

Non-Parametric Evaluation of Lifetime Data

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The proposed large array of small receiving stations for the DSN requires very reliable systems. Reliability can be estimated from life tests of the critical components of the system. This article is a tutorial discussion of the application of a non-parametric approach, known as the product-limit estimate, to lifetime data.

I. Introduction

The development of a very large array of small antennas for the DSN is under consideration. The proposed array may contain thousands of receiving stations, with initial estimates of 3,600 at each of three longitudes. As currently envisioned, each station will use a closed-cycle refrigerator (CCR) to cool the low-noise amplifiers and microwave feed components. To aid in the selection process, some statistical methods of evaluating the CCR reliability are necessary. If lifetime data are not available from the manufacturer, a statistically significant number of coolers should be purchased for lifetime evaluation. If an accelerated life test is required, the methods described in this article may be applied to the components of the CCR thought to be critical to its lifetime. In that case, some further analysis must be done to derive the overall system reliability from the component reliability.

This article discusses one approach that might be used to evaluate lifetime data. It is a non-parametric, sometimes called distribution-free, evaluation of the survivor function. We will use the term “survivor function” rather than the equivalent term “reliability function.” This method is widely used in biomedical applications. Non-parametric methods, unlike parametric methods, are not designed around a particular probability distribution. This is advantageous if the population under investigation does not follow a particular distribution. A good statistical analysis may well include both parametric and non-parametric approaches.

We assume the CCR manufacturer can produce thousands of nearly identical units (or components) whose lifetimes will have some variation. These may be due to materials or manufacturing process variations. It is desired to estimate the survivability of the production line population from the survivability of samples drawn from it. In addition to calculating the survivor function, we will address the calculation of confidence intervals and mean lifetime. An attempt has been made to keep the formalism as general as possible. Another goal has been to rely heavily on graphing of the data. Graphs are a useful way of presenting the data, especially if one is looking for trends. We also briefly discuss the problem of multiple causes of failure, sometimes referred to as the competing risk problem.

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The organization of the article is the following. Section II discusses censoring and other basic definitions and concepts of reliability theory. Section III discusses the product-limit estimate for the survivor function. Section IV discusses the confidence interval, and Section V discusses the restricted mean lifetime. Section VI presents an introduction to the analysis of multiple causes of failure. Section VII gives examples of the formalism discussed in the article, and the final section presents some conclusions. Fortran programs, written by the author, were used to calculate the confidence intervals and mean lifetime.

II. Basic Concepts

In life tests, observation of the time of failure may be prevented for some of the samples by the occurrence of some other event. These events may be the result of an accident or a planned decision to terminate testing of the unit before it fails. “Non-failure” events usually are called “losses” or “censoring times.” It usually is assumed that the failure times are independent of the censoring times. Even though the resulting data set is incomplete, it is desirable to estimate the proportion, $S(t)$, of items in the population whose lifetimes would exceed the time t in the absence of such losses.

If all the units run until failure, then the data are called “complete.” If some of the units have not failed by the end of the test, but the running time for all the unfailed units is the same, the data are termed “singly censored.” If the running time of the unfailed units is greater than the last failure time, the data are termed “time censored.” In this case, the number of failures and their times are random, but not the length of the test. If the running time of the unfailed units is the same as the last failure time, the data are termed “failure censored.” In this case, the duration of the test and the failure times are random, but not the number of failures. Time censoring is more common in practice. Failure censoring is mathematically easier to handle. Incomplete data with failure censoring and time censoring intermixed are termed “multiply censored.”

The procedures described in this article may not apply to every situation. A careful examination of the type of data and the approach used to calculate statistical properties of the data is important. Later in this article, we will also associate with each failure time a discrete variable that labels the mode of failure. This leads to the complex subject of “competing risk,” into which we will make a brief excursion. The prospect of multiple failure modes is quite likely for CCRs. They are known to be susceptible to wear, contamination, fatigue, and leakage of helium gas.

We are interested in the random variable, \mathbf{T} , representing the time to failure. Following normal convention, random variables will be denoted by bold-face type. A random variable is a numerical-valued function defined on a sample description space. The sample description space is the set of all possible outcomes of an experiment or trial. In lifetime studies, the outcome is the time to failure of a system or component. The numerical value associated with the random variable \mathbf{T} will be denoted by t .

The cumulative distribution function is the probability that the lifetime \mathbf{T} is less than or equal to t , and is also the integral of the probability density function $f(t)$,

$$\Pr \{ \mathbf{T} \leq t \} = F(t) = \int_0^t f(x)dx \quad (1)$$

The survivor (or reliability) function is the probability that the lifetime is greater than or equal to t and is given by

$$S(t) = \Pr \{ \mathbf{T} \geq t \} = 1 - F(t) = \int_t^\infty f(x)dx \quad (2)$$

The mean time to failure (MTTF) is expressed in terms of the survivor function by

$$MTTF = \mu = \int_0^{\infty} S(t)dt \quad (3)$$

Another useful function for lifetime analysis is the hazard function. It specifies the instantaneous rate of failure given that the component or system has survived to time t . It is a conditional probability and is given by

$$h(t) = \lim_{\Delta t \rightarrow 0} \frac{\Pr \{t \leq \mathbf{T} < t + \Delta t | \mathbf{T} \geq t\}}{\Delta t} = \frac{f(t)}{S(t)} \quad (4)$$

The connection with the physical causes of the lifetime may be more readily tied to the hazard function than the cumulative probability distribution function or survivor function. For example, a failure rate that increases with the age of the CCR usually indicates that the failures are due to wear. A failure rate that decreases with age usually indicates that the failures are due to manufacturing defects.

Another useful function, especially when there are multiple causes of failure, is the cumulative hazard function, given by

$$H(t) = \int_{-\infty}^t h(x)dx = -\ln [1 - F(t)] = -\ln [S(t)] \quad (5)$$

III. The Product-Limit Estimate of the Survivor Function

The estimate used in this article is called the product-limit estimate. It is a limiting case of the actuarial estimate. Actuarial methods are best suited for large samples. The method described here is better suited for small samples, since individual, rather than grouped, failure times are used. We assume there are N individuals under study and associated with each is either a lifetime, t_i , or a loss time, L_i . Lifetimes correspond to actual failures, and loss times correspond to removal of the unit for reasons other than failure. The time scale is broken up into a number of intervals (t_{j-1}, t_j) . For each interval, one estimates the proportion of units alive just after time t_{j-1} that survive to just beyond time t_j . The product-limit estimate of the survivor function is built up as a product. Each term in the product is a conditional probability of survival. The conditional probability used in the product-limit estimate assumes no functional form. If the number of units remaining at risk just after t_{j-1} is n_j , and there are d_j deaths in the interval (t_{j-1}, t_j) , then the conditional probability of survival to time t_j , given survival to time t_{j-1} , is

$$\hat{p}_j = \frac{\hat{S}_j}{\hat{S}_{j-1}} = \frac{n_j - d_j}{n_j} \quad (6)$$

The survivor function is built up as the product of these conditional probabilities:

$$\hat{S}(t) = \prod_{j:t_j < t} \hat{p}_j = \prod_{j:t_j < t} \frac{n_j - d_j}{n_j} \quad (7)$$

If a unit is removed for reasons other than failure, a death, d_j , is not recorded, but the number of units at risk, n_j , is affected. If there are no censored observations (losses), the survivor function can be simply estimated by

$$\hat{S}(t) = \frac{\text{number of lifetimes } \geq t}{N} \quad (8)$$

This is a step function that decreases by d/N if there are d failures at time t . This is a very intuitive definition of the survivor function. The product-limit estimate is discussed in detail by Kaplan and Meier [1].

IV. Confidence Interval Estimate of the Survivor Function

Section III discussed the evaluation of the survivor function without assuming any particular form for the distribution and also allowed for removal of units under test without failure. Equation (7) provides a single estimate. However, this estimate is of limited value without some measure of its uncertainty. The uncertainty of the estimate is provided by a quantity called a “confidence interval.” It is calculated from the sample data. A confidence interval for the survivor function can be calculated for any time t_o . It consists of a lower limit and an upper limit, and these limits depend on the confidence level associated with the interval. We will denote the interval by $[S_L(t_o), S_U(t_o)]$. Its meaning is best thought of in terms of its “coverage.” Coverage probability reflects the uncertainty in the sampling procedure; 90 percent coverage means that if many samples are taken, or lifetime experiments conducted, 90 percent of the calculated confidence intervals will cover the true parameter. This is not the same as saying that the true value of the parameter has a 90 percent probability of lying within the calculated interval. That probability is either zero or one. It either does or does not lie within the interval. Intervals help the experimenter decide if enough data have been taken.

The reason for estimating intervals rather than points is to have some guarantee of capturing the parameter of interest. Nelson [2] also discusses some reasons for calculating confidence intervals. They are usually wider than the experimenter might expect. This can help prevent the experimenter from believing the estimates are closer to the true value than they really are. In fact, the uncertainty may actually be greater than the confidence interval indicates. The confidence interval calculation assumes a random sample and assumes the sample represents the population of interest. If these assumptions are not strictly true, the results are more uncertain than the calculations indicate. Confidence intervals can be computed for any level of confidence. Common levels are 90, 95, and 99 percent.

The problem of calculating a confidence interval can be quite technical. It is closely related to another topic in statistics, that of hypothesis testing. In general, every confidence interval corresponds to a hypothesis test and vice versa. We will not cover this matter in any detail, but only outline the origin of some of the formulas to be presented. Depending on the type of censoring, an exact method may be found, and other times large sample approximations are necessary. Later in this section when we discuss censored data, we will discuss a powerful technique for calculating confidence intervals for moderate to large samples based on the likelihood-ratio test.

We begin by presenting an “exact” confidence interval calculation for the case of complete data. It is exact in the sense that the procedure holds for any number of units under test. The case of complete data is important in its own right, but it also serves as a means of testing formulas developed for censored data. In the limit where the number of censored items falls to zero, the confidence interval for censored data should closely approximate the confidence interval for complete data for reasonably sized samples.

A. Uncensored (Complete) Data

If the data are uncensored, techniques based on the binomial distribution can be applied. The binomial distribution is based on the concept of a Bernoulli trial. A Bernoulli trial is an experiment with exactly two possible outcomes, one and only one of which must occur. A random variable \mathbf{X} with a Bernoulli distribution can take on the value one or zero. The probability of success ($x = 1$) is usually denoted p , and the probability of failure ($x = 0$) is $(1 - p)$. Many experiments can be modeled as a sequence of Bernoulli trials. It's important to note that our calculation for the interval is based only on survival/non-survival at time t_o . We do not use the actual times to failure that may have been observed earlier, only their number. It turns out that for the binomial distribution $\mathbf{Y} = \sum \mathbf{X}_i$ is a sufficient statistic for p . This means that the total number of 1's in the sample, that is, the number of survivors at time t_o , contains all the information about p that is in the data. Other features of the data, such as which x_i are equal to one and which are equal to zero, contain no additional information about p .

Therefore, a confidence interval can be calculated based on the random variable $\mathbf{Y} = \sum \mathbf{X}_i$. It also follows a binomial distribution. That is, its probability density is given by

$$f(Y) = \binom{n}{Y} p^Y (1 - p)^{n-Y} \quad (9)$$

Let the number of observed surviving units at time t_o , based on the sample, be given by y_c . The probability mass function $f(Y)$ is easily calculated for any values of n and p and generally has a bell shape. As the probability of survival increases, the bell-shaped curve will move towards the right, indicating a larger number of survivors. The lower limit on the survivor function is the smallest value of p such that the probability that $Y \geq y_c$ is equal to some small number, usually called α . This implies

$$\Pr\{\mathbf{Y} \geq y_c\} = \sum_{Y=y_c}^n \binom{n}{Y} p^Y (1 - p)^{n-Y} = \alpha \quad (10)$$

The summation in Eq. (10) can be written using the incomplete beta function, I_p . That is,

$$\sum_{i=y_c}^n \binom{n}{i} p^i (1 - p)^{n-i} = I_p(y_c, n - y_c + 1) \quad (11)$$

The smallest value of p such that

$$I_p(y_c, n - y_c + 1) = \alpha \quad (12)$$

is the lower confidence limit on the survivability, i.e., $S_L(t_o) = p$. For the upper confidence limit on the survivability, we need to find the largest value of p such that

$$\Pr\{Y \leq y_c\} = \sum_{i=0}^{y_c} \binom{n}{i} p^i (1 - p)^{n-i} = \alpha \quad (13)$$

Since the total area under the probability density function is unity, we can rewrite Eq. (13) in terms of the incomplete beta function:

$$\Pr\{\mathbf{Y} \geq y_c + 1\} = \sum_{i=y_c+1}^n \binom{n}{i} p^i (1-p)^{n-i} = I_p(y_c + 1, n - y_c) = 1 - \alpha \quad (14)$$

Therefore, the value of p that satisfies Eq. (14) is the upper confidence limit, i.e., $S_U(t_o) = p$. Fortran subroutines available in [3] can be used to evaluate the incomplete beta function. An example of this calculation is presented in Section VII.

By way of introduction, the same problem can be solved using the likelihood-ratio technique. The procedure is to construct the maximum-likelihood function given a hypothesis, called H_o , and a suitable alternative hypothesis, called H_1 . Then form the ratio, L_R , of the two functions,

$$L_R = \frac{L(x_1 \cdots x_n | H_o)}{L(x_1 \cdots x_n | H_1)} \quad (15)$$

Here $L(x_1, \cdots, x_n | H_o)$ and $L(x_1, \cdots, x_n | H_1)$ are the maximum-likelihood functions of the sample when the hypotheses H_o and H_1 , respectively, are true. Then for moderate to large samples, the quantity $\Lambda = -2 \ln(L_R)$ follows a χ^2 distribution with one degree of freedom. This is a powerful result from the theory of statistical inference.

For the problem at hand, let the hypothesis H_o be that $S(t_o) = p_o$, and the hypothesis H_1 be that $S(t_o) = p; p \neq p_o$. The likelihood function for H_o is given by $L = f(x_1 \cdots x_n; p_o) = p_o^{\sum x_i} (1 - p_o)^{n - \sum x_i}$. By the hypothesis, p is restricted to be equal to p_o . The likelihood function for H_1 is given by $L = f(x_1 \cdots x_n; p) = \hat{p}^{\sum x_i} (1 - \hat{p})^{n - \sum x_i}$, where \hat{p} is the maximum-likelihood estimate of p without restrictions. The likelihood ratio is

$$L_R = \frac{p_o^{\sum x_i} (1 - p_o)^{n - \sum x_i}}{\hat{p}^{\sum x_i} (1 - \hat{p})^{n - \sum x_i}} \quad (16)$$

Taking the logarithm of both sides leads to

$$\Lambda = -2 \ln L_R = 2 \left(\sum x_i \right) \ln \left(\frac{\hat{p}}{p_o} \right) + 2 \left(n - \sum x_i \right) \ln \left(\frac{1 - \hat{p}}{1 - p_o} \right) \quad (17)$$

Or, since $\hat{p} = (\sum x_i)/n$, we can write

$$\Lambda = -2 \ln L_R = 2n\hat{p} \ln \left(\frac{\hat{p}}{p_o} \right) + 2n(1 - \hat{p}) \ln \left(\frac{1 - \hat{p}}{1 - p_o} \right) \quad (18)$$

Under H_o , Λ is distributed as $\chi_{(1)}^2$ in large samples, and an approximate α confidence interval for p consists of all values p_o such that $\Lambda \leq \chi_{(1),\alpha}^2$. The quantity $\chi_{(1),\alpha}^2$ is the α th quantile of the chi-square distribution with one degree of freedom. A comparison of this approach and the binomial approach is presented in Section VII. The results are similar.

B. Censored Data

If the data are censored, the binomial distribution approach is not valid. We will follow Thomas and Grunkemeier [4] and present a technique for calculating confidence intervals for $S(t_o)$ based on the likelihood-ratio method discussed above. We are testing the hypothesis $H_o : S(t_o) = S_o$ versus the hypothesis $H_1 : S(t_o) \neq S_o$. The likelihood ratio, L_R , is given by the ratio of the constrained maximum-likelihood function, L_c , to the unconstrained maximum-likelihood function, L_u :

$$\Lambda = -2 \ln L_R = -2 \ln \left(\frac{L_c}{L_u} \right) \quad (19)$$

For the product-limit estimate, L is given in general by

$$L = \prod_{j=1}^k (1 - p_j)^{d_j} p_j^{n_j - d_j} \quad (20)$$

For the unconstrained case, $p_j = \hat{p}_j = 1 - (d_j/n_j)$, as mentioned previously in Eq. (7). L_c is the same likelihood function subject to the constraint that

$$S(t_o) = \prod_{j:t_j < t_o} p_j = S_o \quad (21)$$

A Lagrange multiplier, λ , is used to build the constraint into the likelihood function. This leads to the likelihood function

$$L_c = \prod_{j=1}^k (1 - \tilde{p}_j)^{d_j} \tilde{p}_j^{n_j - d_j} \quad (22)$$

where the \tilde{p}_j satisfy

$$\tilde{p}_j = 1 - \frac{d_j}{n_j + \lambda}, \quad j = 1, \dots, l \quad (23)$$

$$\tilde{p}_j = 1 - \frac{d_j}{n_j}, \quad j = l + 1, \dots, k \quad (24)$$

where l is defined by $t_l < t_o < t_{l+1}$. The ratio given in Eq. (19) is

$$\Lambda = 2 \sum_{j=1}^k \left[n_j \ln \left(1 + \frac{\lambda}{n_j} \right) - (n_j - d_j) \ln \left(1 + \frac{\lambda}{n_j - d_j} \right) \right] \quad (25)$$

To get an α confidence interval for $S(t_o)$, we need to determine the set of all $S(t_o)$ such that $\Lambda \leq \chi_{(1),\alpha}^2$. This can be accomplished by finding the set of all λ in Eq. (25) that make $\Lambda \leq \chi_{(1),\alpha}^2$. These λ values form a closed interval $[\lambda_L, \lambda_U]$ where $\lambda_L < 0 < \lambda_U$. Once these are found, the confidence interval for $S(t_o)$ is of the form $[S_L(t_o), S_U(t_o)]$, where $0 < S_L(t_o) < S_U(t_o) < 1$. They are given by

$$\left. \begin{aligned} S_L(t_o) &= \prod_{i=1}^l \left(1 - \frac{d_i}{n_i + \lambda_L}\right) \\ S_U(t_o) &= \prod_{i=1}^l \left(1 - \frac{d_i}{n_i + \lambda_U}\right) \end{aligned} \right\} \quad (26)$$

A Fortran program for performing this calculation has been written. An example is presented in Section VII.

V. Restricted Mean Lifetime

Up to now we have focused on the calculation of the survivor function and an associated confidence interval. Oftentimes one is interested in a single number characterizing the behavior of a unit. As discussed in Eq. (3), the mean lifetime can be calculated if the survivor function is known. The product-limit estimate of the mean lifetime is defined as the mean of the product-limit estimate of the lifetime distribution, where the symbol “ $\hat{\cdot}$ ” denotes the product-limit estimate. That is,

$$\hat{\mu} = \int_0^{\infty} \hat{S}(t) dt \quad (27)$$

If the life testing is such that not all the units have failed, one may wish to estimate the mean life limited to a time L , where L is chosen by the investigator. In this case, the restricted mean lifetime defined by

$$\hat{\mu}_L = \int_0^L \hat{S}(x) dx \quad (28)$$

may be given along with the estimated value of the survivor function at time L . Again, to be useful, some estimate of the uncertainty of the lifetime is necessary. An estimate of the variance (the square of the standard deviation) of the restricted mean lifetime is given by [5]

$$\text{Var}(\hat{\mu}_L) = \sum_{j:t_j \leq L} \frac{\hat{A}_j^2 d_j}{n_j(n_j - d_j)} \quad (29)$$

where

$$\hat{A}_j = (t_{j+1} - t_j)\hat{S}(t_j) + (t_{j+2} - t_{j+1})\hat{S}(t_{j+1}) + \cdots + (L - t_m)\hat{S}(t_m) \quad (30)$$

with t_m being the largest observed lifetime less than or equal to L . An example is presented in Section VII.

VI. Multiple Causes of Failure

If the system under study is capable of failing from several causes, then the problem is more complex. In statistical analysis, this is often referred to as the “competing risks problem.” If the system is capable of failing from 3 different causes, then there are 3 random variables, \mathbf{T}_1 , \mathbf{T}_2 , and \mathbf{T}_3 , that correspond to the lifetime due to failure associated with causes 1, 2, and 3. In a series-type system, any one of the causes is sufficient to bring down the system. In such a case, only the minimum of $\{T_1, T_2, T_3\}$ is observed.

We follow the approach presented in [5]. We associate with each unit under test a pair of numbers (T, C) , where the continuous random variable T is the lifetime and the discrete random variable C is the mode of death. C is assumed to take on one of k different values.

The joint distribution of T and C can be obtained using cause-specific hazard or survivor functions,

$$h_j(t) = \lim_{\Delta t \rightarrow 0} \frac{\Pr(t \leq T \leq t + \Delta t, C = j | T \geq t)}{\Delta t} = \frac{f_j(t)}{S(t)} \quad (31)$$

$$S_j(t) = \Pr\{T \geq t, C = j\} \quad (32)$$

When there is no censoring, estimates of the survivor function for the j th failure mode, $S_j(t)$, are given by an obvious generalization of Eq. (8),

$$\hat{S}_j(t) = \frac{\text{Number of lifetimes with } T \geq t \text{ and } C = j}{N} \quad (33)$$

When the data are censored, the cause-specific survivor function, $\hat{S}_j(t)$, is given by

$$\hat{S}_j(t) = \sum_{\substack{i: t_i \geq t \\ C_i = j}} \frac{d_{ji}}{n_i} \hat{S}(t_i) = \sum_{\substack{i: t_i \geq t \\ C_i = j}} \frac{d_{ji}}{n_i} \left(\prod_{k: t_k < t_i} \frac{n_k - d_k}{n_k} \right) \quad (34)$$

When evaluating Eq. (34), it's easiest to start at the earliest failure (or loss) time and calculate \hat{S} (the product terms) for all times. Then starting at the last failure (or loss) time, work your way back to the beginning, calculating the sum. An example is given in Section VII. If the data are complete, then

$$\hat{S}(t) = \sum_{j=1}^k \hat{S}_j(t) \quad (35)$$

If the data are censored, $\hat{S}(t)$ will be less than the sum of the $\hat{S}_j(t)$ terms.

The $\hat{S}_j(t)$ are useful for estimating probabilities related to the cause of death. For example,

$$\Pr\{C = j\} = \hat{S}_j(t = 0) \quad (36)$$

If there is censoring, it is better to redefine $\hat{S}_j(0)$ by

$$\hat{S}_j(0) \rightarrow \frac{\hat{S}_j(0)}{\sum_j \hat{S}_j(0)} \quad (37)$$

For general time, t , we use

$$\Pr\{C = j | T \geq t\} = \frac{\hat{S}_j(t)}{\hat{S}(t)} \quad (38)$$

If it is desired to plot cause-specific cumulative hazard functions, we first calculate a pseudo-survivor function, again using a product-limit estimate. It is given by

$$\hat{G}_j(t) = \prod_{\substack{i:t_i < t \\ C_i=j}} \frac{n_i - d_{ji}}{n_i} \quad (39)$$

where d_{ji} is the number of individuals dying from cause j at t_i . Then the cause-specific cumulative hazard function is given by

$$\hat{H}_j(t) = -\ln(\hat{G}_j(t)) \quad (40)$$

These hazard functions are useful for examining the relative behavior of the different causes of death. They show how the relative risk of failure from different causes varies with time. The composite cumulative hazard function is given simply by the sum of the cause-specific hazard functions. That is,

$$\hat{H}(t) = -\ln(\hat{S}(t)) = \sum_{j=1}^k \hat{H}_j(t) \quad (41)$$

It is easier to assess from such plots whether the hazard function is constant, increasing with time, or decreasing with time. The cumulative hazard function, $H(t)$, is linear with time if $h(t)$ is constant. It is convex upward if $h(t)$ is increasing with time and convex downward if $h(t)$ is decreasing with time. The cumulative hazard function also serves to smooth the data.

VII. Examples

We now illustrate the above formalism with an example. Consider 24 CCRs put on life test. Imagine the refrigerators were run until they all failed. Let the failures times in thousands of hours be given by 2.0, 3.25, 4.1, 5.6, 6.05, 8.01, 8.9, 9.25, 9.4, 9.8, 10.1, 10.6, 12.0, 12.5, 12.9, 13.3, 13.6, 14.0, 15.0, 15.5, 16.9, 18.0, 18.5, and 20.0. These appear in column 1 of Table 1. The survivor function is calculated using Eq. (7), and the cumulative hazard function is calculated using Eq. (5). The survivor function and cumulative hazard function are shown in the columns labeled S_1 and H_1 , respectively. A plot of the survivor function is shown in Fig. 1. (For clarity purposes, the line segments are omitted in Fig. 5.)

Next consider the case where the failures times are 2.0, 3.25, 5.6, 6.05, 8.01, 8.9, 9.25, 9.8, 10.1, 12.0, 12.5, 12.9, 13.3, 15.0, 15.5, and 16.9. Let the censored (loss) times be 4.1, 9.4, 10.6, 13.6, 14.0, 18.0, 18.5, and 20.0. That is, for one reason or another, the life tests on these units were stopped, but the unit was still running when the test was terminated. The survivor function and cumulative hazard function are shown in Table 2 under the columns labeled S_2 and H_2 , respectively. A plot of the survivor function is shown in Fig. 2 as the red circles. This is to be contrasted with the behavior seen if the censored data are not used. The survivor function and cumulative hazard function are shown in Table 3, and the survivor function is plotted in Fig. 2 as the blue squares. Ignoring the censored data, especially in the latter stages of the life test, makes a substantial error.

Confidence intervals can also be calculated for the data shown in Table 1 using some computer programs developed by the author. We can calculate lower and upper confidence limits for the survivor function. Suppose we are interested in 90 percent confidence limits at 9,000 hours, 13,000 hours, and 16,000 hours. They are given in the first two rows of Table 4, calculated using Eqs. (12) and (14). The last two rows

Table 1. Survivor and cumulative hazard functions for 24 units with complete failure.

Failure time, kh	S_1	H_1
0.0	1.0	0.000
2.0	0.958	0.043
3.25	0.917	0.087
4.1	0.875	0.134
5.6	0.833	0.183
6.05	0.792	0.233
8.01	0.750	0.288
8.9	0.708	0.345
9.25	0.667	0.405
9.4	0.625	0.470
9.8	0.583	0.540
10.1	0.542	0.612
10.6	0.500	0.693
12.0	0.458	0.781
12.5	0.417	0.875
12.9	0.375	0.981
13.3	0.333	1.100
13.6	0.292	1.231
14.0	0.250	1.386
15.0	0.208	1.570
15.5	0.167	1.790
16.9	0.125	2.080
18.0	0.083	2.489
18.5	0.042	3.170
20.0	0.000	—

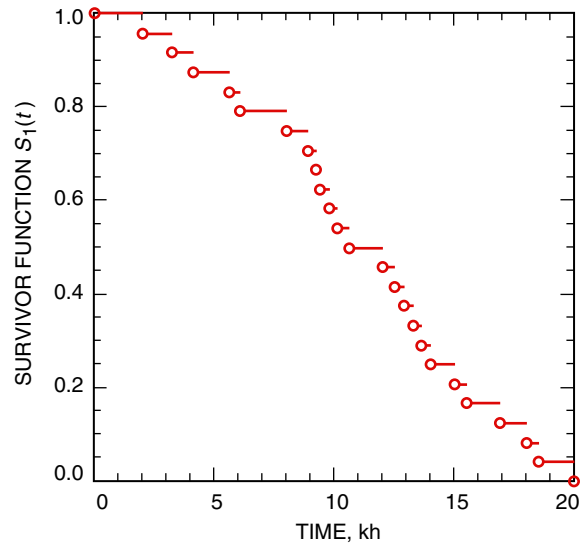


Fig. 1. Survivor function corresponding to the data in Table 1.

Table 2. Survivor and cumulative hazard functions for 24 units, 8 of which are censored units.

Failure time, kh	Loss time, kh	S_2	H_2
0.0	—	1.000	0.000
2.0	—	0.958	0.043
3.25	—	0.917	0.087
—	4.1	0.917	0.087
5.6	—	0.873	0.136
6.05	—	0.829	0.188
8.01	—	0.786	0.241
8.9	—	0.742	0.298
9.25	—	0.698	0.360
—	9.4	0.698	0.360
9.8	—	0.652	0.428
10.1	—	0.605	0.503
—	10.6	0.605	0.503
12.0	—	0.555	0.589
12.5	—	0.504	0.685
12.9	—	0.454	0.790
13.3	—	0.404	0.906
—	13.6	0.404	0.906
—	14.0	0.404	0.906
15.0	—	0.336	1.091
15.5	—	0.269	1.313
16.9	—	0.202	1.600
—	18.0	0.202	1.600
—	18.5	0.202	1.600
—	20.0	0.202	1.600

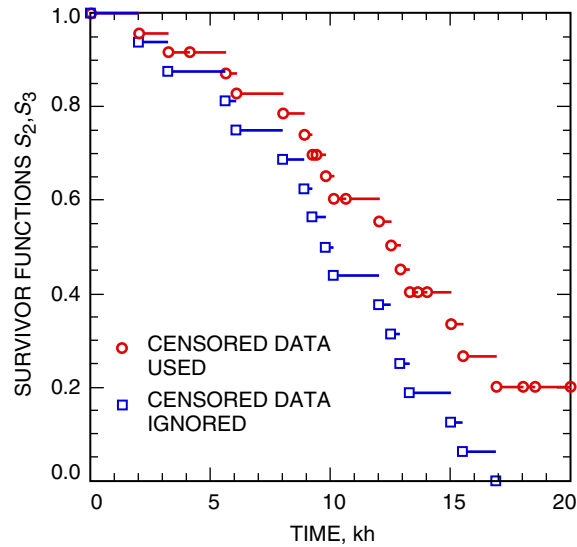


Fig. 2. Survivor functions for the data in Tables 2 and 3.

Table 3. Survivor and cumulative hazard functions for the data of Table 2, when the censored data are ignored.

Failure time, kh	S_3	H_3
0.0	1.000	0.000
2.0	0.938	0.064
3.25	0.875	0.134
5.6	0.813	0.207
6.05	0.750	0.288
8.01	0.688	0.374
8.9	0.625	0.470
9.25	0.563	0.574
9.8	0.500	0.693
10.1	0.438	0.826
12.0	0.375	0.981
12.5	0.313	1.162
12.9	0.250	1.386
13.3	0.188	1.671
15.0	0.125	2.080
15.5	0.063	2.765
16.9	0.000	—

Table 4. Confidence limits of 90 percent for $S(t)$ at 9,000, 13,000, and 16,000 hours for the complete data of Table 1.

9,000 h	13,000 h	16,000 h	Confidence limits
0.521	0.211	0.059	S_L [using Eq. (12)]
0.854	0.563	0.342	S_U [using Eq. (14)]
0.512	0.202	0.056	S_L [using Eq. (18)]
0.862	0.574	0.346	S_U [using Eq. (18)]

of Table 4 give calculated upper and lower limits on the survivor function using the likelihood-ratio method and Eq. (18). The two approaches agree rather closely.

In order to get a better feeling for what these confidence intervals mean, consider the complete data of Table 1 at 13,000 hours. At this point in time, there are 9 survivors. If we plot the binomial probability mass function for $p = 0.211$ (see Table 4), we get the bar chart shown in Fig. 3(a). The area under the curve for 9 or more survivors is 0.0492, very close to 0.05. This corresponds to Eq. (10). If we plot the binomial probability mass function for $p = 0.563$ (see Table 4), we get the bar chart shown in Fig. 3(b). The area under the curve for 10 or more survivors is 0.95. This corresponds to Eq. (14). Finally, we calculate the upper and lower 90 percent confidence limits for the survivor function for the censored data of Table 2 using Eqs. (25) and (26). The results are shown in Table 5.

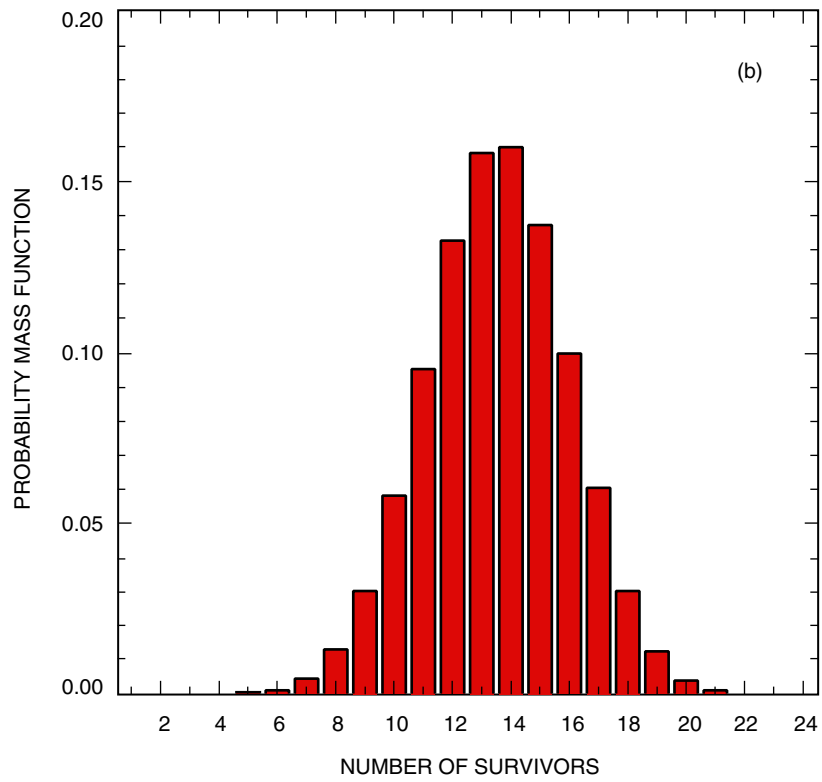
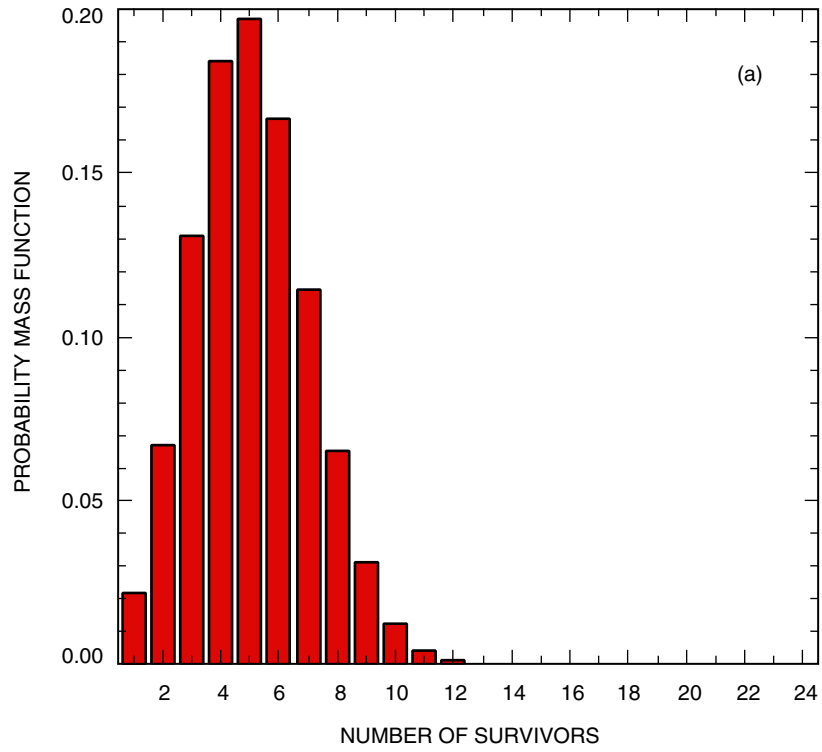


Fig. 3. Probability mass function for (a) $p = 0.211$ and (b) $p = 0.563$. There are 24 units under test.

Table 5. Confidence limits of 90 percent for $S(t)$ at 9,000, 13,000, and 16,000 hours for the censored data of Table 2.

9,000 h	13,000 h	16,000 h	Confidence limits
0.578	0.284	0.120	S_L [using Eqs. (25), (26)]
0.870	0.629	0.455	S_U [using Eqs. (25), (26)]

A mean-time-to-failure example will be presented next. For the data of Table 1, calculating the mean time to failure using Eq. (28) and simply calculating the average of the failures times give the same result. The mean lifetime is 11,219 hours. The variance of the lifetime using Eq. (29) is 951,015 hours², corresponding to a standard deviation equal to 975 hours. This same result can be obtained by using the standard formula

$$\text{variance} = \sum_i \frac{(T_i - T_{\text{mean}})^2}{N^2} \quad (42)$$

For the data in Table 2, which correspond to censored data, the mean time to failure using Eq. (28) is 12,576 hours. The variance using Eq. (29) is 1,296,410 hours², corresponding to a standard deviation of 1,139 hours.

Finally, we examine a multiple-failure-mode example. For this case, we take the same data used in Table 1 and assign a failure code for each observed failure time. The data are shown in Table 6. In this example, we assume there are only 2 modes of failure. The failure mode is given in the column labeled C . The column labeled S contains the product-limit estimate of the survivor function without regard to the cause of failure. It is calculated using Eq. (7). The cause-specific survivor functions are calculated using Eq. (34). In this case, the table is generated starting at the bottom. Figure 4 shows plots of the 3 survivor functions. It is clear that failures due to mode 1 steadily reduce the number of operating refrigerators until about 14,000 hours. Failures due to cause 2 do not begin to reduce the number of operating refrigerators until about 7,000 hours, and then the decrease is roughly linear out to about 18,000 hours. The percentage of failures due to cause 1 is given using Eq. (37) as $S_1(0) = 0.36/(0.36 + 0.48) = 0.43$. Similarly, the percentage of failures due to cause 2 is given by $S_2(0) = 0.48/(0.36 + 0.48) = 0.57$. These numbers agree with the fact that 7 of the 16 failures are due to cause 1 and 9 of the 16 failures are due to cause 2.

The cause-specific pseudo-survivor functions and cause-specific cumulative hazard functions are calculated using Eqs. (39) and (40), respectively. The total cumulative hazard function is calculated using Eq. (41). The 3 cumulative hazard functions are plotted in Fig. 5. The cause 1 cumulative hazard function is slightly convex upward to about 13,000 hours, indicating a slightly increasing rate of failure with time. The hazard rate of cause 2 is steadily increasing after about 7,000 hours. The total cumulative hazard function alone shows a marked change in slope at about 9,000 hours. This can often be an indicator that another failure mode is becoming dominant. This is not so apparent from the total survivor function shown in Fig. 4. The total cumulative hazard function is the sum of the cause-specific cumulative hazards, even if the data are censored.

Table 6. Survivor functions (total and cause-specific), pseudo-survivor functions, and cumulative hazard functions (total and cause-specific) for failure data with 2 modes of failure.

Time, kh	C	S	S_1	S_2	G_1	G_2	H_1	H_2	H
0.0	—	1.000	0.36	0.48	1.000	1	0	0	0
2.0	1	0.958	0.318	0.48	0.958	1	0.0429	0	0.0429
3.25	1	0.917	0.277	0.48	0.917	1	0.0866	0	0.0866
4.1	—	0.917	0.235	0.48	0.917	1	0.0866	0	0.0866
5.6	1	0.873	0.235	0.48	0.873	1	0.136	0	0.136
6.05	2	0.829	0.191	0.48	0.873	0.95	0.136	0.0513	0.188
8.01	1	0.786	0.191	0.436	0.827	0.95	0.190	0.0513	0.241
8.9	2	0.742	0.148	0.436	0.827	0.897	0.190	0.109	0.298
9.25	2	0.698	0.148	0.393	0.827	0.844	0.190	0.17	0.36
9.4	—	0.698	0.148	0.349	0.827	0.844	0.190	0.17	0.36
9.8	1	0.652	0.148	0.349	0.772	0.844	0.259	0.17	0.428
10.1	2	0.605	0.101	0.349	0.772	0.784	0.259	0.243	0.503
10.6	—	0.605	0.101	0.303	0.772	0.784	0.259	0.243	0.503
12.0	1	0.555	0.101	0.303	0.708	0.784	0.345	0.243	0.589
12.5	2	0.504	0.051	0.303	0.708	0.713	0.345	0.338	0.685
12.9	2	0.454	0.051	0.252	0.708	0.642	0.345	0.443	0.790
13.3	1	0.404	0.051	0.202	0.629	0.642	0.464	0.443	0.906
13.6	—	0.404	0	0.202	0.629	0.642	0.464	0.443	0.906
14.0	—	0.404	0	0.202	0.629	0.642	0.464	0.443	0.906
15.0	2	0.336	0	0.202	0.629	0.535	0.464	0.625	1.09
15.5	2	0.269	0	0.135	0.629	0.428	0.464	0.849	1.31
16.9	2	0.202	0	0.067	0.629	0.321	0.464	1.14	1.6
18.0	—	0.202	0	0	0.629	0.321	0.464	1.14	1.6
18.5	—	0.202	0	0	0.629	0.321	0.464	1.14	1.6
20.0	—	—	0	0	0.629	0.321	0.464	1.14	—

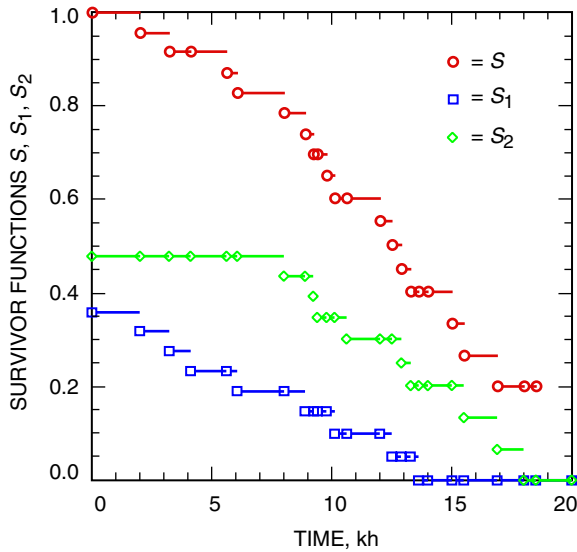


Fig. 4. Cause-specific and total survivor functions for the data in Table 6.

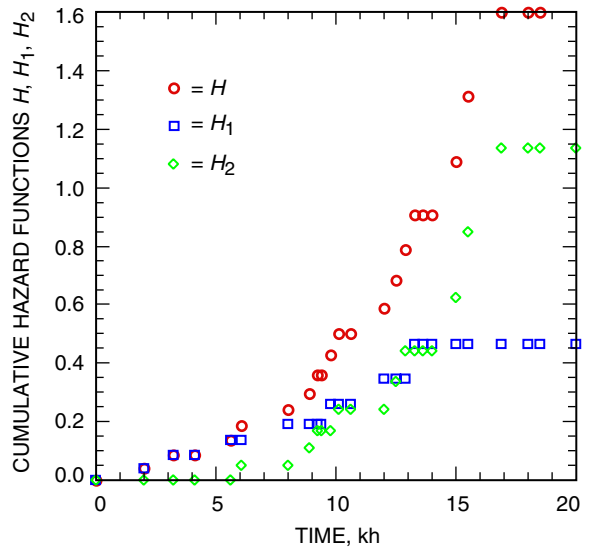


Fig. 5. Total and cause-specific cumulative hazard functions for the data in Table 6.

VIII. Conclusion

The problem of evaluating lifetime data for components of the proposed large array of microwave receiving stations was discussed. We presented a non-parametric analysis, known as the product-limit estimate, that applies to complete and censored data. We chose methods that do not require large samples in general. The calculations of the product-limit estimate of the survivor function, cumulative hazard function, confidence intervals, and mean time to failure were presented. A brief discussion of data with multiple causes of failure was presented. This included cause-specific survivor functions and cause-specific cumulative hazard functions.

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