Voltage Signal-to-Noise Ratio (SNR)  
Nonlinearity Resulting From  
Incoherent Summations  

S. T. Lowe

Simple scaling rules often are used to estimate signal-to-noise ratios (SNRs). For example, one commonly hears that voltage SNR varies as the square root of the number of data samples and linearly as the signal strength. For a variety of reasons, signals often are detected or measured by cross-correlating a small interval of sampled voltage data with a phase model and coherently integrating, followed by incoherently summing the resulting amplitudes over a number of such intervals. When this is done, the usual scaling rules do not always apply and can lead to decidedly incorrect conclusions (an example is given in Section V). This article derives analytic formulas for voltage SNR and some resulting scaling laws when incoherent amplitude sums are performed. In a common low-SNR situation, the correct rule states that voltage SNR varies as the square of the signal strength.

I. Background

Detecting a weak sinusoidal signal in a white noise environment usually involves cross-correlating sampled voltage data, $D(t)$, with a complex phase model, $\phi_m(t)$, and integrating over time:

$$\bar{\rho} = \int_0^T D(t) e^{-i\phi_m(t)} \, dt$$  

(1)

The resulting complex phasor, $\bar{\rho}$, will obtain its maximum amplitude when the phase model is accurate enough to approximately stop the cross-correlated signal’s phase variations over the integration interval, $T$. The phase model usually is parameterized, and finding parameter values that produce a significantly large signal-to-noise ratio (SNR) implies the existence of a signal and approximately determines its phase characteristics.

The simplest and most common phase model is linear, corresponding to a monochromatic tone:

$$\phi_m(t) = \phi_0 + \nu t$$  

(2)

1 Tracking Systems and Applications Section.
Here, there are two model parameters, \( \nu \) and \( \phi_0 \), which represent the frequency and initial phase; however, the integrated phasor’s amplitude is insensitive to \( \phi_0 \), making this effectively a one-parameter model when the goal is signal detection.

II. Incoherent Sums

The phase model given by Eq. (2) seldom is accurate enough to apply over long integration intervals because \( \nu \) is often time dependent. The model can be improved by creating additional parameters, but this requires handling higher dimensional parameter spaces, resulting in a much greater number of trial models. This both raises the threshold for false signal detections and increases the computational requirements. When large data sets and weak signals are involved, this option may not be feasible. A common approach is to apply Eq. (2) over short enough time intervals to ensure its accuracy and to sum the resulting integrated amplitudes over all intervals. This incoherent sum results in a lower overall SNR than does applying an accurate model over the entire interval and summing coherently, and it is this effect on the overall SNR that now is examined in detail.

Assume a sinusoidal signal is added to complex white Gaussian noise, where the real and imaginary components correspond to the in-phase and quadrature-phase noise components, respectively, and each has zero mean and variance \( \sigma^2 \). If the phase model exactly stops the signal’s phase variations, so that the signal voltage is a constant, \( v \), the resulting integrated complex voltage has amplitude \( V \). The probability density function (PDF) of \( V \) is Ricean\(^2\) [1], where the PDF, mean, and variance are given by

\[
PDF(V) = \frac{V}{\sigma^2} e^{-(V^2 + v^2)/2\sigma^2} I_0 \left( \frac{vV}{\sigma^2} \right), \quad 0 \leq V < \infty \tag{3a}
\]

\[
\langle V \rangle = \sqrt{\frac{\pi}{2}} \sigma e^{-v^2/4\sigma^2} \left[ \left( 1 + \frac{v^2}{2\sigma^2} \right) I_0 \left( \frac{v^2}{4\sigma^2} \right) + \frac{v^2}{2\sigma^2} I_1 \left( \frac{v^2}{4\sigma^2} \right) \right] \tag{3b}
\]

\[
\sigma_V^2 \equiv \langle V^2 \rangle - \langle V \rangle^2 = v^2 + 2\sigma^2 - \langle V \rangle^2 \tag{3c}
\]

and \( I_0 \) and \( I_1 \) are modified Bessel functions of the first kind. Assume a data set having \( t \) samples divided into \( k \) pieces, each having \( n \) samples (so \( kn = t \)), a sinusoidal signal with peak voltage \( R \), and single-sample variance \( \sigma_1^2 \). The quantities \( \langle V \rangle \) and \( \sigma_V^2 \) for each \( n \)-sample piece are calculated using Eq. (3) with \( v = nR \) and \( \sigma = \sqrt{n}\sigma_1 \). This assumes the \( n \)-sample sum is completely coherent. Assuming errors on each \( n \)-sample piece are uncorrelated, these quantities are multiplied by \( k \) for the incoherent sum over all \( k \) pieces, leading to the overall amplitude, \( A \), and amplitude variance, \( \sigma_A^2 \), for the entire \( t \)-sample data set:

\[
\langle A \rangle = k \sqrt{\frac{\pi}{2}} \frac{v}{\sqrt{n}} e^{-(nS^2/4)} \left[ \left( 1 + \frac{nS^2}{2} \right) I_0 \left( \frac{nS^2}{4} \right) + \frac{nS^2}{2} I_1 \left( \frac{nS^2}{4} \right) \right] \tag{4a}
\]

\[
\sigma_A^2 = k (n^2 S^2 + 2n - \langle A \rangle_k^2) \tag{4b}
\]

where \( S \equiv R/\sigma_1 \) can be thought of as the single-sample SNR. Removing the no-signal bias \( \langle A \rangle_{S=0} \) and normalizing by \( \sigma_{A|S=0} \) so that the noise has zero mean and unit variance defines the voltage SNR:

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\(^2\)If you are not familiar with the Ricean distribution, it is easy to conceptualize. Imagine a two-dimensional Gaussian distribution centered \( v \) units from the origin and having both \( x \) and \( y \) variances equal to \( \sigma^2 \). Equation (3a) gives the resulting distribution of distances \( V \) from the origin.
$$\text{SNR}_V \equiv \frac{\langle A \rangle - \langle A \rangle_{S=0}}{\sigma_{A|S=0}}$$  \hspace{1cm} (5a)$$

$$= \sqrt{k} \sqrt{\frac{\pi}{4 - \pi}} \left\{ e^{-(\epsilon^2/4)} \left[ \left( 1 + \frac{\epsilon^2}{2} \right) I_0 \left( \frac{\epsilon^2}{4} \right) + \frac{\epsilon^2}{2} I_1 \left( \frac{\epsilon^2}{4} \right) \right] - 1 \right\}$$  \hspace{1cm} (5b)$$

where \( \epsilon \equiv \sqrt{n} S \) can be thought of as the \( n \)-sample SNR. Figure 1 shows \( \text{SNR}_V \) as a function of \( \epsilon \) along with polynomial expansions for small and large \( \epsilon \). Combining these expansions at the crossover point, \( \epsilon = 1.6755 \), leads to an approximate expression for \( \text{SNR}_V \) that is good to better than 1 percent for all \( \epsilon \):

$$\text{SNR}_V \simeq \sqrt{k} \sqrt{\frac{\pi}{4 - \pi}} \left( \frac{\epsilon^2}{4} - \frac{\epsilon^4}{64} + \frac{\epsilon^6}{1024} \right) \quad \epsilon \leq 1.6755$$  \hspace{1cm} (6a)$$

$$\simeq \sqrt{k} \sqrt{\frac{2}{4 - \pi}} \left( \epsilon - \sqrt{\frac{\pi}{2}} + \frac{1}{2\epsilon} + \frac{1}{8\epsilon^3} + \frac{3}{16\epsilon^5} \right) \quad \epsilon > 1.6755$$  \hspace{1cm} (6b)$$

III. Scaling Laws

First, note that the definition of \( \text{SNR}_V \) includes an arbitrary scale factor. Above, \( S \) was defined as the peak voltage of a sinusoidal signal, but another definition for \( S \), for example the signal rms, would change the constant near the front of Eq. (5b). Thus, only relative scaling behavior is meaningful. The scaling laws for \( \text{SNR}_V \), as defined above, fall into two regimes: \( \epsilon \lesssim 1 \) and \( \epsilon \gtrsim 4 \), where, again, \( \epsilon \) is the \( \text{SNR}_V \) from a single \( n \)-sample piece of data.

The regime \( \epsilon \lesssim 1 \) is relevant for signal detection: The signal is too weak to detect in any one \( n \)-sample piece, but one hopes to detect it by incoherently summing \( k \) such pieces. Looking at the lowest order term in Eq. (6a),

$$\text{SNR}_V \simeq \frac{1}{4} \sqrt{\frac{\pi}{4 - \pi}} \sqrt{k} \epsilon^2 \quad \epsilon \ll 1$$  \hspace{1cm} (7a)$$

$$= \frac{1}{4} \sqrt{\frac{\pi}{4 - \pi}} \frac{t S^2}{\sqrt{k}}$$  \hspace{1cm} (7b)$$

This shows that the overall \( \text{SNR}_V \) scales as \( S^2 \) in this regime, and not as \( S \), which indicates that efforts to increase \( S \) are much more effective in increasing \( \text{SNR}_V \) here than in the strong-signal regime, where \( \text{SNR}_V \) scales linearly with \( S \). Also, for a fixed number of pieces, \( k \), \( \text{SNR}_V \) scales linearly with the amount of data, \( t \), and not by the usual \( \sqrt{t} \). Finally, for a given data set (fixed \( t \) and \( S \)), the observed \( \text{SNR}_V \) in this regime scales as \( \sqrt{n} \). In other words, \( n \) should be maximized for signal detection.

When \( \epsilon \gtrsim 4 \), the signal is detectable in a single \( n \)-sample piece, so the problems inherent in weak signals are not found here. The SNR is given approximately by

$$\text{SNR}_V \simeq \sqrt{\frac{2}{4 - \pi}} \sqrt{t} S \quad \epsilon \gg 1$$  \hspace{1cm} (8)$$
The SNR\(_V\) loss, \(L\), associated with breaking the data into \(k\) pieces is defined by the relation

\[
\text{SNR}_V \equiv \text{SNR}_V \bigg|_{k=1} (1 - L) \tag{9a}
\]

Equation (6b) is used to determine \(L\):

\[
L \approx \sqrt{\frac{\pi}{2} \sqrt{k - 1}} \quad \epsilon \gg 1 \tag{9b}
\]

This shows the strong-signal SNR\(_V\) loss associated with splitting a given data set into \(k\) coherent pieces scales as \(\sqrt{k} - 1\) and inversely as the unsplit SNR.

**IV. Numerical Verification**

Equations (5) and (6) were verified by numerical simulation. A signal with \(S = 0.05\) was added to \(2^{20} = 1,048,576\) samples of normalized white noise, coherently integrated over all \(n\)-sample data pieces, and incoherently summed over all \(2^{20}/n\) pieces, where \(n\) was all possible powers of 2. This was repeated 1000 times and the results averaged. Table 1 shows the resulting SNR\(_V\) along with the analytic values. Notice the big drop in SNR\(_V\) around \(\epsilon = 4\), showing its two scaling regimes, \(\epsilon \lesssim 1\) and \(\epsilon \gtrsim 4\). Figure 2 shows the SNR\(_V\) simulation values, the analytic prediction from Eq. (5b), and the asymptotic approximations as a function of \(k\). Figure 3 shows these same quantities as a function of \(\epsilon\).

<table>
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<th>(n)</th>
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<th>(\epsilon)</th>
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Finally, to see $\text{SNR}_V$ as a function of $S$, a second simulation was performed with $n = 2048$, $k = 16,384$, and $S$ varying between 0.001 and 1. The results are plotted in Fig. 4 and clearly show the two scaling regimes. The long-dashed line shows the “usual” $\text{SNR}_V$ scaling rule.

V. Conclusions

It has been shown that $\text{SNR}_V$ derived from incoherent voltage sums cannot always be scaled using the “usual” scaling rules. An example makes this clear. Assume a time series of $33,554 \times 10^6$ sampled voltages is analyzed by continuously dividing the data into 2048-sample subsets and performing a fast Fourier transform (FFT) of each subset. Also assume the coherence time of any signal is approximately equal to the time interval of a 2048-sample subset. The amplitudes of the resulting 16,384 FFTs are summed, and a signal having an $\text{SNR}_V$ of 750 is seen. A second signal, known to have a voltage 50 times smaller than the observed signal could be present. Can this second signal be detected? The usual $\text{SNR}$ scaling would indicate the second signal will have $\text{SNR}_V \approx 750/50 \approx 15$ and can be observed. However,
the correct answer is that it cannot be observed in this manner. Figure 4 shows the curve for $n = 2048$ and $k = 16,384$, indicating this signal has $S = 0.1$. Moving down by a factor of 50 to $S = 0.002$ shows the expected SNR$_V$ is about 0.5, making the signal unobservable. This example shows that Eq. (5) should be used when scaling SNR$_V$s that are derived from incoherent summations. Note that the signal could be detected if longer subsets were used and the signal were coherent over this interval.

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Reference