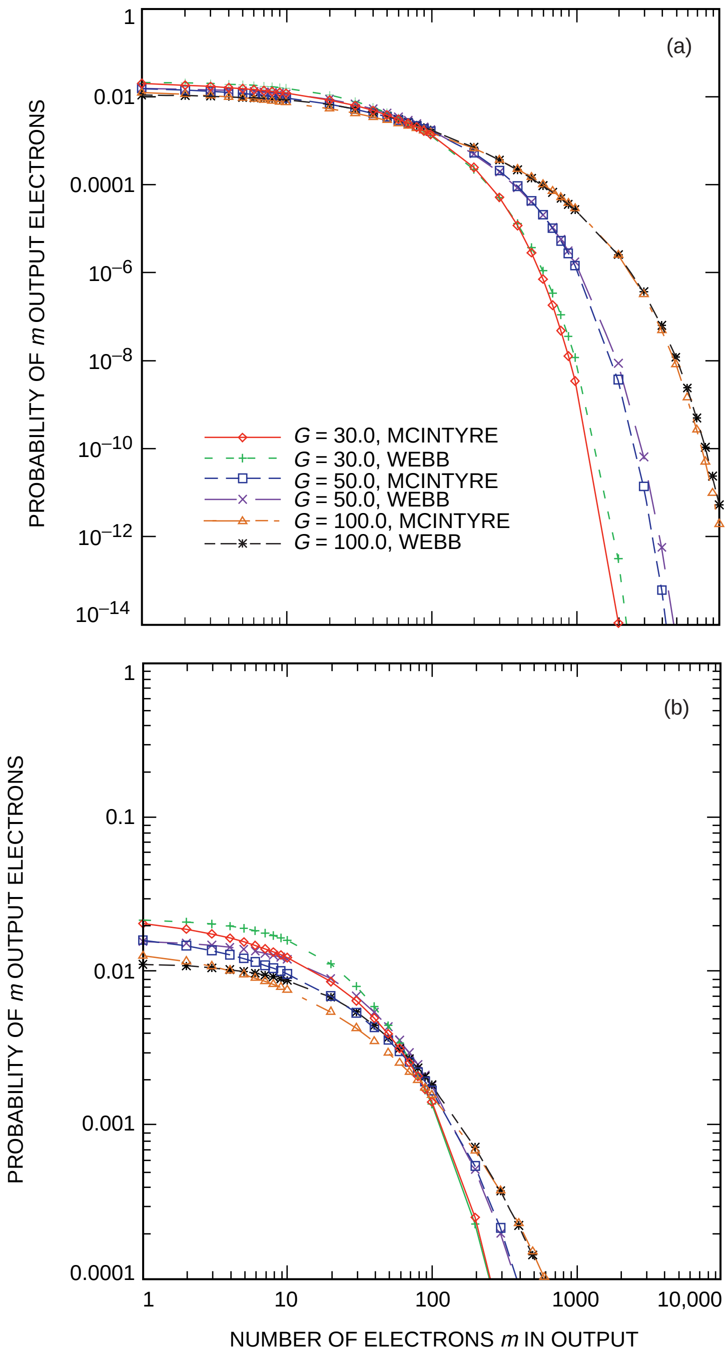

Fig. 3. Exact (McIntyre) and approximate (Webb) APD electron output probability density functions for: (a) λ = 1.0 and *k* = 0.028 and (b) a close-up of Fig. 3(a) for a small number of electrons.

For maximum-likelihood detection, the log-likelihood function corresponding to each hypothesis is computed, and the hypothesis corresponding to the largest value of the log-likelihood function is selected. Note that under equally likely hypotheses, the maximum-likelihood decision also is the maximum a posteriori decision. Suppose hypothesis q is true; that is, a PPM word with a signal pulse in the qth slot is received, and denote this event as H_q. The log-likelihood function given H_q can be expressed as

$$\ln \Lambda(\mathbf{x}|H_q) = \ln p(x_q|\lambda_b + \lambda_s) + \sum_{\substack{i=1 \\ i \neq q}}^{Q} \ln p(x_i|\lambda_b)$$

$$= \ln p(x_q|\lambda_b + \lambda_s) - \ln p(x_q|\lambda_b) + \sum_{i=1}^{Q} \ln p(x_i|\lambda_b) \tag{9}$$

Note that $x_i = m_i - \lambda_b G$, where m_i is the output electron count in slot i. Substituting Eqs. (7) and (8) into Eq. (9), we obtain the following:

$$\ln \Lambda(\mathbf{x}|H_q) = \Psi(x_q) - \ln(\sigma_{sb}/\sigma_b) + \sum_{i=1}^{Q} \ln p(x_i|\lambda_b) \tag{10}$$

where

$$\Psi(x_q) = \frac{x_q^2}{2\sigma_b^2(1 + x_q/\beta_b)} - \frac{(x_q - \lambda_s G)^2}{2\sigma_{sb}^2(1 + (x_q - \lambda_s G)/\beta_{sb})} + \frac{3}{2} \ln \left(\frac{1 + x_q/\beta_b}{1 + (x_q - \lambda_s G)/\beta_{sb}} \right) \tag{11}$$

Note that the last two terms on the right-hand side of Eq. (10) do not depend on q and, hence, cannot contribute any information to the final decision. These terms can, therefore, be ignored, and the decision be based entirely upon the sufficient part of the log-likelihood function, $\Psi(x_q)$, as defined in Eq. (11).

B. Monotonicity of $\Psi(x_q)$

In order to decide which PPM word was received, the decision function $\Psi(x_q)$ is calculated for each time slot q, and the PPM word corresponding to the largest $\Psi(x_q)$, $1 \leq q \leq Q$, is selected. However, $\Psi(x)$ is a somewhat complicated function of its argument that must be computed without creating a bottleneck in a high data rate system—a requirement that may exceed the real-time capabilities of the digital signal-processing assembly. Hence, a simpler decision algorithm would be of value. If it can be shown that the decision function defined in Eq. (11) is a monotonically increasing function of its argument, then the PPM decisions could be based entirely upon the statistics x_i and, hence, on the APD electron counts m_i, avoiding the need for more complicated, time-consuming calculations.

Since $\Psi(x)$ is continuous over its domain, we can show that it is monotonically increasing by showing that its derivative is positive over its domain. After much simplification, the derivative can be expressed as

$$\frac{\partial}{\partial x}\Psi(x) = \frac{3\alpha^3 G((G+\alpha)\lambda_b + \alpha x) - G^2(G+\alpha)^2\lambda_b^2\alpha}{(G(G+\alpha)\lambda_b + \alpha x)^2}$$

$$+ \frac{G^2(G+\alpha)^2(\lambda_b+\lambda_s)^2\alpha - 3\alpha^3(G(G+\alpha)\lambda_b + G^2\lambda_s + \alpha x)}{(G(G+\alpha)\lambda_b + G^2\lambda_s + \alpha x)^2} \tag{12}$$

where $\alpha = (G-1)(1+(G-1)k) > 0$. By cross-multiplying, Eq. (12) may be written as

$$\frac{\gamma^2(\gamma+G^2\lambda_s)^2}{\alpha}\frac{\partial}{\partial x}\Psi(x) = 3\alpha^2\gamma(\gamma+G^2\lambda_s)^2 - G^2(G+\alpha)^2(\lambda_b)^2(\gamma+G^2\lambda_s)^2$$
$$+ G^2(G+\alpha)^2(\lambda_b+\lambda_s)^2\gamma^2 - 3\alpha^2(\gamma+G^2\lambda_s)\gamma^2 \tag{13}$$

where $\gamma = G(G+\alpha)\lambda_b + \alpha x > 0$. Since $\alpha > 0$, we need show only that the right-hand side of Eq. (13) is positive. This quantity may be written as

$$RHS = 3\alpha^2\gamma G^2\lambda_s(\gamma+G^2\lambda_s) + 2G^3(G+\alpha)^2\lambda_b^2\lambda_s\alpha\gamma + 2G^2(G+\alpha)^2\lambda_b\lambda_s\alpha^2x^2$$
$$+ G^4(G+\alpha)^2\lambda_b^2\lambda_s^2(2G\alpha+\alpha^2) + G^2(G+\alpha^2)^2\lambda_s^2\alpha x(2G(G+\alpha)\lambda_b+\alpha x) \tag{14}$$

Substituting $x = m - G\lambda_b$ into Eq. (14) yields

$$RHS = 3\alpha^2G^2\lambda_s(G^2\lambda_b+\alpha m)(G^2(\lambda_b+\lambda_s)+\alpha m)$$
$$+ 2\alpha G^3(G+\alpha)^2(G^2\lambda_b+\alpha m)\lambda_b^2\lambda_s + 2\alpha^2G^2(G+\alpha)^2\lambda_b\lambda_s(m-G\lambda_b)^2$$
$$+ \alpha G^2(G+\alpha)^2(2G^2\lambda_b+\alpha m)\lambda_s^2 \tag{15}$$

For $m \geq 0$ (or $x \geq -G\lambda_bT_S$), the right-hand side of Eq. (15) is positive. Therefore, $\Psi(x)$ is monotonically increasing, and it suffices to base the PPM word decision upon its argument x, or equivalently, upon m, the number of electrons generated by the APD. Thus, the maximum-likelihood receiver selects the PPM word corresponding to the time slot containing the greatest APD electron count.

C. Numerical Results

Using the continuous version of the Webb density, it was shown above that the optimum receiver counts the total number of electrons in each slot and makes a decision by selecting the PPM word corresponding to the slot with the greatest electron count. The PPM word is decoded correctly if the electron count in the time slot containing the signal exceeds all others by at least one. However, correct detection also could occur if several, perhaps all, slots contained the same maximum number and an equiprobable random selection was made among the maximal slots. For example, if no electrons were observed in any slot, pure guessing among the Q slots would lead to a correct decision $1/Q$ times on the average. Similarly, if there were two equal maximum counts out of Q, one of them corresponding to the signal slot, the tossing of a fair coin would result in a correct decision half the time. Considering all possible ways that equalities can occur, and accounting for the probabilities of correct resolution in each case, the probability of correct decision can be expressed as

$$P_c = \frac{1}{Q}e^{-(Q\lambda_b+\lambda_s)} + \sum_{r=0}^{Q-1}\sum_{m=1}^{\infty} p(m|\lambda_b+\lambda_s)\left[\sum_{i=0}^{m-1} p(i|\lambda_b)\right]^{Q-1-r} p(m|\lambda_b)^r \frac{(Q-1)!}{r!(Q-1-r)!(r+1)} \tag{16}$$

The corresponding error probability is simply $P_e = 1 - P_c$. The conditional densities in Eq. (16) should be McIntyre–Conradi for an exact solution but could be replaced by the Webb density for an approximate solution.

Although exact, calculations using Eq. (16) may require too much computing time. For this reason, a bound may be used in which all of the equal-count events are ignored in the probability calculation, except for the event in which all slots have zero electron counts. Since every term in Eq. (16) is positive, this tends to underestimate the probability of correct detection somewhat but results in the following expression, which reduces the computational burden:

$$P_c \geq \frac{1}{Q} e^{-(Q\lambda_b + \lambda_s)} + \sum_{m=1}^{\infty} p(m|\lambda_b + \lambda_s) \left[\sum_{i=0}^{m-1} p(i|\lambda_b) \right]^{Q-1} \tag{17}$$

The probability of PPM word error has been computed using McIntyre–Conradi, Webb, and Gaussian densities in Eq. (17) for an APD with parameters $k = 0.007$ and $G = 50.0$. In the computations with the Webb and Gaussian approximations, the probability mass below zero was mapped to $m = 0$, while the values for $m > 0$ were taken by sampling the continuous Webb and Gaussian density functions at the integers. The results of the calculations are shown in Fig. 4 as a function of the average absorbed signal photon count, λ_s, with average absorbed background photon counts, λ_b, of 1, 5, and 10 photons per slot. Note that for $\lambda_s < 50$, the error probabilities obtained with the Webb and McIntyre–Conradi densities are in good agreement, but the results of the Gaussian approximation differ substantially from both. In Fig. 4, the Gaussian approximation underbounds the performance of the Webb and McIntyre models for lower background photon counts but appears to cross over and overbound the Webb and McIntyre curves for higher photon counts. In [4], it was shown that the Gaussian approximation approaches the performance of the Webb approximation under high background noise conditions but overbounds the word-error probability when the background noise count decreases. However, note that the results in [4] were based upon the inclusion of Gaussian thermal noise as well as APD output statistics, whereas the results here concentrate solely on modeling of the APD output electron count.

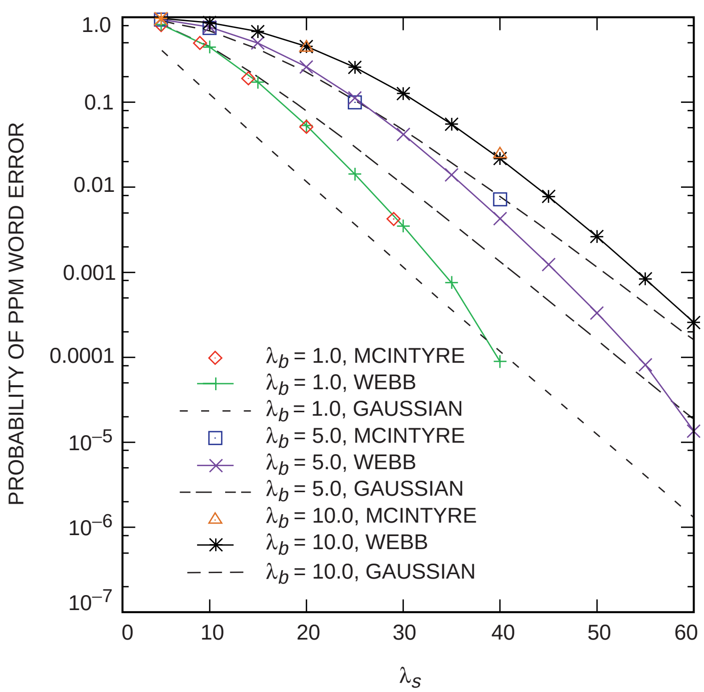

Fig. 4. Word error probability for 256-ary PPM calculated using a McIntyre density function and Webb and Gaussian approximations, with *G* = 50.0 and *k* = 0.007.

IV. Conclusion

In this article, we discussed the distribution of APD output electrons and approximations to the output density function in the absence of thermal noise. It was shown that when using the Webb function to approximate the APD output density, the maximum-likelihood decision for PPM signals is to pick the PPM word corresponding to the slot with the maximum electron count. PPM word-error probabilities then were calculated using this decision rule and both the Webb and the Gaussian approximations for APD output electrons. From these results, it was concluded that the Webb function provides a very good approximation to the true APD output statistics when used to calculate PPM word-error probabilities in the absence of thermal noise and that the Gaussian approximation does not perform as well in this context. Future work will include similar performance analysis when Gaussian thermal noise is added to the APD ouptut current.

References

[1] R. J. McIntyre, "The Distribution of Gains in Uniformly Multiplying Avalanche Photodiodes: Theory," *IEEE Transactions on Electron Devices*, vol. ED-19, no. 6, pp. 703–713, June 1972.

[2] J. Conradi, "The Distribution of Gains in Uniformly Multiplying Avalanche Photodiodes: Experimental," *IEEE Transactions on Electron Devices*, vol. ED-19, no. 6, pp. 713–718, June 1972.

[3] P. P. Webb, R. J. McIntyre, and J. Conradi, "Properties of Avalanche Photodiodes," *RCA Review*, vol. 35, pp. 234–278, June 1974.

[4] F. M. Davidson and X. Sun, "Gaussian Approximation Versus Nearly Exact Performance Analysis of Optical Communications Systems With PPM Signaling and APD Receivers," *IEEE Transactions on Communications*, vol. 36, no. 11, pp. 1185–1192, November 1988.