Intersymbol Interference in Pulse-Amplitude Modulation Signaling Systems Satisfying a Spectral Mask Constraint

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The Space Frequency Coordination Group (SFCG) has recently recommended a set of restrictions on the power spectral densities of space-to-Earth telemetry signals, which may apply to signals sent from Mars to Earth by future JPL missions. These restrictions take the form of spectral "masks" restricting the shape of the power spectral densities. In this article, which is semi-tutorial, we investigate the impact such masks have on the issue of intersymbol interference. For example, we shall give a necessary and sufficient condition for a given spectral mask to be compatible with non-intersymbol interference, i.e., Nyquist, signaling. We shall also see that the important class of root-raised-cosine Nyquist pulses is compatible with the current SFCG recommendations.

I. Introduction

In 1998, the Space Frequency Coordination Group (SFCG) recommended a set of restrictions on the power spectral densities (PSDs) of certain spaceborne transmitted signals, including some of those that may be sent from Mars to Earth by future JPL missions. These restrictions are of the general form

$$\frac{S(f)}{S(0)} \leq M\left(\frac{f}{f_0}\right) \quad \text{for all } f \tag{1}$$

where $S(f)$ is the PSD of the transmitted signal, $M(x)$ is a specified spectral "mask," satisfying

$$0 \leq M(x) \leq M(0) = 1$$

and f_0 is the symbol frequency [1].[3]

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[3] J. Hamkins, "Summary of SFCG Masks," JPL Interoffice Memorandum 331.2002.3.002 (internal document), Jet Propulsion Laboratory, Pasadena, California, March 28, 2002.

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In this article, we shall discuss some of the communications-theoretic issues raised by the existence of masks of this type; and in particular we will focus on the issue of intersymbol interference (ISI). We shall begin by reviewing the basics of pulse-amplitude modulation (PAM) in Sections II and III. In Section II, we will review PAM signaling in the presence of additive white Gaussian noise (AWGN), as well as the Nyquist criterion for avoiding ISI. In Section III, we will consider the power spectral densities of the signals introduced in Section II. In Section IV, we present our main results. For example, we will derive a necessary and sufficient condition on a mask $M(x)$ that guarantees that it can support a Nyquist (non-ISI) pulse shape (Theorem 1). We will also conclude that any of the SFCG masks will permit the use of a root-raised-cosine Nyquist pulse shape with arbitrary roll-off factor. Finally, in Section V, we will venture some conclusions.

II. Baseband Pulse-Amplitude Modulation

In a baseband pulse-amplitude modulation communication system, the transmitted signal is of the form

$$s(t) = \sum_{k=-\infty}^{\infty} a[k] g(t - kT_0) \tag{2}$$

where $a[k]_{k=-\infty}^{\infty}$ is the information signal,[4] modeled as a discrete-time stationary random process (real or complex), T_0 is the symbol time, and $g(t)$ is the transmit pulse shape.[5] We assume that $s(t)$ is transmitted over an AWGN channel and received as $x(t)$, i.e.,

$$x(t) = \mu s(t) + w(t)$$

where $w(t)$ is a sample function of the white Gaussian noise process, with PSD $P_x(f) = N_0/2$, and μ is a scale factor, which reflects a possible signal attenuation that may occur during transmission.

The noisy signal $x(t)$ is then passed through a receive filter (matched to $g(t)$, about which we say more in Section III) with impulse response $c(t)$, resulting in a filter output

$$y(t) = \mu \sum_k a[k] p(t - kT_0) + v(t)$$

where $p(t) = g(t) \star c(t)$, and $v(t)$ is the filtered noise process, which has PSD

$$P_v(f) = \frac{N_0}{2} |C(f)|^2$$

where $C(f)$ is the Fourier transform of $c(t)$.

To estimate the nth transmitted symbol $a[n]$, the receive filter output $y(t)$ is sampled at $t = nT_0$, yielding

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[4] The set of values assumed by the signal $a[k]$ is called the signal constellation. Thus, for binary phase-shift keying (BPSK), the constellation is $\{+1, -1\}$; for quadrature phase-shift keying (QPSK), it is $\{+1 + j, +1 - j, -1 + j, -1 - j\}$, etc.

[5] The envelope of $s(t)$ is defined to be $|s(t)|$. The transmitted signal is said to have constant envelope if there exists a constant K such that $|s(t)| = K$ for all t.

$$\widehat{a[n]} = \mu \sum_k a[k]p\big((n-k)T_0\big) + v(nT_0)$$

$$= \mu p(0)a[n] + \sum_{k \neq n} a[k]p\big((n-k)T_0\big) + z[n] \tag{3}$$

where $\{z[n] = v(nT_0)\}$ are identically distributed Gaussian random variables with mean zero and common variance

$$\mathrm{Var}(z[n]) = \frac{N_0}{2} \int_{-\infty}^{\infty} |C(f)|^2 df = \frac{N_0}{2} \int_{-\infty}^{\infty} |c(t)|^2 dt$$

The second and third terms on the right side of Eq. (3) are clearly unwanted. While nothing can be done about the third term (the noise), the second term, which is called intersymbol interference, can be eliminated entirely if we require that the signal $p(t)$ has zero crossings at the non-zero integer multiples of T_0, i.e.,

$$p(nT_0) = 0 \quad \text{if } n \neq 0 \tag{4}$$

in which case Eq. (3) becomes

$$\widehat{a[n]} = \mu p(0)a[n] + z[n] \tag{5}$$

The "no-ISI" criterion, Eq. (4), can be translated into the frequency domain, as follows. If $P(f)$ denotes the Fourier transform of $p(t)$, i.e.,

$$p(t) = \int_{-\infty}^{\infty} P(f)e^{j2\pi ft} df$$

$$P(f) = \int_{-\infty}^{\infty} p(t)e^{-j2\pi ft} dt$$

then Nyquist's Theorem [2, Section 7.5] says that Eq. (4) is equivalent to the existence of a real constant K such that

$$\sum_{n=-\infty}^{\infty} P(f - nf_0) = K, \quad \text{for all } f \tag{6}$$

where $f_0 = 1/T_0$ is the symbol frequency. (If Eq. (6) holds, then $p(0) = K/T_0$.)

We conclude this section with a few examples of "Nyquist pulses," i.e., signals $p(t)$ satisfying Eq. (4) or, equivalently, Eq. (6). We will define our Nyquist pulses in terms of the following standard functions:

(1) The boxcar function (see Fig. 1):

$$\Pi(x) = \begin{cases} 1 & \text{if } |x| < \dfrac{1}{2} \\ 0 & \text{if } |x| \geq \dfrac{1}{2} \end{cases}$$

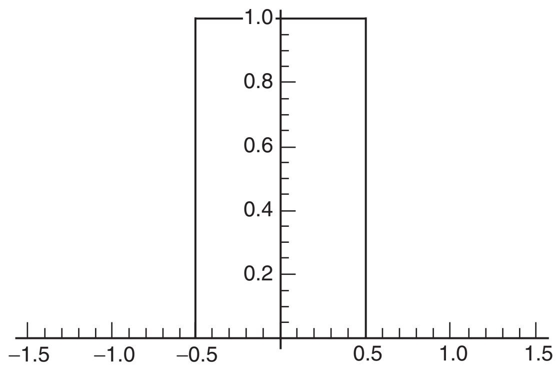

Fig. 1. The boxcar function $\Pi(x) = \mathrm{Rc}(x, 0)$**.**

(2) The sinc function (see Fig. 2):

$$\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$$

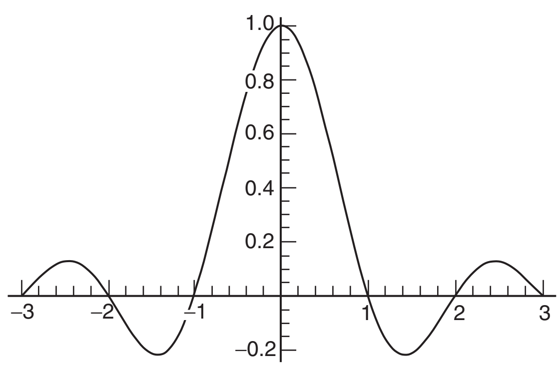

Fig. 2. The function $\mathrm{sinc}(x) = \mathrm{sinc}(x, 0)$**.**

(3) The sinc function, with roll-off factor α, $0 \le \alpha \le 1$ (see Fig. 3):

$$\operatorname{sinc}(x,\alpha) = \operatorname{sinc}(x)\frac{\cos(\pi\alpha x)}{1-(2\alpha x)^2}$$

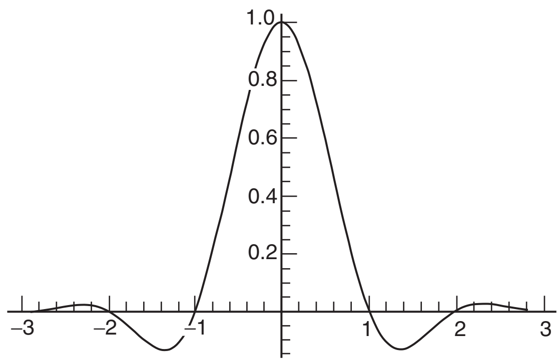

Fig. 3. The function sinc (x, 1/2)**.**

(4) The raised cosine (see Fig. 4):

$$\operatorname{Rc}(x) = \begin{cases} \dfrac{(1+\cos\pi x)}{2} & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

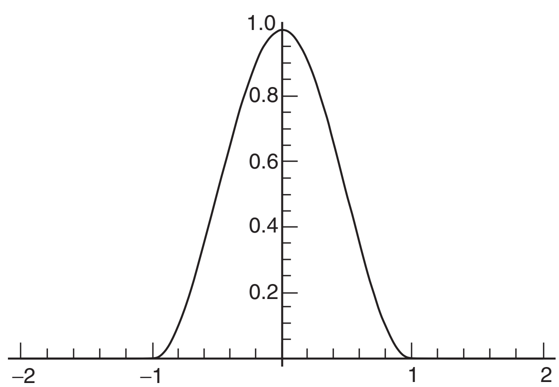

Fig. 4. The raised-cosine function Rc(x) = Rc(x, 1)**.**

(5) The raised cosine, with roll-off factor α, $0 \le \alpha \le 1$ (see Fig. 5):

$$\mathrm{Rc}(x,\alpha) = \begin{cases} 1 & \text{if } |x| < \dfrac{(1-\alpha)}{2} \\ \mathrm{Rc}\left(\dfrac{|x| - (1-\alpha/2)}{\alpha}\right) & \text{if } \dfrac{(1-\alpha)}{2} \le |x| \le \dfrac{(1+\alpha)}{2} \\ 0 & \text{if } \dfrac{(1+\alpha)}{2} < |x| \end{cases}$$

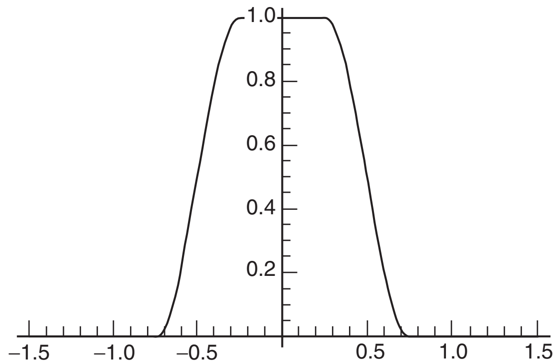

Fig. 5. The raised-cosine function Rc (x, 1/2)**.**

Here then are some examples of Nyquist pulses, expressed both in the time domain and the frequency domain.

Example 1. The squarewave:

$$p(t) = \Pi\left(\frac{t}{T_0}\right) = \mathrm{Rc}\left(\frac{t}{T_0, 0}\right)$$

$$P(f) = T_0\,\mathrm{sinc}\left(\frac{f}{f_0}\right) = T_0\,\mathrm{sinc}\left(\frac{f}{f_0, 0}\right)$$

(For this $p(t)$, if the signal constellation satisfies $|a[n]|^2 = K$, e.g., a phase-shift keyed (PSK) constellation, then the transmitted signal $s(t)$ has constant envelope.)

Example 2. The (time-domain) raised cosine with roll-off factor α:

$$p(t) = \mathrm{Rc}\left(\frac{t}{T_0, \alpha}\right)$$

$$P(f) = T_0\,\mathrm{sinc}\left(\frac{f}{f_0, \alpha}\right)$$

Example 3. The sinc pulse (frequency-domain raised cosine with roll-off factor 0):

$$p(t) = \operatorname{sinc}\left(\frac{t}{T_0}\right)$$

$$P(f) = T_0 \Pi\left(\frac{f}{f_0}\right)$$

Example 4. The frequency-domain raised cosine (roll-off factor 1):

$$p(t) = \operatorname{sinc}\left(\frac{t}{T_0, 1}\right)$$

$$P(f) = T_0 \operatorname{Rc}\left(\frac{f}{f_0, 1}\right)$$

Example 5. The frequency-domain raised cosine with roll-off factor α:[6]

$$p(t) = \operatorname{sinc}\left(\frac{t}{T_0, \alpha}\right)$$

$$P(f) = T_0 \operatorname{Rc}\left(\frac{f}{f_0, \alpha}\right)$$

III. Power Spectral Densities

In this section, we will consider the power spectral densities of the signals introduced in Section II. We begin with the transmitted signal $s(t)$ defined in Eq. (2). According to [4, Section 8.2], we have

$$S_s(f) = \frac{1}{T_0} S_a(f) |G(f)|^2 \tag{7}$$

where $S_a(f)$ is defined as

$$S_a(f) = \sum_n R_a[n] e^{-j2\pi n f T_0}$$

where $R_a[n]$ is the autocorrelation function of the discrete-time process $\{a[n]\}$, i.e.,

$$R_a[n] = E\big(a[k]^* a[k+n]\big)$$

[6] In most texts, e.g., [4, Section 8.3], what we have called the frequency-domain raised cosine is simply called the raised cosine, and what we have called the time-domain raised cosine is not mentioned at all.

and $G(f)$ is the Fourier transform of the transmit pulse $g(t)$.[7]

From now on, however, we will make the simplifying assumption that the process $\{a[n]\}$ is independent, identically distributed (i.i.d.) and has unit variance, so that $S_a(f) \equiv 1$, and so Eq. (7) becomes

$$S_s(f) = \frac{1}{T_0}|G(f)|^2 \tag{8}$$

As mentioned in Section II, the receive filter is matched to $g(t)$, which means it has impulse response

$$c(t) = g^*(\tau - t)$$

and Fourier transform

$$C(f) = G^*(f)e^{-2\pi j\tau}$$

for an arbitrary constant τ [4, Section 7.5]. With no loss in generality, we shall take $\tau = 0$, so that since $p(t) = g(t) \star c(t) = g(t) \star g^*(-t)$, we have

$$P(f) = |G(f)|^2 \tag{9}$$

Thus, if $p(t)$ is a Nyquist pulse, Eq. (6) becomes

$$\sum_n |G(f - nf_0)|^2 = K \tag{10}$$

As an immediate corollary to Eq.(10), we have

$$|G(0)|^2 \leq K \tag{11}$$

Finally we note that Eq. (9) implies that if $p(t)$ is Nyquist, then $P(f)$ must be a non-negative function of f, which eliminates some Nyquist pulses from consideration, e.g., the time-domain raised cosines. On the other hand, the frequency-domain raised cosines have positive Fourier transforms, and so can be "factored" as $p(t) = g(t) \star g(-t)$ for an appropriate pulse shape $g(t)$. For example, if we define the root-raised-cosine pulse by its Fourier transform

$$G(f) = T_0\sqrt{\mathrm{Rc}\left(\frac{f}{f_0, \alpha}\right)}$$

then the corresponding time-domain pulse shape is

$$g(t) = \mathrm{RRc}\left(\frac{t}{T_0, \alpha}\right)$$

[7] Note that $(1/T_0)S_a(f)$ is the PSD of a continuous-time version of $a[n]$, i.e., $a(t) = \sum_n a[n]\delta(t - nT_0)$.

where the function $\mathrm{RRc}(x,\alpha)$ is defined as [3, Eq. 6.105]

$$\mathrm{RRc}(x,\alpha) = \frac{1}{1-(4\alpha x)^2}\left(\frac{4\alpha}{\pi}\cos\big((1+\alpha)\pi x\big) + (1-\alpha)\,\mathrm{sinc}\big((1-\alpha)x\big)\right)$$

This signal is illustrated in Fig. 6 for $\alpha = 1$. The peak of $g(t) = \mathrm{RRc}(t/T_0, 1)$ is $g(0) = 4/\pi$, and the zero-crossings are at $t = \pm(3/4)T_0, \pm(5/4)T_0, \cdots$.

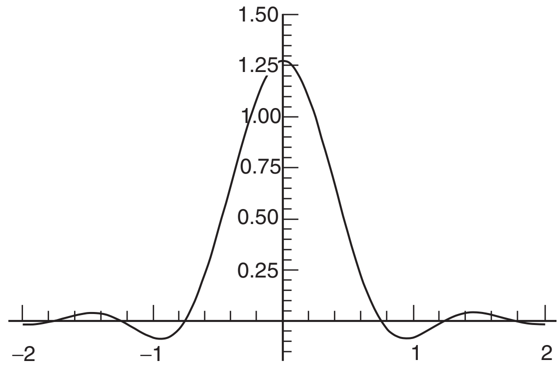

Fig. 6. The root-raised-cosine function
$\mathrm{RRc}(x,1) = \frac{4}{\pi}\frac{\cos(2\pi x)}{(1-16x^2)}$.

IV. Spectral Masks

Suppose now that the PSD of the transmitted signal is required to satisfy

$$\frac{S_s(f)}{S_s(0)} \le M\left(\frac{f}{f_0}\right) \quad \text{for all } f \tag{12}$$

where $M(x)$ is a "mask" specified by some regulatory agency. We assume that the mask satisfies

$$0 \le M(x) \le M(0) = 1$$

By Eq. (8), Eq. (12) implies

$$\frac{|G(f)|^2}{|G(0)|^2} \le M\left(\frac{f}{f_0}\right) \quad \text{for all } f \tag{13}$$

The question is: Is it possible to satisfy Eqs. (10) and (13) simultaneously? That is, does the mask requirement preclude using a Nyquist pulse? To answer the question, we introduce the function

$$\widetilde{M}(x) = \sum_{n=-\infty}^{\infty} M(x-n)$$

which is periodic of period 1. Note that $\widetilde{M}(0) \ge M(0) = 1$.

Theorem 1. *It is possible to find a transmit pulse $g(t)$ so that Eqs. (10) and (13) are satisfied simultaneously if and only if the mask satisfies*

$$\widetilde{M}(x) \geq 1 \quad \text{for all } -\frac{1}{2} \leq x < \frac{1}{2} \tag{14}$$

(Since $\widetilde{M}(x)$ is periodic of period 1, this condition is equivalent to $\widetilde{M}(x) \geq 1$ for all x.)

Proof. (Only if.) If Eqs. (10) and (13) are satisfied, then we have

$$\begin{aligned} K &= \sum_n |G(f - nf_0)|^2 \quad \text{[by Eq. (10)]} \\ &\leq |G(0)|^2 \sum_n M\left(\frac{f - nf_0}{f_0}\right) \quad \text{[by Eq. (13)]} \\ &= |G(0)|^2 \widetilde{M}\left(\frac{f}{f_0}\right) \quad \text{(definition of } \widetilde{M}) \end{aligned}$$

Thus, substituting x for f/f_0, we have

$$\widetilde{M}(x) \geq \frac{K}{|G(0)|^2}$$

Finally, we note that Eq. (11) implies

$$\frac{K}{|G(0)|^2} \geq 1$$

which completes the proof of the "only if" part of the theorem.

Conversely, suppose that Eq. (14) holds. Suppose that there exists a function $h(x)$ satisfying

$$\left.\begin{aligned} 0 \leq h(x) \leq M(x) \\ h(0) = M(0) = 1 \end{aligned}\right\} \tag{15}$$

such that

$$\widetilde{h}(x) = \sum_n h(x - n) = 1 \tag{16}$$

Then if $g(t)$ is chosen so that

$$|G(f)|^2 \propto h\left(\frac{f}{f_0}\right) \tag{17}$$

Equations (10) and (13) and are satisfied simultaneously.

To see that the existence of a function $h(x)$ satisfying Eqs. (15) and (16) implies that Eq. (10) holds, note that by Eq. (17) there exists a constant K such that

$$\sum_n |G(f-nf_0)|^2 = K\sum_n h\left(\frac{f}{f_0-n}\right) = K\widetilde{h}\left(\frac{f}{f_0}\right) = K$$

(The last equality follows from Eq. (16).) Similarly, by Eqs. (17) and (15), we have

$$\frac{|G(f)|^2}{|G(0)|^2} = \frac{h(f/f_0)}{h(0)} = h\left(\frac{f}{f_0}\right) \leq M\left(\frac{f}{f_0}\right)$$

which proves Eq. (13).

All that remains is to prove the existence of a function $h(x)$ satisfying Eqs. (15) and (16). This is almost obvious, since by decreasing $M(x)$ appropriately we expect to be able to change $\widetilde{M}(x)$, which is originally $\geq$1 everywhere, to =1 everywhere. Admittedly this argument leaves something to be desired, so we conclude with a rigorous proof.

Lemma 1. *If the mask $\widetilde{M}(x)$ satisfies Eq. (14), then there exists a function $h(x)$ satisfying Eqs. (15) and (16).*

Proof. For each x_0 satisfying $-1/2 \leq x_0 < 1/2$, we have by Eq. (14) that

$$\sum_{n=-\infty}^{\infty} M(x_0-n) \geq 1$$

Let us define the power series

$$f(y) = \sum_{n=-\infty}^{\infty} M(x_0-n)y^{|n|}$$

Clearly $f(y)$ is a continuous and increasing function of y, with $f(0)=0$ and $f(1)=\widetilde{M}(x_0)\geq 1$. Therefore, by the Intermediate Value Theorem, there exists a real number $0 \leq y = y(x_0) \leq 1$ such that

$$\sum_{n=-\infty}^{\infty} M(x_0-n)y(x_0)^{|n|} = 1 \tag{18}$$

Now define a function $b(x)$, for all real x, as follows. Any given real number x can be written uniquely in the form

$$x = x_0 - n$$

where $-1/2 \leq x_0 < 1/2$ and n is an integer. (Indeed, $n = -\lfloor x+1/2\rfloor$, and $x_0 = x+n$.) Then $b(x)$ is defined to be $y(x_0)^{|n|}$. Now define $h(x) = M(x)b(x)$. Then for $-1/2 \leq x_0 < 1/2$, we have

$$\widetilde{h}(x_0) = \sum_n M(x_0 - n)b(x_0 - n)$$

$$= \sum_n M(x_0 - n)y(x_0)^{|n|}$$

$$= 1$$

❐

We conclude with some examples.

Example 6. Any mask that satisfies

$$M(x) = 1 \quad \text{for } |x| \leq \frac{1}{2}$$

automatically satisfies Eq. (14). In fact, such a mask supports a pulse shape characterized in the frequency domain by

$$P(f) = \Pi\left(\frac{f}{f_0}\right)$$

which in the time domain is the sinc pulse $\text{sinc}(t/T_0)$. (All the SFCG recommendations except the obsolete high-rate 17-2R1 meet this criterion.) ❐

Example 7. Suppose the mask is

$$M(x) = e^{-\lambda|x|} \tag{19}$$

for some positive constant λ. A simple calculation shows that

$$\widetilde{M}(x) = \frac{e^{-\lambda(1-|x|)} + e^{-\lambda|x|}}{1 - e^{-\lambda}} \quad \text{for } |x| \leq \frac{1}{2}$$

It is easy to see that $\widetilde{M}(x)$ is monotonically decreasing in $|x|$ for $|x| \leq 1/2$, so its minimum occurs at $x = \pm 1/2$, which is $2e^{-\lambda/2}/(1-e^{-\lambda})$. Thus, in order for the mask $M(x)$ as defined in Eq. (19) to be compatible with a Nyquist pulse, by Theorem 1 we must have $2e^{-\lambda/2}/(1-e^{-\lambda}) \geq 1$, i.e., $\lambda > -2\ln(\sqrt{2}-1) = 1.7627$. ❐

Example 8. Let us determine conditions under which a given mask will support raised-cosine pulses $\text{Rc}(f/f_0, \alpha)$ for *all* $0 \leq \alpha \leq 1$. Clearly a necessary and sufficient condition is

$$M(x) \geq \max_{0 \leq \alpha \leq 1} \text{Rc}(x, \alpha) \quad \text{for all } x$$

But a routine calculation shows that

$$\max_{0\le\alpha\le 1} \mathrm{Rc}(x,\alpha) = \begin{cases} 1 & \text{if } 0 \le |x| < \dfrac{1}{2} \\ \mathrm{Rc}(x) & \text{if } \dfrac{1}{2} < x \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

Thus, a necessary and sufficient condition for $M(x)$ to support all raised cosines is

$$M(x) \ge \begin{cases} 1 & \text{if } 0 \le |x| < \dfrac{1}{2} \\ \mathrm{Rc}(x) & \text{if } \dfrac{1}{2} < x \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

Figures 7 and 8 show this function on a linear and a dB scale. Figure 8 also shows a portion of two high-rate SFGC masks (21-2 and 21-4-3) compared to the universal raised cosine, and shows that all raised cosines fit under these masks. ❐

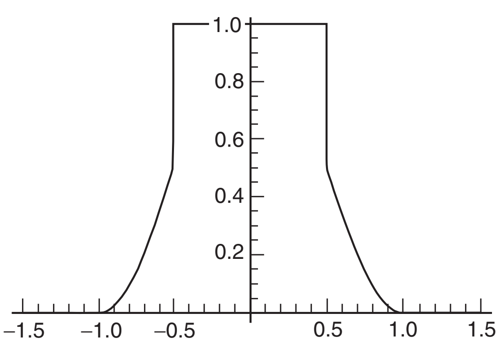

Fig. 7. The universal raised-cosine function.

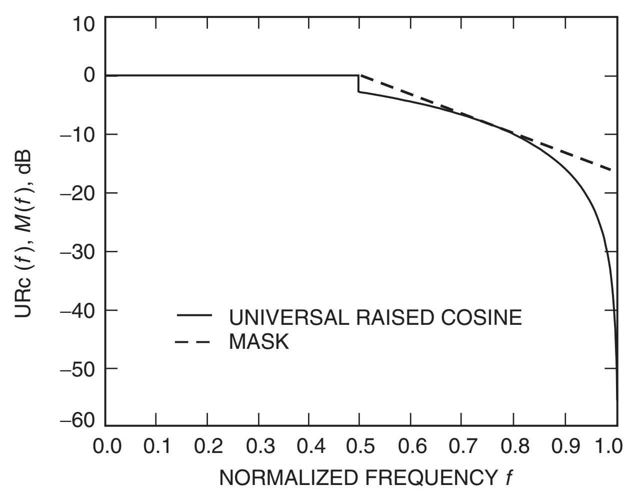

Fig. 8. The universal raised-cosine function compared to the SFCG masks 21-2 and 21-4-3.

V. Conclusions

In this article, we have investigated the problem of avoiding intersymbol interference in a pulse-amplitude modulation communication system that is subject to a mask constraint of the form given in Eq. (12). Since in an unconstrained PAM system ISI is avoided if and only if the pulse shape $p(t)$ satisfies Nyquist's criterion [Eq. (6)], the problem boils down to asking whether a Nyquist pulse $p(t)$ fits under a mask $M(x)$. Our main result (Theorem 1) allows us to determine whether or not *some* Nyquist pulse fits under a given mask. On a more practical note (Example 8), we also determine necessary and sufficient conditions which guarantee that given mask $M(x)$ permits the use of the class of raised-cosine Nyquist pulses, and show that all SFGC masks have this desirable property.

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