Performance of Coherent Binary Phase-Shift Keying (BPSK) with Costas-Loop Tracking in the Presence of Interference

M. K. Simon[1]

The bit-error probability performance of coherent binary phase-shift keying (BPSK) in the presence of narrowband (tone) and wideband (modulated tone) is investigated. The impact of the interference on both the carrier-tracking loop (assumed to be a conventional Costas loop) and the data detection are considered both individually and in combination. It is shown that, for loop parameters of practical interest, the dominant effect is the degradation induced on the data-detection process, which, depending on the relative frequency offset between the interferer and the desired signal as well as their relative power ratio, can be quite significant.

I. Introduction

The tracking performance of a Costas-type loop and its impact on the data detection of digital modulations in a purely additive white Gaussian noise (AWGN) environment are well documented in the literature [1–4]. When, in addition to the AWGN, co-channel interference [e.g., narrowband (unmodulated tone) or wideband (modulated tone)] is present, then additional degradation takes place both in the tracking performance of the loop and in the data-detection process itself. The severity of this degradation depends to a large extent on the strength (power) of the interferer as well as the spectral location (carrier frequency) of the interfering signal relative to that of the desired signal. Also important is the relative phase between the desired and interfering signals, which, in the absence of any side information, must be assumed unknown and, thus, can be modeled as being random with a uniform distribution. With these parameters in mind, we analyze in this article both the individual tracking and data-detection performances for binary phase-shift-keying (BPSK), including in addition the impact of the former on the latter. As is traditional, for the tracking performance of the loop, we describe performance in terms of the mean-square phase jitter whereas, for data detection, performance is measured by bit-error probability (BEP). The results will be expressed as functions of the ratio of interference to desired signal power and the normalized (by the bit rate) frequency separation between the interferer and desired signal carrier frequencies.

The article will be structured into two major sections corresponding to the narrowband and wideband interferer cases. Each of these two sections will be divided into two subsections corresponding to the tracking performance of the loop and the data-detection performance of the matched-filter receiver, first assuming perfect tracking and then combined with the actual tracking loop itself. The tracking loop model

[1] Communications Systems and Research Section.

used throughout will be a so-called I-Q Costas loop, which refers to a Costas loop with integrate-and-dump (I&D) arm filters. To avoid additional complication, we shall assume perfect bit synchronization of these arm filter I&Ds, one of which also serves as the matched filter for data detection.

II. Performance in the Presence of Narrowband Interference

A. Tracking Performance of the Costas Loop

Consider the BPSK receiver illustrated in Fig. 1, where the demodulation reference signal is provided by an I-Q Costas loop as shown.[2] Input to this receiver is the sum of a desired signal, $s_s(t)$, and a narrowband (tone) interference signal, $s_I(t)$, which are mathematically modeled as

$$\left.\begin{aligned} s_s(t) &= \sqrt{2P_s}m(t)\sin(\omega_c t+\theta_s) \\ s_I(t) &= \sqrt{2P_I}\sin((\omega_c+\Delta\omega)t+\theta_I) \end{aligned}\right\} \tag{1}$$

where P_s, ω_c, θ_s and $P_I, \omega_c+\Delta\omega, \theta_I$ are, respectively, the power, radian carrier frequency, and phase of the desired and interference signals, and

$$m(t) = \sum_{n=-\infty}^{\infty} a_n p(t-nT_b) \tag{2}$$

is the binary data modulation with $\{a_n\}$ an independent and identically distributed (i.i.d.) sequence taking on equiprobable values ± 1, and $p(t)$ is a unit amplitude rectangular pulse of duration equal to the bit time, T_b. Adding to $s_s(t)$ and $s_I(t)$ is the AWGN noise:

$$n(t) = \sqrt{2}\left[N_c(t)\cos(\omega_c t+\theta_s) - N_s(t)\sin(\omega_c t+\theta_s)\right] \tag{3}$$

where $N_c(t)$ and $N_s(t)$ are in-phase (I) and quadrature-phase (Q) low-pass noise components that are independent and have single-sided power spectral density (PSD) N_0 W/Hz. As such, the total received signal is then

$$r(t) = s_s(t) + s_I(t) + n(t) \tag{4}$$

Demodulating $r(t)$ with the I and Q reference signals derived by the loop produces the baseband signals:

[2] In this subsection, we shall first consider the ideal case, wherein the demodulation reference for data-detection purposes is assumed to be perfect.

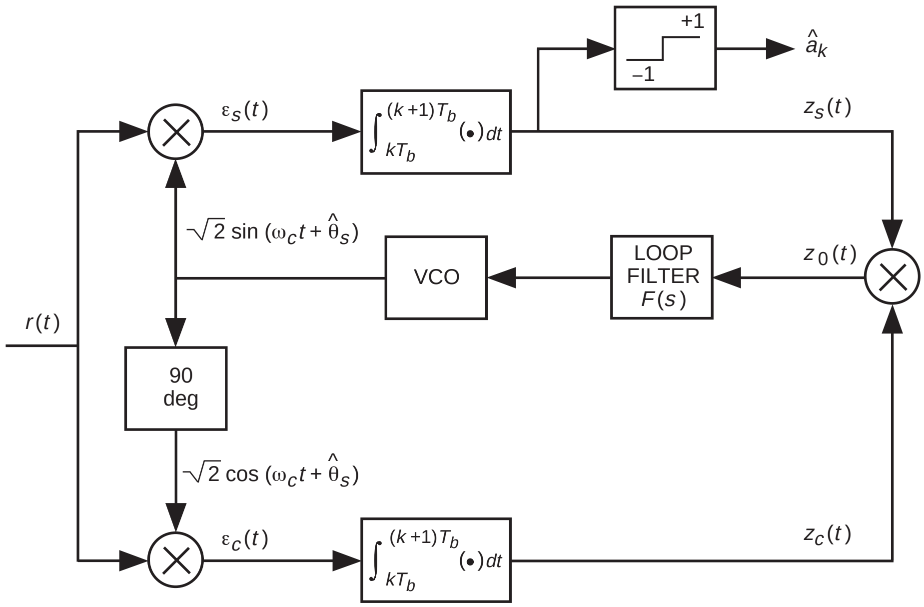

Fig. 1. Block diagram of the I-Q Costas loop receiver.

$$\left.\begin{aligned}
\varepsilon_c(t) &= r(t)\sqrt{2}\cos\left(\omega_c t + \hat{\theta}_s\right) \\
&= \left[\sqrt{P_s}m(t) - N_s(t)\right]\sin\phi_s + N_c(t)\cos\phi_s + \sqrt{P_I}\sin(\Delta\omega t + \phi_s + \Delta\theta) \\
\varepsilon_s(t) &= r(t)\sqrt{2}\sin\left(\omega_c t + \hat{\theta}_s\right) \\
&= \left[\sqrt{P_s}m(t) - N_s(t)\right]\cos\phi_s - N_c(t)\sin\phi_s + \sqrt{P_I}\cos(\Delta\omega t + \phi_s + \Delta\theta)
\end{aligned}\right\} \tag{5}$$

where $\phi_s \triangleq \theta_s - \hat{\theta}_s$ is the loop phase error and $\Delta\theta \triangleq \theta_I - \theta_s$ is the phase difference between the desired and interference signals. After passing through the I and Q arm filters, we obtain the sample-and-hold values at the end of the kth bit interval:

$$\left.\begin{aligned}
z_s(t) &= \int_{kT_b}^{(k+1)T_b} \varepsilon_s(t)\,dt = T_b\sqrt{P_s}a_k\cos\phi_s - N_1\sin\phi_s - N_2\cos\phi_s \\
&\quad + T_b\sqrt{P_I}\left\{A_{ck}\cos(\phi_s + \Delta\theta) - A_{sk}\sin(\phi_s + \Delta\theta)\right\} \\
z_c(t) &= \int_{kT_b}^{(k+1)T_b} \varepsilon_c(t)\,dt = T_b\sqrt{P_s}a_k\sin\phi_s + N_1\cos\phi_s - N_2\sin\phi_s \\
&\quad + T_b\sqrt{P_I}\left\{A_{sk}\cos(\phi_s + \Delta\theta) + A_{ck}\sin(\phi_s + \Delta\theta)\right\}, \qquad (k+1)T_b \le t \le (k+2)T_b
\end{aligned}\right\} \tag{6}$$

where

$$\left.\begin{aligned}A_{sk} &\triangleq \frac{1}{T_b}\int_{kT_b}^{(k+1)T_b}\sin\Delta\omega t dt = \frac{\cos\Delta\omega kT_b - \cos\Delta\omega\left(k+1\right)T_b}{\Delta\omega T_b}\\ A_{ck} &\triangleq \frac{1}{T_b}\int_{kT_b}^{(k+1)T_b}\cos\Delta\omega t dt = \frac{-\sin\Delta\omega kT_b + \sin\Delta\omega\left(k+1\right)T_b}{\Delta\omega T_b}\end{aligned}\right\} \tag{7}$$

and N_1 and N_2 are independent zero-mean Gaussian random variables with variance $\sigma_{N_1}^2 = \sigma_{N_2}^2 = N_0 T_b/2$. Defining the complex amplitude

$$A_k = A_{ck} + jA_{sk} = \frac{1}{T_b}\int_{kT_b}^{(k+1)T_b} e^{j\Delta\omega t}dt = \frac{1}{T_b}\int_0^{T_b} e^{j\Delta\omega(t+kT_b)}dt \tag{8}$$

then we can rewrite Eq. (6) as

$$\left.\begin{aligned}z_s\left(t\right) &= \int_{kT_b}^{(k+1)T_b}\varepsilon_s\left(t\right)dt = T_b\sqrt{P_s}a_k\cos\phi_s - N_1\sin\phi_s - N_2\cos\phi_s + T_b\sqrt{P_I}\ \mathrm{Re}\left\{A_k e^{j(\phi_s+\Delta\theta)}\right\}\\ z_c\left(t\right) &= \int_{kT_b}^{(k+1)T_b}\varepsilon_c\left(t\right)dt = T_b\sqrt{P_s}a_k\sin\phi_s + N_1\cos\phi_s - N_2\sin\phi_s + T_b\sqrt{P_I}\ \mathrm{Im}\left\{A_k e^{j(\phi_s+\Delta\theta)}\right\},\\ &\qquad \left(k+1\right)T_b \le t \le \left(k+2\right)T_b\end{aligned}\right\} \tag{9}$$

Further note that

$$|A_k| = \left|\frac{1}{T_b}\int_0^{T_b} e^{j\Delta\omega(t+kT_b)}dt\right| = \left|\frac{1}{T_b}\int_0^{T_b} e^{j\Delta\omega t}dt\right| = \left|\frac{\sin\frac{\Delta\omega T_b}{2}}{\frac{\Delta\omega T_b}{2}}\right| \tag{10}$$

which is independent of k.

Multiplying the two I&D outputs produces the dynamic error signal in the loop, which is given by

$$\begin{aligned}z_0\left(t\right) = z_c\left(t\right)z_s\left(t\right) &= \left(\frac{1}{2}P_sT_b^2 - \sqrt{P_s}T_b a_k N_2 + \frac{1}{2}N_2^2 - \frac{1}{2}N_1^2\right)\sin 2\phi_s\\ &+ \frac{1}{2}P_I T_b^2\left|A_k\right|^2\sin\left(2\left(\phi_s+\Delta\theta+\alpha_k\right)\right)\\ &+ \left(\sqrt{P_s}T_b a_k - N_2\right)N_1\cos 2\phi_s + \sqrt{P_sP_I}T_b^2 a_k\left|A_k\right|\sin\left(2\phi_s+\Delta\theta+\alpha_k\right)\\ &- \sqrt{P_I}T_b N_2\left|A_k\right|\sin\left(2\phi_s+\Delta\theta+\alpha_k\right) + \sqrt{P_I}T_b N_1\left|A_k\right|\cos\left(2\phi_s+\Delta\theta+\alpha_k\right)\end{aligned} \tag{11}$$

where

$$\alpha_k \triangleq \arg A_k = \tan^{-1} \frac{A_{sk}}{A_{ck}} \tag{12}$$

Unlike $|A_k|$, the argument α_k is a function of the index of the bit interval, k. To see this, we first write A_{sk} and A_{ck} in terms of their values in the zeroth interval, namely, A_{s0} and A_{c0}, as

$$\left.\begin{aligned} A_{sk} &\triangleq \frac{1}{T_b}\int_0^{T_b} \sin\Delta\omega\,(t+kT_b)\,dt = A_{s0}\cos k\Delta\omega T_b + A_{c0}\sin k\Delta\omega T_b \\ A_{ck} &\triangleq \frac{1}{T_b}\int_0^{T_b} \cos\Delta\omega\,(t+kT_b)\,dt = A_{c0}\cos k\Delta\omega T_b - A_{s0}\sin k\Delta\omega T_b \end{aligned}\right\} \tag{13}$$

where

$$\left.\begin{aligned} A_{s0} &\triangleq \frac{1}{T_b}\int_0^{T_b} \sin\Delta\omega t dt = \frac{1-\cos\Delta\omega T_b}{\Delta\omega T_b} \\ A_{c0} &\triangleq \frac{1}{T_b}\int_0^{T_b} \cos\Delta\omega t dt = \frac{\sin\Delta\omega T_b}{\Delta\omega T_b} \end{aligned}\right\} \tag{14}$$

Then,[3]

$$\begin{aligned} \alpha_k \triangleq \arg A_k = & \tan^{-1}\frac{A_{s0}\cos k\Delta\omega T_b + A_{c0}\sin k\Delta\omega T_b}{A_{c0}\cos k\Delta\omega T_b - A_{s0}\sin k\Delta\omega T_b} \\ = & \tan^{-1}\frac{|A_0|\sin(k\Delta\omega T_b + \alpha_0)}{|A_0|\cos(k\Delta\omega T_b + \alpha_0)} = k\Delta\omega T_b + \alpha_0 \end{aligned} \tag{15}$$

where

$$\alpha_0 \triangleq \arg A_0 = \tan^{-1}\frac{\frac{1}{T_b}\int_0^{T_b}\sin\Delta\omega t dt}{\frac{1}{T_b}\int_0^{T_b}\cos\Delta\omega t dt} = \tan^{-1}\left(\frac{1-\cos\Delta\omega T_b}{\sin\Delta\omega T_b}\right) = \tan^{-1}\left(\frac{\dfrac{\sin^2(\eta/2)}{\eta/2}}{\dfrac{\sin\eta}{\eta}}\right) \tag{16}$$

and we have further introduced the shorthand notation for normalized frequency offset,

$$\eta \triangleq \Delta\omega T_b = 2\pi\Delta f T_b \tag{17}$$

As we shall soon see, the linear dependence of α_k on k as exhibited in Eq. (15) is important in determining the effect of the interference on the loop's ability to lock. In particular, the behavior of the loop in the

[3] Unless otherwise noted, all arctangent functions are assumed to be taken in the four-quadrant sense, i.e., $\tan^{-1}(X/Y) = (\text{sgn } X)(\tan^{-1}[X/Y])_{P.V.} + \pi([1 - \text{sgn } X]/2)$, where $(\tan^{-1}[X/Y])_{P.V.}$ denotes the principal value (angle in first and fourth quadrants) of the arctangent of X divided by Y.

presence of tone interference is reminiscent of the false-lock behavior of Costas loops in the absence of interference[4] [5–8] in that the loop potentially can lock at frequencies other than that of the desired signal carrier.

The signal component of the dynamic error signal (which results in the so-called loop S-curve) is the statistical and time average, i.e., the dc component of the expression in Eq. (11), where the statistical average is taken over both the noise components and the binary data sequence. Performing first the expectation results in

$$g(\phi_s;k) \triangleq \overline{z_0(t)} = \left(\frac{1}{2}P_s T_b^2\right)\sin 2\phi_s + \frac{1}{2}P_I T_b^2 \left|\frac{\sin(\eta/2)}{\eta/2}\right|^2 \sin\left(2\left(\phi_s + \Delta\theta + k\eta + \alpha_0\right)\right) \tag{18}$$

Now taking the time average (i.e., the average over the index k) of Eq. (18), we observe that the second term (the one due to the interference) will be zero except when η is an integer multiple of π, i.e., $\eta = n\pi$, or, equivalently from Eq. (17),

$$\Delta f = \frac{n}{2T_b} \tag{19}$$

That is, if the tone interference occurs at integer multiples of half the data rate away from and either side of the desired signal carrier frequency,[5] then the loop S-curve may be affected by the interference. Otherwise, it will not. The S-curves corresponding to these scenarios are as follows. For $\Delta f = n/(2T_b)$,

$$g(\phi_s) \triangleq \left\langle \overline{z_0(t)} \right\rangle_k = \begin{cases} \left(\frac{1}{2}P_s T_b^2\right)\sin 2\phi_s + \frac{1}{2}P_I T_b^2 \left(\frac{2}{n\pi}\right)^2 \sin\left(2\left(\phi_s + \Delta\theta + \alpha_0\right)\right), & n \text{ odd} \\ \left(\frac{1}{2}P_s T_b^2\right)\sin 2\phi_s, & n \text{ even } (n \neq 0) \\ \left(\frac{1}{2}P_s T_b^2\right)\sin 2\phi_s + \frac{1}{2}P_I T_b^2 \sin\left(2\left(\phi_s + \Delta\theta + \alpha_0\right)\right), & n = 0 \end{cases} \tag{20}$$

with

$$\alpha_0 = \begin{cases} \tan^{-1}\left(\dfrac{2\sin^2\left(\dfrac{\pi n}{2}\right)}{\sin \pi n}\right) = \dfrac{\pi}{2}, & n \text{ odd} \\ 0, & n \text{ even } (n \neq 0) \\ 0, & n = 0 \end{cases} \tag{21}$$

For $\Delta f \neq n/(2T_b)$,

$$g(\phi_s) \triangleq \left\langle \overline{z_0(t)} \right\rangle_k = \left(\frac{1}{2}P_s T_b^2\right)\sin 2\phi_s \tag{22}$$

---

[4] M. K. Simon, "False Lock Behavior of Costas Receivers," Appendix J of *Integrated Source and Channel Encoded Digital Communication System Design Study*, Final Report, Axiomatix Report R7607-3, July 31, 1976.

[5] Note that for other data formats, e.g., Manchester coding, or pulse shapes, the frequencies and extent to which the interference affects the loop S-curve will be different, as was the case for the false-lock phenomena [6,8].

which is the well-known result for a Costas-loop error signal in the absence of interference and in the true-lock condition. Also, note that the S-curve of Eq. (20) also occurs when the interference tone is located at an even integer multiple of half the data rate away from the desired signal carrier frequency.

One of the quantities needed to evaluate the tracking performance of the loop is the slope of the S-curve at its lock point. Before differentiating Eq. (20) [or Eq. (22)], we first combine the two terms of the S-curve for n odd and $n = 0$ in Eq. (20), which after some algebra results in

$$g(\phi_s) = \begin{cases} \left(\frac{1}{2}P_s T_b^2\right)\sqrt{1 - 2\frac{P_I}{P_s}\left(\frac{2}{n\pi}\right)^2 \cos 2\Delta\theta + \left[\frac{P_I}{P_s}\left(\frac{2}{n\pi}\right)^2\right]^2}\, \sin\left(2\left(\phi_s + \beta_n\right)\right), & n \text{ odd} \\ \left(\frac{1}{2}P_s T_b^2\right)\sin 2\phi_s, & n \text{ even } (n \neq 0) \\ \left(\frac{1}{2}P_s T_b^2\right)\sqrt{1 + 2\frac{P_I}{P_s}\cos 2\Delta\theta + \left(\frac{P_I}{P_s}\right)^2}\, \sin\left(2\left(\phi_s + \beta_0\right)\right), & n = 0 \end{cases} \tag{23}$$

where

$$\beta_n = \begin{cases} \frac{1}{2}\tan^{-1}\dfrac{-\frac{P_I}{P_s}\left(\frac{2}{n\pi}\right)^2 \sin 2\Delta\theta}{1 - \frac{P_I}{P_s}\left(\frac{2}{n\pi}\right)^2 \cos 2\Delta\theta}, & n \text{ odd} \\ \frac{1}{2}\tan^{-1}\dfrac{\frac{P_I}{P_s}\sin 2\Delta\theta}{1 + \frac{P_I}{P_s}\cos 2\Delta\theta}, & n = 0 \end{cases} \tag{24}$$

Note that for n odd and $n = 0$, the loop locks with a static phase error, i.e., $\phi_s = -\beta_n$, whereas for n even or no interference, the loop locks at $\phi_s = 0$. Now letting $\phi_l = \phi_s$ or $\phi_l = \phi_s + \beta_n$ as appropriate, then differentiating Eqs. (20) and (22) with respect to ϕ_l and evaluating the result at $\phi_l = 0$ gives

$$K_g \triangleq \frac{dg(\phi_s)}{\phi_l}\Big|_{\phi_l = 0} = \begin{cases} P_s T_b^2 \sqrt{1 - 2\frac{P_I}{P_s}\left(\frac{2}{n\pi}\right)^2 \cos 2\Delta\theta + \left[\frac{P_I}{P_s}\left(\frac{2}{n\pi}\right)^2\right]^2}, & n \text{ odd} \\ P_s T_b^2, & n \text{ even } (n \neq 0) \\ P_s T_b^2 \sqrt{1 + 2\frac{P_I}{P_s}\left(\frac{2}{n\pi}\right)^2 \cos 2\Delta\theta + \left(\frac{P_I}{P_s}\right)^2}, & n = 0 \end{cases} \tag{25}$$

where the result for n even $(n \neq 0)$ also holds for the case $\Delta f \neq (n/[2T_b])$.

The next consideration is the effect of the noise terms contributed by the interference in Eq. (11) on the tracking performance of the loop. Since the loop tracks a $2\phi_s$ process, then, as in previous interference-free analyses of Costas loops, we define the equivalent noise by $N(t) \triangleq -2z_0(t)\Big|_{\substack{\text{noise}\\ \text{terms}}}$, which from Eq. (11) becomes

$$N(t) = \left(N_1^2 - N_2^2 + 2\sqrt{P_s}T_b a_k N_2\right)\sin 2\phi_s - 2\left(\sqrt{P_s}T_b a_k N_1 - N_1 N_2\right)\cos 2\phi_s$$

$$+ 2\sqrt{P_I}T_b N_2\left(\frac{\sin\eta/2}{\eta/2}\right)\sin\left(2\phi_s + \Delta\theta + \alpha_k\right) - 2\sqrt{P_I}T_b N_1\left(\frac{\sin\eta/2}{\eta/2}\right)\cos\left(2\phi_s + \Delta\theta + \alpha_k\right) \quad (26)$$

This noise process, $N(t)$, is piecewise constant (over the duration of a data bit) and can be modeled as a delta-correlated process with triangular correlation function

$$R_N(\tau) = E\left\{N(t)N(t+\tau)\right\} = \begin{cases} \sigma_N^2\left(1 - \dfrac{|\tau|}{T_b}\right), & |\tau| \le T_b \\ 0, & |\tau| > T_b \end{cases} \quad (27)$$

where σ_N^2 is its variance, which after some algebraic manipulation is given by

$$\sigma_N^2 = E\left\{N^2(t)\right\} = 2P_s N_0 T_b^3\left[1 + \frac{P_I}{P_s}\left(\frac{\sin\eta/2}{\eta/2}\right)^2\right] + N_0^2 T_b^2 \quad (28)$$

The equivalent flat PSD of $N(t)$ is then

$$N_0' = 2\int_{-\infty}^{\infty} R_N(\tau)\,d\tau = 2\sigma_N^2 T_b = 4P_s N_0 T_b^4\left[1 + \frac{P_I}{P_s}\left(\frac{\sin\eta/2}{\eta/2}\right)^2\right] + N_0^2 T_b^3 \quad (29)$$

Ignoring the variance of the data-dependent signal × interference term, $\sqrt{P_s P_I}T_b^2 a_k|A_k|\sin\left(2\left(\phi_s + \Delta\theta\right)\right)$, in Eq. (11),[6] then the mean-squared phase jitter of the $2\phi_s$ process is computed from

$$\sigma_{2\phi_s}^2 = \frac{N_0' B_L}{K_g^2} \quad (30)$$

where B_L is the single-sided loop noise bandwidth. Substituting Eqs. (25) and (29) in Eq. (30) gives the desired result, which can be expressed in the form

$$\sigma_{2\phi_s}^2 = 4\left(\frac{1}{\rho_{PLL} S_L}\right) \quad (31)$$

where $\rho_{PLL} \triangleq P_s/N_0 B_L$ is the equivalent loop signal-to-noise ratio (SNR) of a phase-locked loop (PLL) and $S_L = 4K_g^2/\left(P_s N_0'/N_0\right)$ is a factor traditionally referred to as squaring loss that accounts for the nonlinear (relative to a linear loop such as the PLL) distortions produced in the error signal by the

[6] This assumption is similar to ignoring the variance of the signal self-interference term (the difference between the signal × signal term and its mean value), which only contributes at a very high signal-to-noise ratio (SNR).

multiplication of the I and Q signal, interference, and noise terms. Substituting Eqs. (25) and (29) in the squaring-loss definition gives, for $\Delta f = n/2T_b$ ($\eta = n\pi$),

$$S_L = \begin{cases} \dfrac{1 - 2\dfrac{P_I}{P_s}\left(\dfrac{2}{n\pi}\right)^2 \cos 2\Delta\theta + \left[\dfrac{P_I}{P_s}\left(\dfrac{2}{n\pi}\right)^2\right]^2}{1 + \dfrac{P_I}{P_s}\left(\dfrac{2}{n\pi}\right)^2 + \dfrac{1}{2R_d}}, & n \text{ odd} \\ \dfrac{1}{1 + \dfrac{P_I}{P_s}\left(\dfrac{2}{n\pi}\right)^2 + \dfrac{1}{2R_d}}, & n \text{ even } (n \neq 0) \\ \dfrac{1 + 2\dfrac{P_I}{P_s}\cos 2\Delta\theta + \left(\dfrac{P_I}{P_s}\right)^2}{1 + \dfrac{P_I}{P_s} + \dfrac{1}{2R_d}}, & n = 0 \end{cases} \tag{32a}$$

and, for $\Delta f \neq n/2T_b$ ($\eta \neq n\pi$),

$$S_L = \frac{1}{1 + \dfrac{P_I}{P_s}\left(\dfrac{\sin \eta/2}{\eta/2}\right)^2 + \dfrac{1}{2R_d}} \tag{32b}$$

where $R_d \triangleq P_s T_b / N_0 = E_b/N_0$ is the bit energy-to-noise ratio. Note that for no interference ($P_I = 0$), Eqs. (32a) and (32b) simplify to

$$S_L = \frac{1}{1 + \dfrac{1}{2R_d}} = \frac{2R_d}{1 + 2R_d} \tag{33}$$

which is the well-known result for the squaring loss of an I-Q Costas loop [1–4].

Defining the effective loop SNR of the $2\phi_s$ process by

$$\rho_{2\phi_s} = \frac{1}{\sigma^2_{2\phi_s}} = \frac{1}{4}\rho_{PLL} S_L = \frac{1}{4}\left(\frac{E_b}{N_0}\right)\left(\frac{1}{B_L T_b}\right) S_L \triangleq \frac{1}{4}\left(\frac{E_b}{N_0}\right)\delta S_L \tag{34}$$

where δ is the reciprocal of the bit time-loop bandwidth product (typically a large number in standard applications), then it is customary to model the conditional (on the phase offset between the desired and interference signals) probability density function (PDF) of this process by a Tikhonov distribution, namely,

$$p_{2\phi_s}\left(2\phi_s \left|\Delta\theta\right.\right) = \frac{\exp\left\{\rho_{2\phi_2} \cos\left(2\left(\phi_s + \beta_n\right)\right)\right\}}{2\pi I_0\left(\rho_{2\phi_s}\right)}, \quad 2\left|\phi_s + \beta_n\right| \leq \pi \tag{35}$$

where β_n is determined from Eq. (24) for n integer and $\beta_n = 0$ for n noninteger. Also, in Eq. (35), $I_0(\bullet)$ is the modified Bessel function of the first kind. In the next subsection, we shall discuss the impact of this noisy carrier demodulation reference on the average BEP of the BPSK receiver.

B. Average Bit-Error-Probability Performance of the Receiver

As indicated in the introduction, we start out by considering the direct effect of the interference on the matched-filter output, assuming that the carrier-demodulation reference signal is perfect. So as not to duplicate our effort when we consider the case when the demodulation reference is derived from the Costas loop itself, we shall first evaluate the conditional (on the loop phase error, ϕ_s) BEP from which the result for a perfect carrier reference is obtained simply by setting $\phi_s = 0$.

The matched filter in the I arm of the Costas loop serves as the data detector. We have already specified its sample-and-hold output value, $z_s(t)$, in Eq. (9), which for the arbitrarily selected zeroth bit interval $k = 0$ becomes

$$\begin{aligned} z_s(t) &= T_b\sqrt{P_s}a_0\cos\phi_s + T_b\sqrt{P_I}\left[\frac{\sin\eta}{\eta}\cos(\phi_s+\Delta\theta) - \frac{1-\cos\eta}{\eta}\sin(\phi_s+\Delta\theta)\right] \\ &\quad - N_1\sin\phi_s - N_2\cos\phi_s \\ &= T_b\sqrt{P_s}a_0\cos\phi_s + T_b\sqrt{P_I}\left|\frac{\sin\eta/2}{\eta/2}\right|\cos(\phi_s+\Delta\theta+\alpha_0) \\ &\quad - N_1\sin\phi_s - N_2\cos\phi_s, \qquad T_b \le t \le 2T_b \end{aligned} \tag{36}$$

where α_0 is defined in Eq. (16) and $N \triangleq -N_1\sin\phi_s - N_2\cos\phi_s$ is a zero-mean Gaussian noise process with variance $\sigma^2 = N_0T_b/2$. Comparing this output to a zero threshold results in a decision on a_0. Assuming $a_0 = 1$, the conditional probability of error is given by

$$\begin{aligned} P_b(E|\phi_s,\Delta\theta)|_{a_0=1} &= \Pr\{z_s(t) < 0 | a_0 = 1\} \\ &= \frac{1}{2}\mathrm{erfc}\left\{\sqrt{R_d}\left[\cos\phi_s + \sqrt{\frac{P_I}{P_s}}\left|\frac{\sin\eta/2}{\eta/2}\right|\cos(\phi_s+\Delta\theta+\alpha_0)\right]\right\} \end{aligned} \tag{37a}$$

Similarly, assuming $a_0 = -1$, the conditional probability of error is given by

$$\begin{aligned} P_b(E|\phi_s,\Delta\theta)|_{a_0=-1} &= \Pr\{z_s(t) \ge 0 | a_0 = -1\} \\ &= \frac{1}{2}\mathrm{erfc}\left\{\sqrt{R_d}\left[\cos\phi_s - \sqrt{\frac{P_I}{P_s}}\left|\frac{\sin\eta/2}{\eta/2}\right|\cos(\phi_s+\Delta\theta+\alpha_0)\right]\right\} \end{aligned} \tag{37b}$$

Thus, since the hypotheses $a_0 = -1$ and $a_0 = 1$ are equiprobable, then averaged over the data, the conditional BEP is

$$P_b(E|\phi_s,\Delta\theta) = \frac{1}{2}P_b(E|\phi_s,\Delta\theta)|_{a_0=1} + \frac{1}{2}P_b(E|\phi_s,\Delta\theta)|_{a_0=-1} \tag{38}$$

Assuming a uniform distribution on $\Delta\theta$, which is appropriate in the absence of any a priori information concerning the relative phase between the desired and interference signals, then for ideal carrier tracking, we set $\phi_s = 0$ in Eq. (38), which gives

$$
\begin{aligned}
P_b(E)\left.\right|_{\text{ideal}} &= \frac{1}{8\pi}\int_{-\pi}^{\pi}\operatorname{erfc}\left\{\sqrt{R_d}\left[1+\sqrt{\frac{P_I}{P_s}}\left|\frac{\sin\eta/2}{\eta/2}\right|\cos\left(\Delta\theta+\alpha_0\right)\right]\right\}d\Delta\theta \\
&\quad + \frac{1}{8\pi}\int_{-\pi}^{\pi}\operatorname{erfc}\left\{\sqrt{R_d}\left[1-\sqrt{\frac{P_I}{P_s}}\left|\frac{\sin\eta/2}{\eta/2}\right|\cos\left(\Delta\theta+\alpha_0\right)\right]\right\}d\Delta\theta
\end{aligned} \tag{39}
$$

This result is upper and lower bounded by Eq. (38) evaluated at the worst- and best-case values of $\Delta\theta$, namely,

$$
\left.
\begin{aligned}
\Delta\theta\left.\right|_{\text{worst}} &= -\alpha_0 \\
\Delta\theta\left.\right|_{\text{best}} &= -\alpha_0 \pm \frac{\pi}{2}
\end{aligned}
\right\} \tag{40}
$$

which results in

$$
\left.
\begin{aligned}
P_b(E)\left.\right|_{\max} &= \frac{1}{4}\operatorname{erfc}\left\{\sqrt{R_d}\left[1+\sqrt{\frac{P_I}{P_s}}\left|\frac{\sin\eta/2}{\eta/2}\right|\right]\right\} + \frac{1}{4}\operatorname{erfc}\left\{\sqrt{R_d}\left[1-\sqrt{\frac{P_I}{P_s}}\left|\frac{\sin\eta/2}{\eta/2}\right|\right]\right\} \\
P_b(E)\left.\right|_{\min} &= \frac{1}{2}\operatorname{erfc}\left\{\sqrt{R_d}\right\}
\end{aligned}
\right\} \tag{41}
$$

When the demodulation reference is provided by the Costas loop itself, then there is an additional BEP degradation due to the impact of the noise and interference on the loop tracking performance. This can be evaluated by averaging the conditional BEP of Eq. (38) over both the uniform distribution on $\Delta\theta$ and the Tikhonov distribution on ϕ_s obtained from Eq. (35), which gives

$$
\begin{aligned}
P_b(E) &= \frac{1}{2\pi}\int_{-(\pi/2)-\beta_n}^{(\pi/2)-\beta_n}\int_{-\pi}^{\pi} P_b\left(E\left|\phi_s,\Delta\theta\right.\right) 2p_{2\phi_s}\left(2\phi_s\left|\Delta\theta\right.\right) d\Delta\theta d\phi_s \\
&= \frac{1}{4\pi}\int_{-(\pi/2)-\beta_n}^{(\pi/2)-\beta_n}\int_{-\pi}^{\pi}\operatorname{erfc}\left\{\sqrt{R_d}\left[\cos\phi_s+\sqrt{\frac{P_I}{P_s}}\left|\frac{\sin\eta/2}{\eta/2}\right|\cos\left(\phi_s+\Delta\theta+\alpha_0\right)\right]\right\} \\
&\quad \times \frac{\exp\left\{\rho_{2\phi_s}\cos\left(2\left(\phi_s+\beta_n\right)\right)\right\}}{2\pi I_0\left(\rho_{2\phi_s}\right)} d\Delta\theta d\phi_s \\
&\quad + \frac{1}{4\pi}\int_{-(\pi/2)-\beta_n}^{(\pi/2)-\beta_n}\int_{-\pi}^{\pi}\operatorname{erfc}\left\{\sqrt{R_d}\left[\cos\phi_s-\sqrt{\frac{P_I}{P_s}}\left|\frac{\sin\eta/2}{\eta/2}\right|\cos\left(\phi_s+\Delta\theta+\alpha_0\right)\right]\right\} \\
&\quad \times \frac{\exp\left\{\rho_{2\phi_s}\cos\left(2\left(\phi_s+\beta_n\right)\right)\right\}}{2\pi I_0\left(\rho_{2\phi_s}\right)} d\Delta\theta d\phi_s
\end{aligned} \tag{42}
$$

and is the desired result.

III. Performance in the Presence of Wideband Interference

A. Tracking Performance of the Costas Loop

For the wideband interferer case, the received signal is again given by Eq. (4), where now

$$s_I(t) = \sqrt{2P_I}m_I(t)\sin((\omega_c + \Delta\omega)t + \theta_I) \tag{43}$$

with

$$m_I(t) = \sum_{n=-\infty}^{\infty} a'_n p(t - nT_b - \tau) \tag{44}$$

denoting the interference modulation, which is independent of the desired signal modulation and whose data rate is assumed to be equal to that of the desired signal. Again, in Eq. (44), $\{a'_n\}$ is an independent and identically distributed (i.i.d.) sequence taking on equiprobable values ± 1; $p(t)$ is a unit amplitude rectangular pulse of duration T_b; and now τ denotes the time asynchronism of the interference with respect to the desired signal, which in the absence of any a priori information, is assumed to be uniformly distributed over a T_b-second interval. Analogously to Eq. (5), the I and Q demodulator baseband outputs become

$$\left.\begin{aligned}
\varepsilon_c(t) &= r(t)\sqrt{2}\cos\left(\omega_c t + \hat{\theta}_s\right) \\
&= \left[\sqrt{P_s}m(t) - N_s(t)\right]\sin\phi_s + N_c(t)\cos\phi_s + \sqrt{P_I}m_I(t)\sin(\Delta\omega t + \phi_s + \Delta\theta) \\
\varepsilon_s(t) &= r(t)\sqrt{2}\sin\left(\omega_c t + \hat{\theta}_s\right) \\
&= \left[\sqrt{P_s}m(t) - N_s(t)\right]\cos\phi_s - N_c(t)\sin\phi_s + \sqrt{P_I}m_I(t)\cos(\Delta\omega t + \phi_s + \Delta\theta)
\end{aligned}\right\} \tag{45}$$

After passing these signals through the I and Q I&Ds, the sample-and-hold values for the kth bit interval are still given by Eq. (9), where now

$$\begin{aligned}
A_k = A_{ck} + jA_{sk} &= \frac{1}{T_b}\int_{kT_b}^{(k+1)T_b} \sum_{n=-\infty}^{\infty} a'_n p(t - nT_b - \tau)e^{j\Delta\omega t}dt \\
&= \frac{1}{T_b}\int_0^{T_b} \sum_{n=-\infty}^{\infty} a'_n p(t + (k-n)T_b - \tau)\cos(\Delta\omega(t + kT_b))\,dt \\
&\quad + j\frac{1}{T_b}\int_0^{T_b} \sum_{n=-\infty}^{\infty} a'_n p(t + (k-n)T_b - \tau)\sin(\Delta\omega(t + kT_b))\,dt
\end{aligned} \tag{46}$$

Analogously to Eq. (13), the coefficients in Eq. (46) can be expressed in terms of ones that are independent of k, i.e.,

$$\left.\begin{aligned} A_{sk} &= A_{s0}\cos k\Delta\omega T_b + A_{c0}\sin k\Delta\omega T_b \\ A_{ck} &= A_{c0}\cos k\Delta\omega T_b - A_{s0}\sin k\Delta\omega T_b \end{aligned}\right\} \tag{47}$$

where now[7]

$$\left.\begin{aligned} A_{c0} &\triangleq \frac{1}{T_b}\int_0^{T_b}\sum_{n=-\infty}^{\infty} a'_n p\left(t+(k-n)T_b-\tau\right)\cos\Delta\omega t dt \\ &= \frac{1}{T_b}\int_0^{T_b}\sum_{m=-\infty}^{\infty} a'_m p\left(t-mT_b-\tau\right)\cos\Delta\omega t dt \\ A_{s0} &\triangleq \frac{1}{T_b}\int_0^{T_b}\sum_{n=-\infty}^{\infty} a'_n p\left(t+(k-n)T_b-\tau\right)\sin\Delta\omega t dt \\ &= \frac{1}{T_b}\int_0^{T_b}\sum_{m=-\infty}^{\infty} a'_m p\left(t-mT_b-\tau\right)\sin\Delta\omega t dt \end{aligned}\right\} \tag{48}$$

which are clearly independent of k, the index of the desired signal's bit interval. Furthermore,

$$|A_k|^2 = A_{ck}^2 + A_{sk}^2 = A_{c0}^2 + A_{s0}^2 \tag{49}$$

Letting $\varepsilon \triangleq \tau/T_b$, then for $\tau \geq 0$, Eq. (48) evaluates to

$$\left.\begin{aligned} A_{c0} &= a'_{-1}\frac{\sin(\eta\varepsilon)}{\eta} + a'_0\frac{\sin\eta - \sin(\eta\varepsilon)}{\eta} \\ A_{s0} &= a'_{-1}\frac{1-\cos(\eta\varepsilon)}{\eta} + a'_0\frac{\cos(\eta\varepsilon)-\cos\eta}{\eta} \end{aligned}\right\} \tag{50}$$

Similar results in terms of a'_0 and a'_1 would be obtained for $\tau \leq 0$. Substituting Eq. (50) in Eq. (49) and simplifying the trigonometry gives

$$\begin{aligned} |A_k|^2 = &\left(\frac{\sin(\eta\varepsilon/2)}{\eta/2}\right)^2 + \left(\frac{\sin(\eta(1-\varepsilon)/2)}{\eta/2}\right)^2 \\ &+ a'_0 a'_{-1}\left[\left(\frac{\sin(\eta(1-\varepsilon)/2)}{\eta/2}\right)\left(\frac{\sin(\eta(1+\varepsilon)/2)}{\eta/2}\right) - \left(\frac{\sin(\eta(1-\varepsilon)/2)}{\eta/2}\right)^2\right] \end{aligned} \tag{51}$$

[7] Note that the data bit a'_m in the terms to the right of the second equality in Eq. (48) should really be a'_{m+k}. However, since $\{a'_m\}$ represents an arbitrary doubly infinite random data sequence, then there is no loss in generality in replacing the sequence $\{a'_{m+k}\}$ by $\{a'_m\}$.

Multiplying the two I&D outputs as before produces the dynamic error signal in the loop, which is still given by Eq. (11) with α_k as in Eq. (15) and

$$\alpha_0 \triangleq \arg A_0 = \tan^{-1} \frac{A_{s0}}{A_{c0}} \tag{52}$$

which by virtue of Eq. (48) is also independent of k. Thus, the conditions (frequency offsets) under which the loop S-curve is affected by the presence of wideband interference are identical to those for the narrowband interference case; in particular, the S-curve is now

$$\begin{aligned} g(\phi_s;k) \triangleq \overline{z_0(t)} &= \left(\frac{1}{2}P_s T_b^2\right)\sin 2\phi_s + \frac{1}{2}P_I T_b^2 E_{a'}\left\{|A_k|^2 \sin\left(2\left(\phi_s + \Delta\theta + k\eta + \alpha_0\right)\right)\right\} \\ &= \left(\frac{1}{2}P_s T_b^2\right)\sin 2\phi_s + \frac{1}{2}P_I T_b^2 \overline{|A_k|^2 \cos 2\alpha_0}^{\alpha'} \sin\left(2\left(\phi_s + \Delta\theta + k\eta\right)\right) \\ &\quad + \frac{1}{2}P_I T_b^2 \overline{|A_k|^2 \sin 2\alpha_0}^{\alpha'} \cos\left(2\left(\phi_s + \Delta\theta + k\eta\right)\right) \end{aligned} \tag{53}$$

and the second and third terms contribute to the time-averaged S-curve, $g(\phi_s) = \langle g(\phi_s;k)\rangle_k$, only when the condition of Eq. (19) is met. Using Eq. (50), the statistical averages $\overline{|A_k|^2 \cos 2\alpha_0}^{\alpha'}$ and $\overline{|A_k|^2 \sin 2\alpha_0}^{\alpha'}$ are obtained as

$$\left.\begin{aligned} \overline{|A_k|^2 \sin 2\alpha_0}^{\alpha'} &= 2\overline{A_{s0}A_{c0}}^{\alpha'} = 2\left[\left(\frac{\sin(\eta\varepsilon)}{\eta}\right)\left(\frac{1-\cos(\eta\varepsilon)}{\eta}\right)\right. \\ &\qquad \left. + \left(\frac{\sin\eta}{\eta} - \frac{\sin(\eta\varepsilon)}{\eta}\right)\left(\frac{\cos(\eta\varepsilon)}{\eta} - \frac{\cos\eta}{\eta}\right)\right] \triangleq K_s(\varepsilon;n) \\ \overline{|A_k|^2 \cos 2\alpha_0}^{\alpha'} &= \overline{A_{c0}^2 - A_{s0}^2}^{\alpha'} = \left[\left(\frac{\sin(\eta\varepsilon)}{\eta}\right)^2 + \left(\frac{\sin\eta}{\eta} - \frac{\sin(\eta\varepsilon)}{\eta}\right)^2\right. \\ &\qquad \left. - \left(\frac{1-\cos(\eta\varepsilon)}{\eta}\right)^2 - \left(\frac{\cos(\eta\varepsilon)}{\eta} - \frac{\cos\eta}{\eta}\right)^2\right] \triangleq K_c(\varepsilon;n) \end{aligned}\right\} \tag{54}$$

which for $\Delta f = n/2T_b\,(\eta = n\pi)$ become

$$\left.\begin{array}{l}K_s(\varepsilon;n)=\begin{cases}-2\left(\dfrac{\sin(2n\pi\varepsilon)}{n\pi}\right), & n \text{ odd}\\[2ex] 8\left(\dfrac{\sin(n\pi\varepsilon)}{n\pi}\right)\left(\dfrac{\sin^2(n\pi\varepsilon/2)}{n\pi}\right), & n \text{ even } (n\neq 0)\\[2ex] 0, & n=0\end{cases}\\[8ex] K_c(\varepsilon;n)=\begin{cases}2\left(\dfrac{\sin\varepsilon}{n\pi}\right)^2-4\left(\dfrac{\sin^2(n\pi\varepsilon/2)}{n\pi}\right)^2-4\left(\dfrac{\cos^2(n\pi\varepsilon/2)}{n\pi}\right)^2, & n \text{ odd}\\[2ex] 2\left(\dfrac{\sin(n\pi\varepsilon)}{n\pi}\right)^2-8\left(\dfrac{\sin^2(n\pi\varepsilon/2)}{n\pi}\right)^2, & n \text{ even } (n\neq 0)\\[2ex] \varepsilon^2+(1-\varepsilon)^2, & n=0\end{cases}\end{array}\right\} \tag{55}$$

Finally, conditioned on τ and $\Delta\theta$, the time-averaged S-curves are given by

$$\begin{aligned} g(\phi_s)= &\left(\frac{1}{2}P_sT_b^2\right)\sin 2\phi_s+\frac{1}{2}P_IT_b^2K_c(\varepsilon;n)\sin(2(\phi_s+\Delta\theta))\\ &+\frac{1}{2}P_IT_b^2K_s(\varepsilon;n)\cos(2(\phi_s+\Delta\theta))\\ &\triangleq \frac{1}{2}K_g\sin(2(\phi_s+\beta_n)) \end{aligned} \tag{56}$$

with slope at the lock point [see Eq. (25)],

$$K_g=g(\phi_s)=P_sT_b^2\sqrt{1+2\frac{P_I}{P_s}\left[K_c(\varepsilon;n)\cos 2\Delta\theta-K_s(\varepsilon;n)\sin 2\Delta\theta\right]+\left(\frac{P_I}{P_s}\right)^2\left(K_s^2(\varepsilon;n)+K_c^2(\varepsilon;n)\right)} \tag{57}$$

where

$$\beta_n=\frac{1}{2}\tan^{-1}\frac{\dfrac{P_I}{P_s}\left[K_c(\varepsilon;n)\sin 2\Delta\theta+K_s(\varepsilon;n)\cos 2\Delta\theta\right]}{1+\dfrac{P_I}{P_s}\left[K_c(\varepsilon;n)\cos 2\Delta\theta-K_s(\varepsilon;n)\sin 2\Delta\theta\right]} \tag{58}$$

When $\Delta f\neq n/2T_b$, then the time-averaged S-curve is unaffected by the interference and, as before, is given by $g(\phi_s)=\left([1/2]P_sT_b^2\right)\sin 2\phi_s$ with slope $K_g=P_sT_b^2$.

The equivalent noise perturbing the loop is obtained by analogy with Eq. (26) as

$$\begin{aligned} N(t)= &\left(N_1^2-N_2^2+2\sqrt{P_s}T_ba_kN_2\right)\sin 2\phi_s-2\left(\sqrt{P_s}T_ba_kN_1-N_1N_2\right)\cos 2\phi_s\\ &+2\sqrt{P_I}T_bN_2\left|A_k\right|\sin(2\phi_s+\Delta\theta+\alpha_k)-2\sqrt{P_I}T_bN_1\left|A_k\right|\cos(2\phi_s+\Delta\theta+\alpha_k) \end{aligned} \tag{59}$$

which now has a variance

$$\sigma_N^2 = E\left\{N^2(t)\right\} = 2P_sN_0T_b^3\left[1+\frac{P_I}{P_s}\overline{|A_k|^2}\right] + N_0^2T_b^2 \tag{60}$$

where, from Eq. (51),

$$\overline{|A_k|^2} = \left(\frac{\sin(\eta\varepsilon/2)}{\eta/2}\right)^2 + \left(\frac{\sin(\eta(1-\varepsilon)/2)}{\eta/2}\right)^2 \tag{61}$$

Thus, the mean-squared phase jitter is still given by Eq. (31), where now the squaring-loss factor is, for $\Delta f = n/2T_b$ $(\eta = n\pi)$,

$$S_L = \frac{1 + 2\dfrac{P_I}{P_s}\left[K_c(\varepsilon;n)\cos 2\Delta\theta - K_s(\varepsilon;n)\sin 2\Delta\theta\right] + \left(\dfrac{P_I}{P_s}\right)^2\left(K_s^2(\varepsilon;n) + K_c^2(\varepsilon;n)\right)}{1 + \dfrac{P_I}{P_s}\left[\left(\dfrac{\sin(n\pi\varepsilon/2)}{n\pi/2}\right)^2 + \left(\dfrac{\sin(n\pi(1-\varepsilon)/2)}{n\pi/2}\right)^2\right] + \dfrac{1}{2R_d}} \tag{62a}$$

and, for $\Delta f \neq n/2T_b$ $(\eta \neq n\pi)$,

$$S_L = \frac{1}{1 + \dfrac{P_I}{P_s}\left[\left(\dfrac{\sin(\eta\varepsilon/2)}{\eta/2}\right)^2 + \left(\dfrac{\sin(\eta(1-\varepsilon)/2)}{\eta/2}\right)^2\right] + \dfrac{1}{2R_d}} \tag{62b}$$

Note that, for $\tau = 0$ $(\varepsilon = 0)$ (synchronous interferer and desired user), the squaring loss of Eqs. (62a) and (62b) for the wideband interferer become identical to those for the narrowband (tone) interferer as given by Eqs. (32a) and (32b), respectively. Furthermore, because of the symmetry of the problem, replacing ε by $|\varepsilon|$ in Eqs. (62a) and (62b) makes them, in addition, valid for $\varepsilon < 0$.

B. Average Bit-Error-Probability Performance of the Receiver

Once again, as indicated in the introduction, we start out by first evaluating the conditional (on the loop phase error, ϕ_s) BEP from which the result for a perfect carrier reference is obtained simply by setting $\phi_s = 0$.

The matched filter in the I arm of the Costas loop serves as the data detector. We have already specified its sample-and-hold output value, $z_s(t)$, in Eq. (9), which, for the arbitrarily selected zeroth bit interval $k = 0$, becomes

$$\begin{aligned} z_s(t) = {} & T_b\sqrt{P_s}a_0\cos\phi_s + T_b\sqrt{P_I}\left[A_{c0}\cos(\phi_s+\Delta\theta) - A_{s0}\sin(\phi_s+\Delta\theta)\right] \\ & - N_1\sin\phi_s - N_2\cos\phi_s, \qquad T_b \le t \le 2T_b \end{aligned} \tag{63}$$

where $N \triangleq -N_1\sin\phi_s - N_2\cos\phi_s$ is again a zero-mean Gaussian noise process with variance $\sigma^2 = N_0T_b/2$. Comparing this output to a zero threshold results in a decision on a_0. Assuming $a_0 = 1$, the conditional probability of error is given by

$$P_b\left(E\left|\phi_s,\Delta\theta,\varepsilon\right.\right)|_{a_0=1} = \overline{\Pr\left\{z_s\left(t\right)<0\left|a_0=1\right.\right\}}^{a_0',a_{-1}'}$$

$$= \overline{\frac{1}{2}\mathrm{erfc}\left\{\sqrt{R_d}\left[\cos\phi_s+\sqrt{\frac{P_I}{P_s}}\left|A_k\right|\cos\left(\phi_s+\Delta\theta+\alpha_0\right)\right]\right\}}^{a_0',a_{-1}'} \tag{64}$$

where $|A_k|$ is determined from Eq. (51), and from Eqs. (50) and (52),

$$\alpha_0 = \tan^{-1}\frac{\cos\left(\eta\varepsilon\right)-\cos\eta+a_0'a_{-1}'\left(1-\cos\left(\eta\varepsilon\right)\right)}{\sin\eta-\sin\left(\eta\varepsilon\right)+a_0'a_{-1}'\sin\left(\eta\varepsilon\right)}+\pi\frac{\left(1-\mathrm{sgn}\ a_0'\right)}{2} \tag{65}$$

Noticing that $|A_k|$ depends only on the interference bit product $a_0'a_{-1}'$ but that α_0 depends both on the product $a_0'a_{-1}'$ and on a_0' itself, then Eq. (64) can be written as

$$\begin{aligned}
P_b\left(E\left|\phi_s,\Delta\theta,\varepsilon\right.\right)|_{a_0=1} &= \frac{1}{8}\mathrm{erfc}\left\{\sqrt{R_d}\left[\cos\phi_s+\sqrt{\frac{P_I}{P_s}}\left|A_k^{(1)}\right|\cos\left(\phi_s+\Delta\theta+\alpha_0^{(1)}\right)\right]\right\} \\
&\quad+\frac{1}{8}\mathrm{erfc}\left\{\sqrt{R_d}\left[\cos\phi_s+\sqrt{\frac{P_I}{P_s}}\left|A_k^{(-1)}\right|\cos\left(\phi_s+\Delta\theta+\alpha_0^{(-1)}\right)\right]\right\} \\
&\quad\times\frac{1}{8}\mathrm{erfc}\left\{\sqrt{R_d}\left[\cos\phi_s-\sqrt{\frac{P_I}{P_s}}\left|A_k^{(1)}\right|\cos\left(\phi_s+\Delta\theta+\alpha_0^{(1)}\right)\right]\right\} \\
&\quad+\frac{1}{8}\mathrm{erfc}\left\{\sqrt{R_d}\left[\cos\phi_s-\sqrt{\frac{P_I}{P_s}}\left|A_k^{(-1)}\right|\cos\left(\phi_s+\Delta\theta+\alpha_0^{(-1)}\right)\right]\right\}
\end{aligned} \tag{66}$$

where

$$\left.\begin{aligned}
\left|A_k^{(1)}\right|^2 &= \left(\frac{\sin\left(\eta\varepsilon/2\right)}{\eta/2}\right)^2+\left(\frac{\sin\left(\eta\left(1-\varepsilon\right)/2\right)}{\eta/2}\right)\left(\frac{\sin\left(\eta\left(1+\varepsilon\right)/2\right)}{\eta/2}\right) \\
\left|A_k^{(-1)}\right|^2 &= 2\left(\frac{\sin\left(\eta\left(1-\varepsilon\right)/2\right)}{\eta/2}\right)^2+\left(\frac{\sin\left(\eta\varepsilon/2\right)}{\eta/2}\right)^2-\left(\frac{\sin\left(\eta\left(1-\varepsilon\right)/2\right)}{\eta/2}\right)\left(\frac{\sin\left(\eta\left(1+\varepsilon\right)/2\right)}{\eta/2}\right)
\end{aligned}\right\} \tag{67a}$$

and

$$\left.\begin{aligned}
\alpha_0^{(1)} &\triangleq \tan^{-1}\frac{1-\cos\eta}{\sin\eta} \\
\alpha_0^{(-1)} &\triangleq \tan^{-1}\frac{2\cos\left(\eta\varepsilon\right)-\cos\eta-1}{\sin\eta-2\sin\left(\eta\varepsilon\right)}
\end{aligned}\right\} \tag{67b}$$

For $\Delta f = n/2T_b$ $(\eta = n\pi)$, these quantities explicitly evaluate to

$$
\left.
\begin{aligned}
\left|A_k^{(1)}\right|^2 &= \begin{cases} \left(\dfrac{2}{n\pi}\right)^2, & n \text{ odd} \\ 0, & n \text{ even } (n \neq 0) \\ 1, & n = 0 \end{cases} \\
\left|A_k^{(-1)}\right|^2 &= \begin{cases} \left(\dfrac{2}{n\pi}\right)^2, & n \text{ odd} \\ 4\left(\dfrac{\sin n\pi\varepsilon/2}{n\pi/2}\right)^2, & n \text{ even } (n \neq 0) \\ (1-2\varepsilon)^2, & n = 0 \end{cases}
\end{aligned}
\right\} \tag{68a}
$$

and

$$
\left.
\begin{aligned}
\alpha_0^{(1)} &= \begin{cases} \dfrac{\pi}{2}, & n \text{ odd} \\ 0, & n \text{ even } (n \neq 0) \\ 0, & n = 0 \end{cases} \\
\alpha_0^{(-1)} &= \begin{cases} \dfrac{\pi}{2} + n\pi\varepsilon, & n \text{ odd} \\ \pi + \tan^{-1}\left(\dfrac{1-\cos(n\pi\varepsilon)}{\sin(n\pi\varepsilon)}\right), & n \text{ even } (n \neq 0) \\ 0, & n = 0 \end{cases}
\end{aligned}
\right\} \tag{68b}
$$

Similarly to Eq. (64), for $a_0 = -1$, the conditional probability of error is given by

$$
\begin{aligned}
P_b\left(E\left|\phi_s, \Delta\theta, \varepsilon\right.\right)\left|_{a_0=-1}\right. &= \overline{\Pr\left\{z_s(t) \geq 0 \left|a_0 = -1\right.\right\}}^{a'_0, a'_{-1}} \\
&= \overline{\frac{1}{2}\operatorname{erfc}\left\{\sqrt{R_d}\left[\cos\phi_s - \sqrt{\frac{P_I}{P_s}}\left|A_k\right|\cos\left(\phi_s + \Delta\theta + \alpha_0\right)\right]\right\}}^{a'_0, a'_{-1}}
\end{aligned} \tag{69}
$$

Thus, averaging over the equiprobable desired signal bits $a_0 = 1$ and $a_0 = -1$, the conditional BEP, $P_b\left(E\left|\phi_s, \Delta\theta, \varepsilon\right.\right)$, is also given by Eq. (66). Again, for $\varepsilon = 0$,

$$
\left|A_k^{(1)}\right| = \left|A_k^{(-1)}\right| = \left|\frac{\sin(\eta/2)}{\eta/2}\right|
$$

$$
\alpha_0^{(1)} = \alpha_0^{(-1)} = \tan^{-1}\frac{1-\cos\eta}{\sin\eta}
$$

and Eq. (66) reduces to the previously obtained result [see the average of Eqs. (37a) and (37b)] for the tone interferer.

Considering first the case of ideal carrier tracking wherein we set $\phi_s = 0$, then averaging Eq. (66) over uniform distributions for $\Delta\theta$ and ε, we obtain the average BEP:

$$
\begin{aligned}
P_b\left(E\right) = & \frac{1}{16\pi}\int_{-\pi}^{\pi}\int_{-1/2}^{1/2}\operatorname{erfc}\left\{\sqrt{R_d}\left[1+\sqrt{\frac{P_I}{P_s}}\left|A_k^{(1)}\right|\cos\left(\Delta\theta+\alpha_0^{(1)}\right)\right]\right\}d\varepsilon d\Delta\theta \\
& + \frac{1}{16\pi}\int_{-\pi}^{\pi}\int_{-1/2}^{1/2}\operatorname{erfc}\left\{\sqrt{R_d}\left[1+\sqrt{\frac{P_I}{P_s}}\left|A_k^{(-1)}\right|\cos\left(\Delta\theta+\alpha_0^{(-1)}\right)\right]\right\}d\varepsilon d\Delta\theta \\
& + \frac{1}{16\pi}\int_{-\pi}^{\pi}\int_{-1/2}^{1/2}\operatorname{erfc}\left\{\sqrt{R_d}\left[1-\sqrt{\frac{P_I}{P_s}}\left|A_k^{(1)}\right|\cos\left(\Delta\theta+\alpha_0^{(1)}\right)\right]\right\}d\varepsilon d\Delta\theta \\
& + \frac{1}{16\pi}\int_{-\pi}^{\pi}\int_{-1/2}^{1/2}\operatorname{erfc}\left\{\sqrt{R_d}\left[1-\sqrt{\frac{P_I}{P_s}}\left|A_k^{(-1)}\right|\cos\left(\Delta\theta+\alpha_0^{(-1)}\right)\right]\right\}d\varepsilon d\Delta\theta \qquad (70)
\end{aligned}
$$

Next, for the more realistic case when the demodulation reference is provided by the Costas loop itself, one requires as before an additional average of Eq. (66) over a Tikhonov distribution on ϕ_s, analogous to Eq. (42). In particular,

$$
\begin{aligned}
P_b\left(E\right) = & \frac{1}{16\pi}\int_{-(\pi/2)-\beta_n}^{(\pi/2)-\beta_n}\int_{-\pi}^{\pi}\int_{-1/2}^{1/2}\operatorname{erfc}\left\{\sqrt{R_d}\left[\cos\phi_s+\sqrt{\frac{P_I}{P_s}}\left|A_k^{(1)}\right|\cos\left(\phi_s+\Delta\theta+\alpha_0^{(1)}\right)\right]\right\} \\
& \times p_{2\phi_s}\left(2\phi_s\left|\Delta\theta,\varepsilon\right.\right)d\varepsilon d\Delta\theta d\phi_s \\
& + \frac{1}{16\pi}\int_{-(\pi/2)-\beta_n}^{(\pi/2)-\beta_n}\int_{-\pi}^{\pi}\int_{-1/2}^{1/2}\operatorname{erfc}\left\{\sqrt{R_d}\left[\cos\phi_s+\sqrt{\frac{P_I}{P_s}}\left|A_k^{(-1)}\right|\cos\left(\phi_s+\Delta\theta+\alpha_0^{(-1)}\right)\right]\right\} \\
& \times p_{2\phi_s}\left(2\phi_s\left|\Delta\theta,\varepsilon\right.\right)d\varepsilon d\Delta\theta d\phi_s \\
& + \frac{1}{16\pi}\int_{-(\pi/2)-\beta_n}^{(\pi/2)-\beta_n}\int_{-\pi}^{\pi}\int_{-1/2}^{1/2}\operatorname{erfc}\left\{\sqrt{R_d}\left[\cos\phi_s-\sqrt{\frac{P_I}{P_s}}\left|A_k^{(1)}\right|\cos\left(\phi_s+\Delta\theta+\alpha_0^{(1)}\right)\right]\right\} \\
& \times p_{2\phi_s}\left(2\phi_s\left|\Delta\theta,\varepsilon\right.\right)d\varepsilon d\Delta\theta d\phi_s \\
& + \frac{1}{16\pi}\int_{-(\pi/2)-\beta_n}^{(\pi/2)-\beta_n}\int_{-\pi}^{\pi}\int_{-1/2}^{1/2}\operatorname{erfc}\left\{\sqrt{R_d}\left[\cos\phi_s-\sqrt{\frac{P_I}{P_s}}\left|A_k^{(-1)}\right|\cos\left(\phi_s+\Delta\theta+\alpha_0^{(-1)}\right)\right]\right\} \\
& \times p_{2\phi_s}\left(2\phi_s\left|\Delta\theta,\varepsilon\right.\right)d\varepsilon d\Delta\theta d\phi_s \qquad (71)
\end{aligned}
$$

where $p_{2\phi_s}\left(2\phi_s\left|\Delta\theta,\varepsilon\right.\right)$ is characterized by Eq. (35) with $\rho_{2\phi_s}$ determined from the squaring loss of Eqs. (62a) and (62b) and β_n is given by Eq. (58).

IV. Numerical Results

Illustrated in Figs. 2(a) and 2(b) is the squaring-loss behavior of the Costas loop in the presence of narrowband (tone) interference corresponding, respectively, to the cases when $\Delta f T_b \neq n/2$ and $\Delta f T_b = n/2$ and n is integer. The curves are plotted with the ratio of interference to desired signal power, P_I/P_s, and $\Delta f T_b$ as parameters. Also shown for comparison is the squaring-loss performance for no interference ($P_I/P_s = 0$), which corresponds to the well-known result in Eq. (33).

In the case of the former, i.e., $\Delta f T_b \neq n/2$, we see that the squaring loss in dB [as computed from Eq. (32b)] is always negative (implying a true loss or degradation in loop SNR), with the severity of this loss increasing as P_I/P_s increases and $\Delta f T_b$ becomes smaller (i.e., the frequency of the interfering tone approaches that of the desired signal). Alternately, as the interfering tone moves further and further outside the bandwidth of the desired signal modulation, its effect on loop tracking performance diminishes.

In the case of the latter, i.e., $\Delta f T_b = n/2$, the potential effect of the interfering tone on the signal $\times$ signal term (numerator) in the squaring loss (as per the same behavior in the false-lock problem) results in a somewhat different phenomenon. First of all, for n an even integer (not including $n = 0$), the interferer has no effect on the signal $\times$ signal term and, thus, the squaring-loss behavior is similar to that for the case of $\Delta f T_b \neq n/2$ described above. However, when n is an odd integer or $n = 0$ (i.e., the interfering tone falls exactly on the desired signal carrier frequency), then the signal $\times$ signal term is affected [see Eq. (32a)] and, furthermore, depends on the phase difference, $\Delta\theta$, between the interferer and desired signal. The worst squaring loss for these two cases corresponds to $\Delta\theta = 0$ or $\Delta\theta = \pi$ for n an odd integer and $\Delta\theta = \pi/2$ for $n = 0$. [Of course, one should remember that, in a true situation, $\Delta\theta$ is a uniformly distributed parameter and, thus, the effect of the loop's ability to track on the average error-probability performance would be obtained by averaging over this parameter—see Eq. (42).] The curves plotted in Fig. 2(b) correspond to this worst-case value of $\Delta\theta$. Interestingly enough, we observe that, when the interference-to-desired-signal-power ratio is sufficiently large and the tone occurs at the desired signal carrier frequency (i.e., $n = 0$), the squaring "loss" in fact becomes a gain! The reasoning behind this is that a Costas loop is quite capable (moreover it prefers this to a modulated tone) of tracking a pure tone when it occurs at the correct frequency. Thus, in this situation, the additional power provided by the interfering signal at the desired signal carrier frequency aids the loop's ability to track. In fact, if this interfering tone becomes sufficiently large (in power) relative to the desired signal, the loop will instead track the tone. Even if the interfering tone is not at the desired signal carrier frequency but is in fact offset from it by an odd half integer multiple of the data rate, then analogously to the false-lock problem, a dc component is produced at the error-signal point in the loop and again this additional error-signal voltage can contribute a positive effect on the loop's ability to track. This can be observed in Fig. 2(b) by comparing the squaring-loss curves corresponding to $\Delta f T_b = 0.5$ for $P_I/P_s = 2.0$ and $P_I/P_s = 4.0$. For small values of P_I/P_s, the loop will always exhibit a squaring *loss*.

Figure 3 illustrates the average BEP in the presence of narrowband interference for the case when the carrier tracking is assumed to be perfect and is computed from Eq. (39). The interference-to-desired-signal-power ratio is held fixed at $P_I/P_s = 1.0$, whereas the relative frequency offset between the two signals is varied. Also shown is the curve corresponding to zero interference, $P_I/P_s = 0$, which corresponds to the ideal BEP performance of PSK, i.e., $P_b(E) = 1/2 \ \mathrm{erfc}\sqrt{R_d}$. The results in this figure illustrate the importance of the frequency location of the tone relative to that of the desired signal. In particular, when the tone is close (in frequency) to the desired signal, then it has a major impact on the BEP performance, whereas its effect is considerably reduced when it is further out in the signal spectrum. Also, note that the performance for $\Delta f T_b = 1.2$ is better than that for $\Delta f T_b = 1.5$, which reflects the $\sin x/x$ behavior (between the first and second nulls of this function) of the BEP on the parameter $\Delta f T_b$ [see Eq. (39)]. Finally, when the interfering tone occurs at integer values of the data rate away from the desired signal carrier frequency, i.e., $\Delta f T_b$ is integer (n is even), then $\sin(\eta/2)/(\eta/2) = \sin n\pi/n\pi = 0$, whereupon from Eq. (39), the interferer has no effect and the performance is given by the zero interference curve.

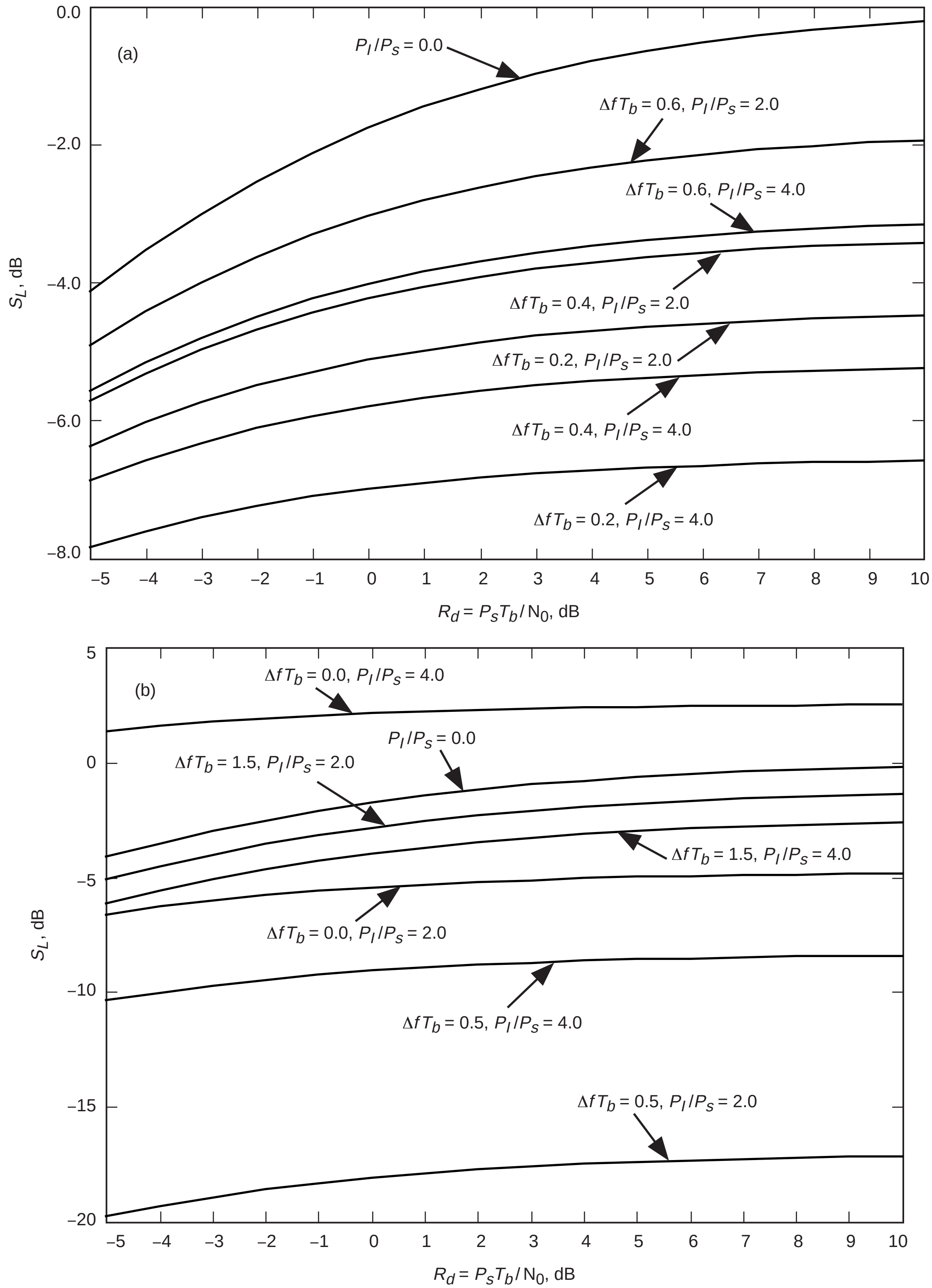

Fig. 2. Squaring loss versus bit SNR in the presence of narrowband (tone) interference: (a) $\Delta fT_b \neq n/2$ and (b) $\Delta fT_b = n/2$ (n integer).

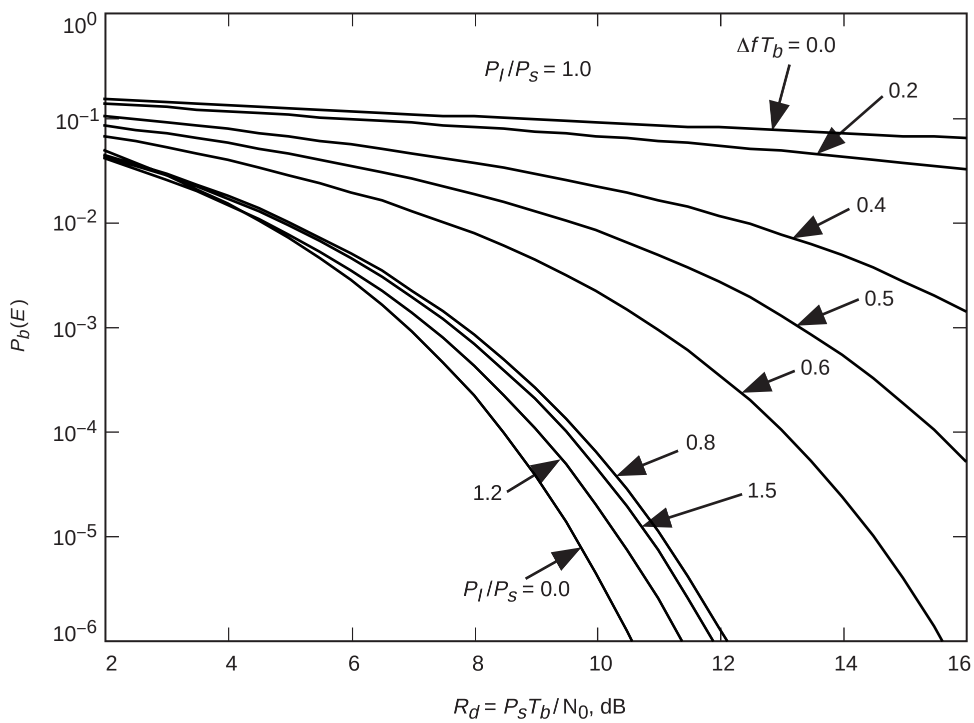

Fig. 3. Average bit-error probability versus bit SNR in the presence of narrowband (tone) interference (perfect carrier synchronization assumed).

Figure 4 is the companion to Fig. 3 when the carrier-phase tracking is provided by the Costas loop as per the discussion in Subsection 2.A. A value of $\delta = 10$ [see Eq. (34)] has been chosen for all of the curves. Even for this relatively small value for the reciprocal of the bit time-loop bandwidth product (typical values of practical systems are in the hundreds), the degradation due to the Costas-loop tracking relative to an ideal phase-coherent demodulation reference is very small and hardly noticeable on the graphs themselves (the raw data that generated these curves show the small difference). Thus, we conclude that the dominant effect of the interferer on average BEP is that produced on the data detector itself rather than on the phase tracking loop.

For wideband interference, Fig. 5 is the analogous plot to Fig. 3. Comparing these two figures, we observe that the wideband interferer has a more deleterious effect on performance than that produced by the narrowband interferer. We further note in Fig. 5 that, unlike the tone interference case, when the center frequency of the interferer occurs at integer values of the data rate away from the desired signal carrier frequency, i.e., $\Delta f T_b$ is an integer (n is even), the interferer indeed still has an effect on the BEP performance of the receiver. The reason for this can be gleaned from Eqs. (70) and (71) in combination from Eq. (68a), where we observe that the interference degradation is not simply a $\sin x/x$ function dependent only on $\Delta f T_b$ (or equivalently η), but rather depends on the combination of η and the normalized time offset, ε, between the desired and interfering signals. Thus, only when the two signals are perfectly time aligned (i.e., $\varepsilon = 0$) does the interference effect disappear. Finally, when the carrier synchronization is provided by the I-Q Costas loop, then analogous to what was true for the narrowband interferer case, the additional degradation due to such nonideal tracking is again very small. As such, we shall not show the numerical plots for this case since, as was true for Fig. 4 relative to Fig. 3, the effect would be hardly noticeable on the graphs themselves. Thus, we once again conclude that the dominant effect of the interferer on average BEP is that produced on the data detector itself rather than on the phase tracking loop.

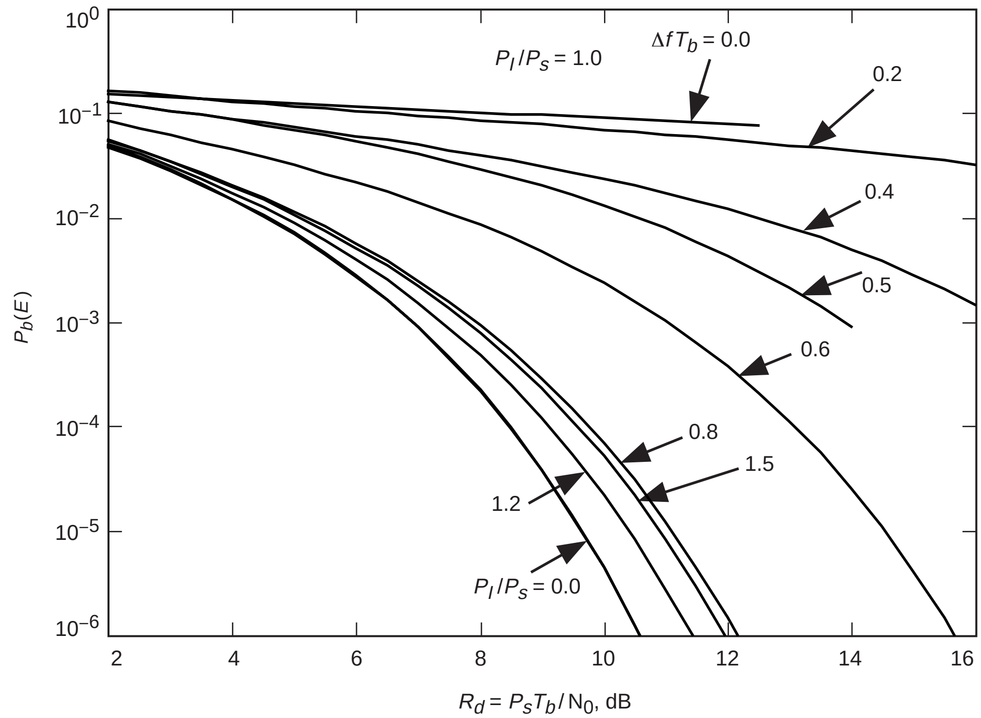

Fig. 4. Average bit-error probability versus bit SNR in the presence of narrow-band (tone) interference (carrier synchronization provided by a Costas loop); $\delta = 1/B_L T_B = 10$.

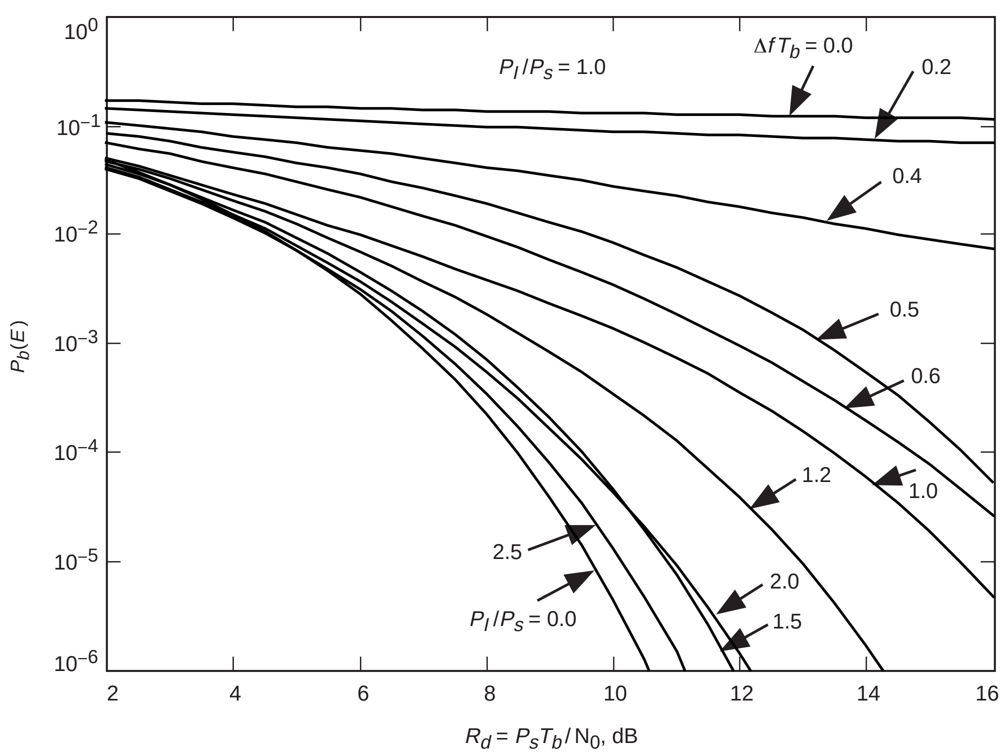

Fig. 5. Average bit-error probability versus bit SNR in the presence of wideband interference (perfect carrier synchronization assumed).

References

[1] W. C. Lindsey and M. K. Simon, "Optimum Design and Performance of Suppressed Carrier Receivers with Costas Loop Tracking," *IEEE Transactions on Communications*, vol. COM-25, no. 2, pp. 215–227, February 1977.

[2] W. C. Lindsey and M. K. Simon, *Telecommunication Systems Engineering*, Englewood Cliffs, New Jersey: Prentice Hall, Inc., 1973.

[3] J. H. Yuen, ed., *Deep Space Telecommunications Systems Engineering*, New York: Plenum Press, 1983.

[4] J. K. Holmes, *Coherent Spread Spectrum Systems*, New York: John Wiley & Sons, Inc., 1981.

[5] G. Hedin, J. K. Holmes, W. C. Lindsey, and K. T. Woo, "Theory of False Lock in Costas Loops," *IEEE Transactions on Communications*, vol. COM-26, no. 1, pp. 1–12, January 1978.

[6] M. K. Simon, "The False Lock Performance of Polarity-Type Costas Loops with Manchester Coded Input," *Proceedings of the 1977 National Telecommunications Conference*, Los Angeles, California, pp. 30:1-1–30:1-5, 1977.

[7] G. Hedin, J. K. Holmes, W. C. Lindsey, and K. T. Woo, "False Lock Phenomenon in Costas and Squaring Loops," *Proceedings of the 1977 National Telecommunications Conference*, Los Angeles, California, pp. 34:4-1–34:4-6, 1977.

[8] M. K. Simon, "False Lock Performance of I-Q Costas Loops for Pulse-Shaped Binary Phase Shift Keying," *The Telecommunications and Data Acquisition Progress Report 42-131, July–September 1997*, Jet Propulsion Laboratory, Pasadena, California, pp. 1–8, November 15, 1997.
http://tmo.jpl.nasa.gov/tmo/progress_report/42-131/131C.pdf