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Parametric Evaluation of Lifetime Data

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The proposed large array of small antennas for the DSN requires very reliable systems. Reliability can be estimated from life tests of the critical components of the system. This article is a tutorial introduction of a commonly used parametric distribution for lifetime analysis, known as the Weibull distribution.

I. Introduction

The development of a very large array of small antennas for the DSN is under consideration. Each receiving station will likely be equipped with a closed-cycle refrigerator (CCR) to cryogenically cool the low-noise amplifiers and microwave feed components. A previous article examined a non-parametric approach, known as the product-limit (PL) estimate, to the analysis of lifetime data [1]. This article considers the application of a parametric probability distribution, known as the Weibull distribution, to the analysis of lifetime data. Others have analyzed CCR lifetime data using this distribution [2,3]. The Weibull distribution is popular because of its versatility. Depending on the values of the parameters, an increasing, constant, or decreasing hazard rate can be modeled. There is both a two-parameter and a three-parameter version of the Weibull distribution. This article will be concerned with the two-parameter version.

The article is organized as follows. Section II presents the Weibull distribution and a closely related distribution known as the extreme-value distribution. Section III discusses graphical analysis and goodness-of-fit tests. These techniques can indicate if the use of a Weibull distribution is not appropriate. Section IV presents maximum-likelihood point estimates for the Weibull parameters. Section V discusses the calculation of confidence intervals for the survivor function. A few examples are presented in Section VI using computer programs developed by the author. Section VII presents some conclusions.

II. Definition of the Weibull and the Extreme-Value Distribution

The Weibull probability density function (pdf) has the form

$$f(t,\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \exp\left(-\left(\frac{t}{\alpha}\right)^{\beta}\right), \quad t \geq 0, \beta > 0, \alpha > 0 \tag{1}$$

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where t represents time; α is called the scale parameter and determines the spread of the distribution. It has the dimensions of time, and $t = \alpha$ corresponds to the 63rd percentile of the distribution. This property is seen by integrating Eq. (1):

$$\int_0^{\alpha} f(t,\alpha,\beta)dt = -\exp\left(-\frac{t}{\alpha}\right)^{\beta}\Bigg|_0^{\alpha} = 1 - \frac{1}{e} = 0.632$$

β is called the shape parameter and is dimensionless. For $\beta = 1$, the Weibull distribution reduces to the exponential distribution. For $3 \leq \beta \leq 4$, the Weibull distribution resembles the normal distribution.

From the basic definition of the survivor (reliability) function,

$$S(t) = \Pr\{\mathbf{T} \geq t\} = \int_t^{\infty} f(y)dy$$

the survivor function $S(t)$ for the Weibull distribution is given by

$$S(t,\alpha,\beta) = \exp\left(-\left(\frac{t}{\alpha}\right)^{\beta}\right), \quad t \geq 0 \tag{2}$$

From the definition

$$h(t,\alpha,\beta) = \frac{f(t,\alpha,\beta)}{S(t,\alpha,\beta)}$$

the hazard function for the Weibull distribution is given by

$$h(t,\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}, \quad t \geq 0 \tag{3}$$

If $\beta > 1$, the hazard function increases with time; if $\beta = 1$, it is independent of time; and if $\beta < 1$, it decreases with time.

Weibull-distributed data can be converted to extreme-value-distributed data using

$$\left.\begin{aligned} x &= y^{-1}(t) = \ln(t) \\ b &= \frac{1}{\beta} \\ u &= \ln(\alpha) \end{aligned}\right\} \tag{4}$$

The new probability density function can be calculated from the Weibull pdf using

$$g(x) = f\big(y(x)\big)\left|\frac{dy}{dx}\right|$$

This leads to the extreme-value probability density function

$$g(x,u,b) = \frac{1}{b}\exp\left(\frac{x-u}{b}\right)\exp\left(-\exp\left(\frac{x-u}{b}\right)\right), \quad -\infty < x < \infty, -\infty < u < \infty, b > 0 \tag{5}$$

Now the shape parameter is gone, scale and location parameters remain. The goodness-of-fit tests and confidence intervals presented later are based on this extreme-value distribution. The probability density is a function of only one variable,

$$w = \frac{x-u}{b}$$

where u is a location parameter and b is a scale parameter. The survivor function is given by

$$S(w) = \exp\big(-\exp(w)\big) \tag{6}$$

The hazard function is given by

$$h(w,b) = \frac{1}{b}\exp(w), \quad -\infty \le w \le \infty \tag{7}$$

III. Checking for Weibull Behavior

Before using a particular parametric model, some determination must be made as to whether or not the sample could realistically have come from a population governed by the chosen model. Two approaches used for this are graphing the data and goodness-of-fit tests. In general, it is also a good idea to try another model to see how much the conclusions change. A comparison with a non-parametric analysis may be worthwhile. It should be stressed that these procedures are designed to reject a model as inappropriate; they cannot "prove" that the model chosen is correct. The decision of whether or not a given model is valid is highly dependent on the amount of data. The more data, the better are the chances of rejecting an inappropriate model. Generally, if only a few data points are available, even poorly fitting models cannot be strongly rejected.

A. Graphing the Data

We begin with a common technique for checking whether the data follow a Weibull distribution. From Eq. (2), it is seen that

$$\ln\big(-\ln\big[S(t,\alpha,\beta)\big]\big) = \beta\ln t - \beta\ln\alpha \tag{8}$$

Thus, if the data follow a Weibull distribution, a plot of $\ln(-\ln[S(t)])$ versus $\ln(t)$ should approximate a straight line. One approach is to calculate the product-limit estimate of the survivor function, $S_{PL}(t)$, and plot $\ln(-\ln[S_{PL}(t)])$ versus $\ln(t)$. If the plot bows up or down significantly from a straight line, then another distribution may be appropriate. Or, the three-parameter Weibull distribution may be necessary to fit the data. In that case, the Weibull pdf is given by

$$f(t,\alpha,\beta,\mu)=\frac{\beta}{\alpha}\left(\frac{t-\mu}{\alpha}\right)^{\beta-1}\exp\left(-\left(\frac{t-\mu}{\alpha}\right)^{\beta}\right),\quad t\geq\mu$$

The location parameter μ is the time before which no failures occur. A non-zero value for this parameter may cause bowing of the plot [4]. Another explanation for deviation from a straight line is that there is more than one failure mechanism. Plotting also helps indicate if some of the data are suspect and provides a first indication of the parameter values. The slope of the line is β, and $-\beta\ln\alpha$ is the y-intercept. An excellent discussion of probability plotting and its use in statistical model selection can be found in [5].

B. Goodness-of-Fit Test

The objective of the test is to determine if the sample is consistent with being drawn from a population characterized by a certain distribution, rather than to demonstrate it beyond all doubt. In this procedure, a random variable, called the test statistic, is evaluated from the observed data. The probability of obtaining a given value of the test statistic, assuming the sample is drawn from a population that follows the distribution, must be known. A quantity α, called a "significance level," is specified for the test. It has the following meaning: If the population does indeed follow the assumed distribution, the probability of the value of the test statistic falling outside a specified range is α. The value for α is usually chosen less than or equal to 0.10. In other words, the range is usually chosen so that the probability of rejecting the assumed distribution, when it is in fact correct, is relatively small.

Another point regarding goodness-of-fit tests should be mentioned. There exist several test statistics, such as the Kolmogorov–Smirnov, Cramer–von Mises, and Anderson–Darling, whose distributions do not depend on the choice of parametric distribution that is being fit to the data. In that sense, they are "distribution free." However, the distribution percentiles of these test statistics require that the parameters of the fitting distribution be known. Unfortunately, in most cases arising in engineering, the parameters are also unknown and must be estimated from the data. Modified versions of the above tests can be used if estimates of unknown parameters are used, but then the test statistic is no longer distribution free. This complicates the calculation of their percentiles. Furthermore, the percentiles are generally not accurate for small samples. We choose instead to discuss goodness-of-fit tests that do not require the parameters of the distribution be known and that are specific to the Weibull (or extreme-value) distribution.

Mann, Scheuer, and Fertig [6] have developed a test for the extreme-value distribution. It is based on the fact that the left tail of an extreme-value distribution is longer than most other distributions, and its right tail is shorter. The test of Mann, Scheuer, and Fertig is applicable to censored data.

More recently, Shapiro and Brain have developed a goodness-of-fit test [7]. It is similar to the W-test for normal behavior as discussed in [5]. It is thought to be an effective test against a wide class of alternative distributions. Their test, like that of Mann, Scheuer, and Fertig, is for an extreme-value distribution. Therefore, the logarithm of the failure times must be taken before applying the test. The equations for calculating the test statistic, denoted W, for the case of n failures are given below:

$$
\left.\begin{aligned}
x_i &= \ln t_i \\
w_i &= \ln\left(\frac{n+1}{n+1-i}\right), \quad i = 1, 2, \cdots, n-1, \quad w_n = n - \sum_{i=1}^{n-1} w_i \\
w_{n+i} &= w_i(1 + \ln w_i) - 1, \quad i = 1, 2, \cdots, n-1, \quad w_{2n} = 0.4228n - \sum_{i=1}^{n-1} w_{n+i} \\
L_1 &= \sum_{i=1}^{n} w_i x_i, \quad L_2 = \sum_{i=1}^{n} w_{n+i} x_i \\
b &= \frac{0.6079L_2 - 0.2570L_1}{n} \\
S^2 &= \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2, \quad W = \frac{nb^2}{S^2}
\end{aligned}\right\} \tag{9}
$$

The subscript defines the order number, assuming the data have been ordered with the earliest failure time first, that is, $t_1 < t_2 < \cdots t_n$. The equations are appropriate for complete data. A few percentiles of the distribution for the W-test statistic are given in [7]. An example using this approach is presented in Section VI.

IV. Maximum-Likelihood Estimation of α, β

Cohen has derived the maximum-likelihood estimates for progressively censored data [8], and they are presented below. (His notation has been altered slightly to be consistent with the above equations.) Consider a sample of N units, of which n are run to failure. Suppose the censoring occurs in k stages at times T_i, and at those times r_i units are removed from test. The failure times are denoted by t_i. The maximum-likelihood estimate (MLE) for β, denoted by $\hat{\beta}$, can be found by solving

$$
\frac{\sum^{**} t_i^{\beta} \ln t_i}{\sum^{**} t_i^{\beta}} - \frac{1}{\beta} = \frac{1}{n} \sum_{i=1}^{n} \ln t_i \tag{10}
$$

The MLE for α, denoted by $\hat{\alpha}$ then is determined from

$$
\hat{\alpha} = \left(\sum^{**} \frac{t_i^{\hat{\beta}}}{n}\right)^{1/\hat{\beta}} \tag{11}
$$

In these equations,

$$
\sum^{**} t_i^{\beta} \ln t_i = \sum_{i=1}^{n} t_i^{\beta} \ln t_i + \sum_{i=1}^{k} r_i T_i^{\beta} \ln T_i
$$

and

$$\sum\nolimits^{**} t_i^\beta = \sum_{i=1}^{n} t_i^\beta + \sum_{i=1}^{k} r_i T_i^\beta$$

An example of the use of these equations will be given in Section VI.

V. Confidence Interval for the Survivor Function

As in the case for non-parametric analysis, different methods may be used to calculate confidence intervals for the survivor function for complete and censored data. One approach for the case of complete data or failure-censored data is called the conditional method. This is the approach used by Lawless [9]. In principle, it applies to any size of sample. It is a Bayesian approach in which one conditions on ancillary statistics. The classical (unconditional) approach generally requires Monte Carlo simulations. The results of such simulations are published in tabular form for various numbers of items under test with corresponding variable numbers of failures. We will follow Lawless' treatment since tables are not required. Generally, Bayesian and classical approaches give similar results [10]. We discuss this point further in Section VI. The time-censored data are analyzed using the likelihood-ratio technique (LRT). Some discussion of the LRT is presented in [1].

A. Conditional Method

Calculation of confidence intervals for Weibull-distributed data is a fairly involved process. However, computer programs can be written to perform the calculations. The author has written computer programs to calculate maximum-likelihood estimates and confidence intervals. A brief description of the conditional method for finding the confidence limits of the survivor function follows. We use an argument for probability distributions whose pdf's are functions of a single variable, i.e., $(x-u)/b$. From Eq. (5), the extreme-value distribution is such a distribution. If we are calculating confidence intervals for Weibull-distributed data, the logarithm of the failure time data is used as discussed in Eq. (4).

From Eq. (5), the probability density function has the form

$$g(x,u,b) = \frac{1}{b} f\left(\frac{x-u}{b}\right)$$

The pth quantile of this distribution, x_p, satisfies the equation

$$\int_{-\infty}^{x_p} g(x,u,b)dx = \int_{-\infty}^{x_p} f\left(\frac{x-u}{b}\right)\frac{dx}{b} = p \tag{12}$$

Let S be the survivor function associated with this distribution. If we define a change of random variable, $\mathbf{w} = (\mathbf{x}-u)/b$, then we can write

$$\int_{x_p}^{\infty} f\left(\frac{x-u}{b}\right)\frac{dx}{b} = \int_{w_p}^{\infty} f(w)dw = S(w_p)$$

Using the fact that $\int_{-\infty}^{\infty} f(w)dw = 1$ and $\int_{-\infty}^{w_p} f(w)dw = p$, it follows that

$$(1-p) = S(w_p)$$

For the case of the extreme-value distribution, from Eq. (6),

$$(1-p) = \exp\big(-\exp(w_p)\big) \tag{13}$$

or

$$\ln\big(-\ln(1-p)\big) = w_p$$

Suppose we seek a γ lower confidence bound on the survivor function at the "time" $x_o = \ln t_o$. This can be obtained from a lower confidence limit on x_p in the following way. Suppose $l_L(\mathbf{x})$ is a γ lower confidence limit on x_p based on data $\mathbf{x}$. Then

$$\Pr\{l_L(\mathbf{x}) \le x_p\} = \gamma$$

Since the survivor function is a monotone decreasing function of x, this implies

$$\Pr\big\{S\big(l_L(\mathbf{x})\big) \ge S(x_p)\big\} = \gamma$$

But, from Eq. (6),

$$S(x_p) = \exp\left(-\exp\left(\frac{x_p - u}{b}\right)\right) = \exp\big(-\exp(w_p)\big) = 1-p$$

Thus, if one can determine p such that $l_L(\mathbf{x}) = x_o$, then

$$\Pr\big\{S(x_o) \ge (1-p)\big\} = \gamma \tag{14}$$

and we have the lower bound we seek.

Now the lower confidence limit, $l_L(\mathbf{x})$, of x_p can be obtained if a random variable containing x_p whose distribution is known can be found. The quantity $\mathbf{Z}_p$, given by

$$\mathbf{Z}_p = \frac{x_p - \hat{u}(\mathbf{x})}{\hat{b}(\mathbf{x})} \tag{15}$$

satisfies this requirement. For any given γ and p, we can find $z_{p,\gamma}$ such that

$$\Pr\big\{\mathbf{Z}_p \ge z_{p,\gamma} | \mathbf{a}\big\} = \gamma \tag{16}$$

The vertical bar followed by the "**a**" indicates that this is a conditional probability. It is dependent on the ancillary statistic **a**. This is a characteristic feature of the Bayesian approach to calculating confidence intervals. We will drop the notation, $|\mathbf{a}\}$, writing instead for simplicity,

$$\Pr\{\mathbf{Z}_p \geq z_{p,\gamma}\} = \gamma$$

Now, substituting from Eq. (15),

$$\Pr\left\{\frac{x_p - \hat{u}(\mathbf{x})}{\hat{b}(\mathbf{x})} \geq z_{p,\gamma}\right\} = \gamma$$

or

$$\Pr\left\{z_{p,\gamma}\hat{b}(\mathbf{x}) + \hat{u}(\mathbf{x}) \leq x_p\right\} = \gamma$$

Thus, $z_{p,\gamma}\hat{b}(\mathbf{x}) + \hat{u}(\mathbf{x})$ is the lower confidence limit, $l_L(\mathbf{x})$, that we seek. So we need only determine p such that

$$z_{p,\gamma}\hat{b}(\mathbf{x}) + \hat{u}(\mathbf{x}) = x_o$$

or

$$z_{p,\gamma} = \frac{x_o - \hat{u}(\mathbf{x})}{\hat{b}(\mathbf{x})} \tag{17}$$

Once we have the data and select a "time," x_o, the value of $z_{p,\gamma}$ is determined. Once the confidence level is selected, we need only find the value of p such that Eq. (16) is true. If p satisfies Eq. (16), then $(1-p)$ is a lower bound for $S(x_o)$. A similar argument holds for the upper bound. In summary, the steps to find the confidence interval for the survivor function at time t_o are

(1) Calculate $x_o = \ln(t_o)$.

(2) Calculate the maximum-likelihood estimates $\hat{u}(\mathbf{x}), \hat{b}(\mathbf{x})$ of the extreme-value distribution.

(3) Calculate $z_{p,\gamma}$ using Eq. (17).

(4) Choose a value for γ; then perform a computer search to find the value of p that satisfies Eq. (16).

(5) Assign the value $(1-p)$ as the lower confidence limit of $S(t_o)$.

B. Likelihood-Ratio Method

We next present a method based on the likelihood-ratio technique. The philosophy behind this technique is discussed in [1]. The present treatment follows that of [11]. This approach applies to both time-censored and failure-censored data. It works best for moderate-to-large sample sizes. Suppose that a confidence interval is desired for

$$S(t_o) = \exp\left(-\left(\frac{t_o}{\alpha}\right)^{\beta}\right) \tag{18}$$

We consider testing the hypothesis $H_o : S(t_o) = S_o$ versus the hypothesis $H_1 : S(t_o) \neq S_o$. The log-likelihood function for the Weibull distribution is

$$\log L(\alpha,\beta) = n\log\beta - n\beta\log\alpha + (\beta-1)\sum_{\text{failures}}\log t_i - \sum_{\text{all times}}\left(\frac{t_i}{\alpha}\right)^{\beta} \tag{19}$$

Following our previous notation, n is the number of failure times whereas N is the total number of units. The likelihood ratio for testing H_o versus H_1 is

$$L_R = \frac{L(\tilde{\alpha},\tilde{\beta})}{L(\hat{\alpha},\hat{\beta})} \tag{20}$$

The numerator of L_R is maximized under the hypothesis H_o. This is achieved by setting $S(t_o) = S_o$ in Eq. (18) and solving for α. The result is

$$\alpha = \frac{t_o}{(-\ln S_o)^{1/\beta}} \tag{21}$$

This expression is substituted into the log-likelihood function, differentiated with respect to β, and set equal to zero. The result is

$$\frac{n}{\beta} - n\ln t_o + \sum_{\text{failures}}\ln t_i + \ln(S_o)\sum_{\text{all times}}\left(\frac{t_i}{t_o}\right)^{\beta}\ln\left(\frac{t_i}{t_o}\right) = 0 \tag{22}$$

We denote the root of this equation $\tilde{\beta}$. Putting this value into Eq. (21) determines $\tilde{\alpha}$. These values, substituted into Eq. (19), form the numerator of L_R. The denominator of L_R is the likelihood function maximized without any restrictions on α and β. In other words, α and β are the maximum-likelihood values given by Eqs. (10) and (11). The set of values S_o such that $\Lambda = -2\ln(L_R) \le \chi^2_{(1),\gamma}$ forms a γ confidence interval for $S(t_o)$. Here, $\chi^2_{(1),\gamma}$ is the γth percentile of the chi-square distribution with one degree of freedom.

VI. Examples

We now illustrate some of the above results. We begin by taking the data from Table 1 of [1]. It is reproduced here in Table 1 for convenience. Following Section III.A, we construct a plot of $\ln(-\ln[S_{PL}(t)])$ versus $\ln(t)$. This is shown in Fig. 1. The data appear as though they may follow a Weibull distribution. To obtain a less subjective estimate, we use the goodness-of-fit test of Shapiro and Brain. The W-test statistic is calculated to be 0.4930. The percentiles of the W-distribution for $n = 24$ can be estimated. We find $W_{0.05} = 0.4095$ and $W_{0.95} = 0.9217$. Since W lies between these two values, the test is not rejected at the 10 percent significance level.

From Fig. 1 we also can estimate α and β. For a less subjective estimate, we can use Eqs.(10) and (11). The maximum-likelihood point estimates for these data are $\hat{\alpha} = 12.62$ and $\hat{\beta} = 2.54$. That line also is shown in Fig. 1. Figure 2 compares the survivor functions using these values of α, β and the product-limit estimate. The greatest deviation is for the earlier time data, as also was indicated in Fig. 1.

Confidence intervals for the data in Table 1 can be calculated using the conditional method described above because the data are complete. The values at 9,000 hours, 13,000 hours, and 16,000 hours are shown in Table 2. The results are shown along with the values calculated using the binomial distribution from [1]. It is noteworthy that the 90 percent confidence interval is significantly smaller assuming the

Table 1. Survivor functions for 24 units with complete failure (from [1]).

Failure time, kh	$S(t)$
0.0	1.0
2.0	0.958
3.25	0.917
4.1	0.875
5.6	0.833
6.05	0.792
8.01	0.750
8.9	0.708
9.25	0.667
9.4	0.625
9.8	0.583
10.1	0.542
10.6	0.500
12.0	0.458
12.5	0.417
12.9	0.375
13.3	0.333
13.6	0.292
14.0	0.250
15.0	0.208
15.5	0.167
16.9	0.125
18.0	0.083
18.5	0.042
20.0	0.000

data follow a Weibull distribution, as seen by comparing the values in the columns labeled "Delta." A possible explanation is that the binomial estimate uses only the number of survivors at time t_o to estimate the confidence interval. The actual failure times are not used. Fitting the data to a Weibull distribution uses the known failure times.

As a check of the conditional approach to calculating confidence intervals, we will compare the lower bound of the survivor function for the completely censored data obtained using the program written by the author to results based on Monte Carlo simulations [12]. The results from [12] are actually for complete failure of 25 units, but this is deemed close enough to the 24 units presented above to make the comparison meaningful. Figure 3 shows the comparison of the lower bounds for the survivor functions for the two cases. The survivor (reliability) plot begins at 0.5 because this is where the tabular data from [12] begin. The agreement is rather good.

Next we consider the censored data from [1]. They are reproduced here as Table 3. The data are for a total of 24 units, 16 of which are run until failure, and 8 of which are removed before failure. The maximum-likelihood point estimates, using Eqs. (10) and (11), for these data are $\hat{\alpha} = 14.88$ and $\hat{\beta} = 2.13$. A plot of $\ln(-\ln[S_{PL}(t)])$ versus $\ln(t)$ is shown in Fig. 4 along with the maximum-likelihood fit to a Weibull distribution. Figure 5 compares the survivor functions using these values of α, β and the product-limit estimate.

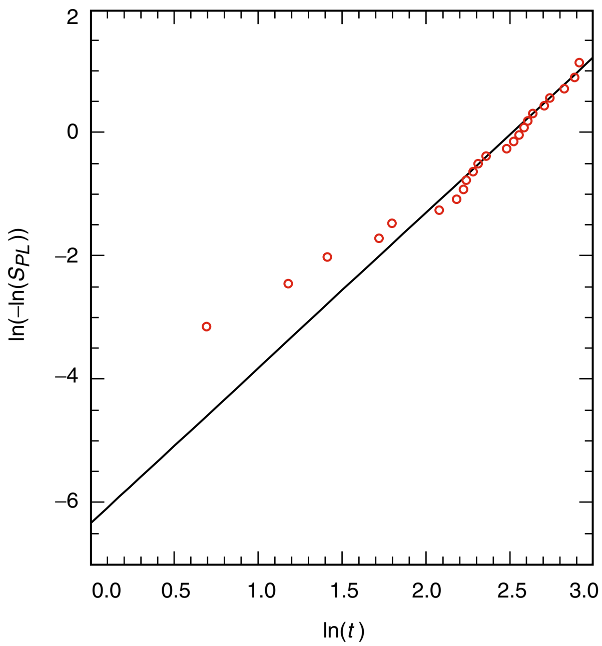

Fig. 1. ln(−ln[$S_{PL}(t)$]) **versus** ln(t) **for the data in Table 1. The line drawn corresponds to** $\beta = 2.54$ **and** $\alpha = 12.62$**.**

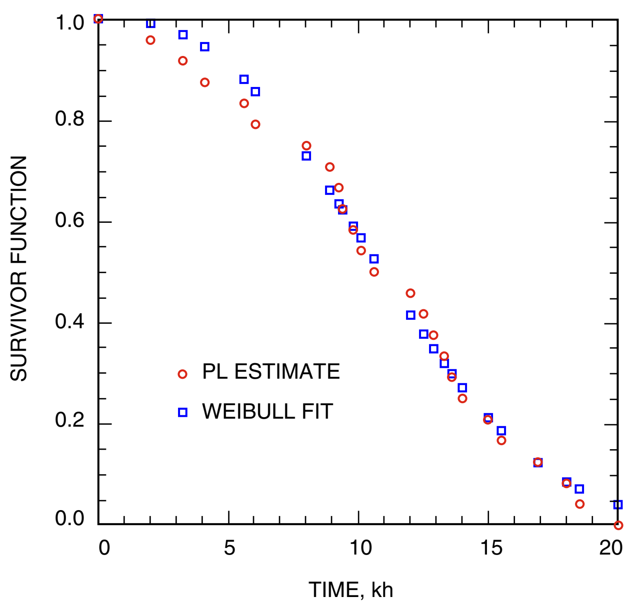

Fig. 2. Product-limit estimate and Weibull survivor functions for the data in Table 1. For the Weibull plot, $\beta = 2.54$ **and** $\alpha = 12.62$**.**

Table 2. Confidence limits of 90 percent at 9,000, 13,000, and 16,000 hours for the complete data of Table 1.

9,000 h	Delta	13,000 h	Delta	16,000 h	Delta	90 percent confidence limits
0.540	0.205	0.251	0.197	0.101	0.160	S_L (using conditional method)
0.745		0.448		0.261		S_U (using conditional method)
0.521	0.333	0.211	0.352	0.059	0.283	S_L (from [1])
0.854		0.563		0.342		S_U (from [1])

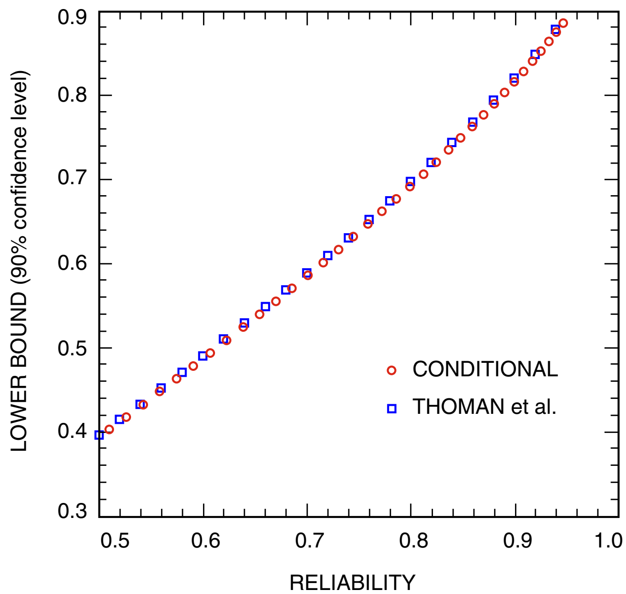

Fig. 3. A comparison of the lower bound (90 percent confidence level) of the reliability using the conditional approach and the Monte Carlo approach of Thoman, Bain, and Antle.

Confidence intervals for the data in Table 3 can be calculated using the likelihood-ratio method, assuming the data follow a Weibull distribution. The values at 9,000 hours, 13,000 hours, and 16,000 hours are shown in Table 4. The results are shown along with the values calculated using the likelihood- ratio method of Thomas and Grunkemeier [13] for a non-parametric fit. The interval sizes are nearly the same. At 9,000 hours, the Weibull interval is centered a little lower than the corresponding non-parametric interval. At 13,000 and 16,000 hours, the Weibull interval is centered a little higher than the corresponding non-parametric interval. This is probably due to the fact that the Weibull fit to the data is slightly below the product-limit estimate at 9,000 hours and slightly above at 13,000 and 16,000 hours.

VII. Conclusions

We have presented a discussion of the Weibull probability distribution and the related extreme-value distribution. Graphical and analytical means for testing whether the data could have been drawn from a Weibull-distributed population are presented. Graphical and analytical techniques for determining the shape and scale parameter were presented. We discussed the calculation of confidence intervals for the survivor function for complete and censored data. The formalism was illustrated by comparing the results of a parametric analysis to a non-parametric analysis using data from [1].

Survivor functions based on the parametric and non-parametric approaches for the complete and censored data are shown in Figs. 2 and 5. Comparison of the confidence intervals is interesting. For the complete data, the confidence intervals for the parametric fit are significantly smaller than for the non-parametric fit. Published lower bounds using Monte Carlo simulations agree quite well with the lower bounds calculated using the conditional approach. For the censored data, where both methods are based on the likelihood-ratio approach, the confidence intervals are nearly the same size.

Table 3. Survivor functions for 24 units, 8 of which are censored units (from [1]).

Failure time, kh	Loss time, kh	S_2
0.0	—	1.000
2.0	—	0.958
3.25	—	0.917
—	4.1	0.917
5.6	—	0.873
6.05	—	0.829
8.01	—	0.786
8.9	—	0.742
9.25	—	0.698
—	9.4	0.698
9.8	—	0.652
10.1	—	0.605
—	10.6	0.605
12.0	—	0.555
12.5	—	0.504
12.9	—	0.454
13.3	—	0.404
—	13.6	0.404
—	14.0	0.404
15.0	—	0.336
15.5	—	0.269
16.9	—	0.202
—	18.0	0.202
—	18.5	0.202
—	20.0	0.202

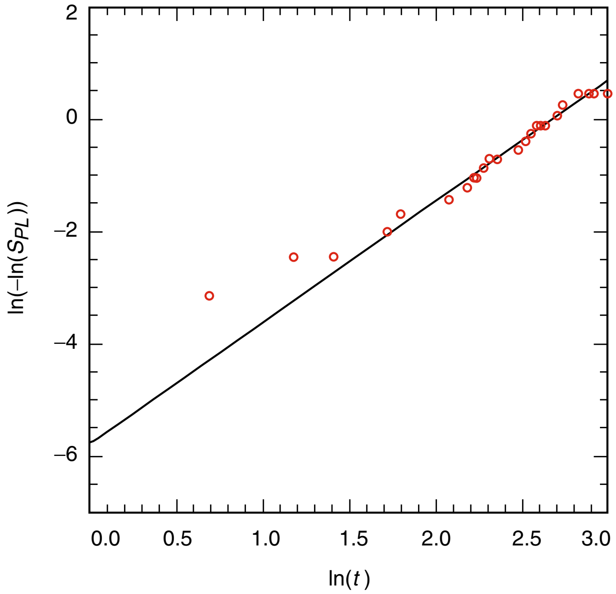

Fig. 4. $\ln(-\ln[S_{PL}(t)])$ **versus** $\ln(t)$ **for the data in Table 3. The line drawn corresponds to** $\beta = 2.13$ **and** $\alpha = 14.88$**.**

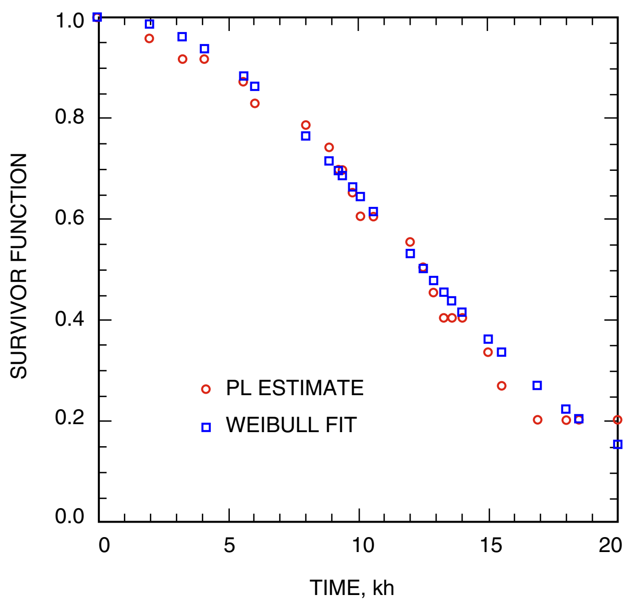

Fig. 5. Product-limit estimate and Weibull survivor functions for the data in Table 3. For the Weibull plot, $\beta = 2.13$ **and** $\alpha = 14.88$**.**

Table 4. Confidence limits of 90 percent at 9,000, 13,000, and 16,000 hours for the censored data of Table 3.

9,000 h	Delta	13,000 h	Delta	16,000 h	Delta	90 percent confidence limits
0.543	0.299	0.306	0.339	0.156	0.351	S_L (using LRM [Weibull])
0.842		0.645		0.507		S_U (using LRM [Weibull])
0.578	0.292	0.284	0.345	0.120	0.335	S_L (from [1])
0.870		0.629		0.455		S_U (from [1])

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