Computing the Apparent Elevation of a Near-Earth Spacecraft at Low Elevation Angles for an Arbitrary Refraction Model

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This article gives an exact expression for parallax adjustments to apparent elevation of objects at finite distances beyond the Earth atmosphere suitable for use at the very low elevation angles required by the Deep Space Network (DSN) Services Preparation Subsystem (SPS) Metric Prediction Generator (MPG). The parallax depends entirely on the astronomical refraction function and does not require a separate model, as was used in the Network Support Subsystem (NSS) Metric Prediction (MP) application. Rather, the effects of refraction and finite target distance are combined into a single computation. Results are compared with the parallax model used by NSS MP software, where a sizable difference at low elevation angles is noted. The NSS model is known to break down at low elevations, so this behavior was expected. The study also displays the performance of the method using a hybrid Lanyi/Berman–Rockwell refraction model being prepared for MPG use.

I. Introduction

The Deep Space Network (DSN) is currently engaged in the replacement of its Network Support Subsystem (NSS) with a Services Preparation Subsystem (SPS) that upgrades operational autonomy, accuracy, and network services. Specifically, the NSS Metric Prediction (MP) software is being replaced by the Metric Prediction Generator (MPG), whose key requirements are increased accuracy (by three orders of magnitude in some metrics), autonomous operation, superior robustness, and incorporation of the Navigation Ancillary Information Facility (NAIF) SPICE functions. Among the predictions generated in both subsystems are view periods keyed to rises and sets of targets above arbitrary horizon masks. Such event predictions require computation of apparent elevation angles at each Deep Space Station (DSS) above and below the masks, within an accuracy of a few millidegrees.

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The NSS MP makes use of two separate atmospheric corrections in the generation of view-period event predictions. One of these is the refraction model used to compute apparent elevation angles at DSSs for objects at stellar distances, and the other is a separate parallax computation required for objects at finite distances. The stellar refraction component increases the apparent elevation angle over the unrefracted (geometric) target elevation, while the parallax decreases it. The parallax is always lower in magnitude than the stellar component.

The NSS parallax model was developed by Dan Cain [1,2] and dates from 1964, with improvements in 1988.[2] It also was used in the Orbit Determination Program (ODP) and Double Precision ODP (DPODP) of that era. The model defines a rational function least-squares fit to the old Planetary Orbiter Error Analysis Study (POEAS) "ARFC1" function, itself fit to empirical ray tracings of a standard atmosphere. The model proved especially problematic due to its lack of accuracy at low elevation angles. In fact, it predicts an infinite parallax at an elevation of -2.71 deg. Special accommodations are necessary in the NSS MP to avoid problems in this region.

Other parallax models were sought for MPG use to avoid the problem altogether and to make beneficial use of the more accurate atmospheric modeling of recent times. During the course of study, it was learned that the core of refraction modeling was done for the purpose of determining atmospheric composition, temperature and pressure distributions, seasonal variations, and dependence on station latitude and source elevation using Global Positioning System (GPS) and very long baseline interferometry (VLBI). Examples of such studies are [3] and [4]. The only studies seemingly interested in being able to point antennas more accurately were made by those who, like NASA, possess large, narrow-focus instruments (see [5], for example). But none addressed the use of the refraction function to determine the accurate apparent elevation of a target at finite distance.

Because DSN antennas do not point below 6 deg, they do not require an accurate refraction model below this elevation. Moreover, they do not require extreme pointing accuracy for near targets at any elevation, because signal strengths are high in these regions for such powerful instruments.

However, prediction of rise and set events does require accurate modeling, especially at low elevations, in order to meet precision requirements imposed by interface agreements. The MPG therefore imposes the following criteria on candidate solutions to this problem:

(1) Accuracy sufficient to predict rise and set times for arbitrary horizon masks within the specified event precision for all DSN missions.

(2) Uniform behavior at all elevations so as not to indicate false events. This is tantamount to apparent elevation being a monotonic function of geometric elevation over the entire (-90 deg, $+90$ deg) range (no mirages).

(3) Accuracy at elevations that are below *all* DSN horizon masks is not an issue, but should degrade gracefully to promote software robustness.

Searches of JPL and open literature sources did not reveal any models specifically devoted to parallax computation. However, on review of refractive studies, it became clear that parallax and refraction are interrelated phenomena and do not require separate models. After all, a radio wave emanating from a DSS toward a near spacecraft or satellite target eventually proceeds in the direction of some stellar coordinate.

This article develops a formula for the apparent DSS elevation for a given arbitrary refraction model, the geometric elevation of the target, and the geometric target range.

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[2] D. L. Cain, "Refraction Parallactic Angle Corrections," JPL Interoffice Memorandum 3685-88-069 (internal document), Jet Propulsion Laboratory, Pasadena, California, March 14, 1988.

II. Coordinate Reference Frame and Refraction Effects

For specificity (Fig. 1), let a DSS be positioned at the origin (point O) of a topocentric coordinate system, and let $\alpha = \varepsilon + \sigma$ be the *apparent* elevation of a star at *geometric* elevation ε due to refraction, where σ is the stellar refraction correction. By definition, then, a ray viewed (or emitted) at the DSS at angle α enters (or exits) the atmosphere at angle ε and is bent by an amount σ.

MPG criteria assume that there is a one-to-one continuous mapping between α and ε, such that σ can be expressed as a function of either α or ε. Here, we assume the dependency is $\sigma[\varepsilon]$. The variation of σ with ε is assumed to be known or approximated from astrometric observations, meteorological phenomena, and atmospheric models. The apparent stellar elevation function is

$$\left.\begin{aligned} &\alpha(\varepsilon) = \varepsilon + \sigma(\varepsilon) \\ &\sigma(\varepsilon) \geq 0 \end{aligned}\right\} \tag{1}$$

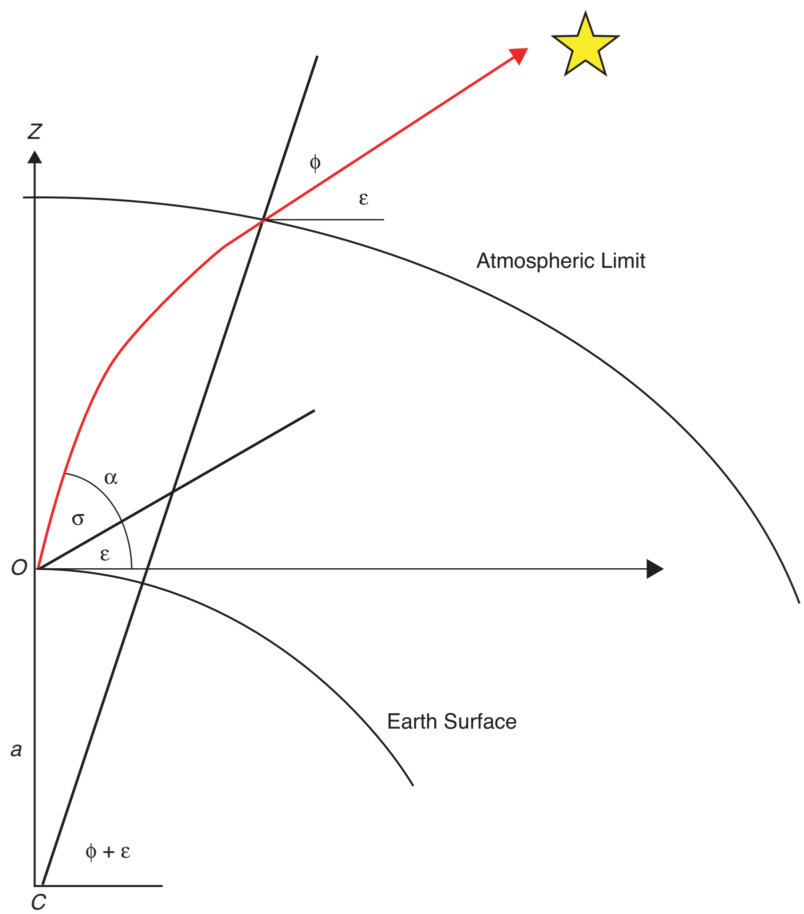

Fig. 1. Atmospheric refraction geometry: The star at geometric elevation ε has apparent elevation α at a DSS located at point *O* of the topocentric coordinate frame. The refraction σ is a given function that depends on elevation. Earth center is at point *C*; *a* is the index of refraction radius of curvature, taken to be Earth's mean radius. The zenith angle φ everywhere along the ray satisfies the refractive invariant. The geocentric elevation is φ + ε.

As depicted in Fig. 2, let S (denoting the spacecraft position) be a point on a ray at geometric distance R and elevation ε_S from O, whose apparent elevation as seen from O is α_S. The coordinates of S are then

$$\left.\begin{aligned} S_h &= R\sin(\varepsilon_S) \\ S_z &= R\cos(\varepsilon_S) \end{aligned}\right\} \tag{2}$$

where S_z is the zenith component and S_h is the horizontal component in the direction of the azimuth of S. If C denotes the geocenter, then the geocentric elevation and geocentric distance are

$$\left.\begin{aligned} \theta &= \arctan(S_h, S_z + a) \\ SC &= \sqrt{S_h^2 + (S_z + a)^2} \end{aligned}\right\} \tag{3}$$

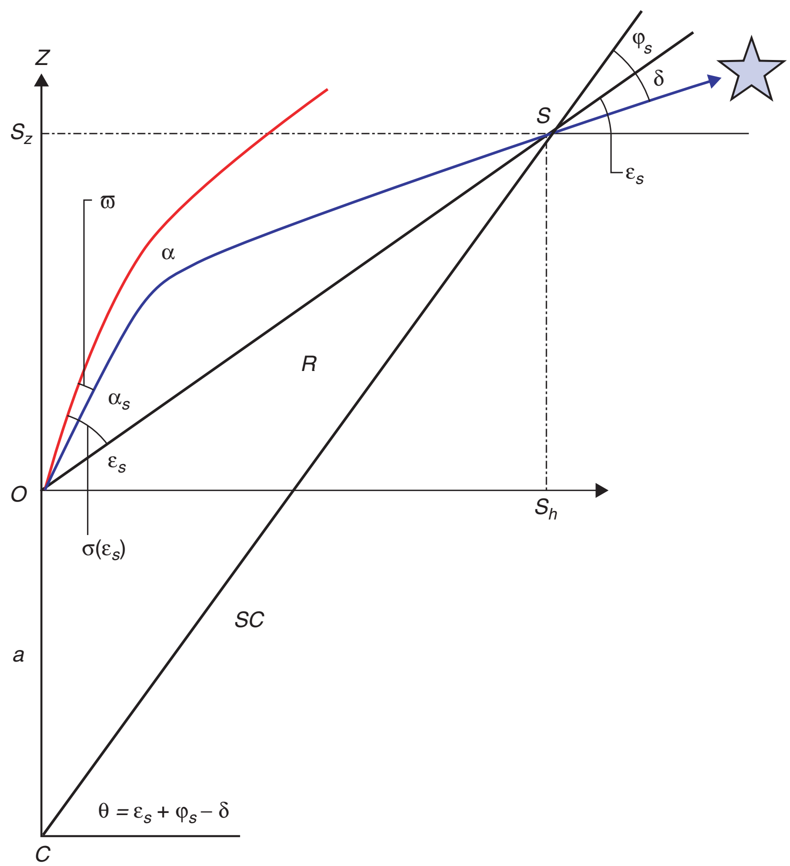

Fig. 2. Atmospheric parallax geometry: The spacecraft at point *S*, being at distance *R* from *O* and at geometric elevation ε_s, has apparent elevation α_s, which differs by the station parallax ϖ from a star at the same geometric elevation. The ray path angle at *S* is less than the geometric elevation by the spacecraft parallax angle δ. The topocentric horizontal and vertical coordinates S_h and S_z, the geocentric elevation θ of *S*, the ray path zenith angle φ_s, and geocentric distance *SC* are as shown.

where a is the local radius of curvature of the reference ellipsoid through O, and the arctan( ) function arguments are, respectively, the horizontal and vertical components of the angle. Here, a will be set equal to the mean radius of Earth.

III. Atmospheric Parallax

Under the assumption that the atmospheric refractive index $n(H)$, where H is the height above the reference surface, has its gradient in the radial direction, then the law of refractive invariance (Snell's law) states that the product $n(H) r \sin(\phi)$ is constant along any given ray path, ϕ being the angle between the radius vector and the ray path direction, and r being the radial distance. For example, when applied to a ray leaving O at angle α_S and reaching S, the invariance principle demands that

$$n(0) a \sin\left(\frac{\pi}{2} - \alpha_S\right) = n(SC - a) \sin(\phi_S) \tag{4}$$

The refractive index at the target is unity, so the ray path angle with the radius vector at S is

$$\phi_S = \arcsin\left(\frac{an(0)}{SC} \cos(\alpha_S)\right) \tag{5}$$

Since $\cos(\sigma_S)$ is always positive in the interval (-90 deg, $+90$ deg), the values accorded to ϕ_S in the above are always positive also.

Now, whereas a ray transmitted toward a star at angle $\alpha = \varepsilon + \sigma$ exits the top of the atmosphere and then travels at the geometric angle ε on toward the star, a ray launched at angle α_S toward S exits the top of the atmosphere and travels toward the target, and thence on toward another star, but at a somewhat lower angle, $\varepsilon_S - \delta$, with δ being the parallax at S. Due to bending, the ray elevation at S is always lower than the geometric elevation, so δ is nonnegative.

However, according to the definition of the stellar refraction function, a ray observed at an apparent elevation angle of $(\varepsilon_S - \delta) + \sigma(\varepsilon_S - \delta)$ at O corresponds to an elevation of $\varepsilon_S - \delta$ at the atmospheric limit and beyond. Therefore, we have two expressions for apparent target elevation: first, as a parallax correction to the stellar refraction,

$$\alpha_S = \alpha(\varepsilon_S) + \varpi = \sigma(\varepsilon_S) + \varepsilon_S + \varpi \tag{6}$$

where the small (negative) correction designated as ϖ (curly-π) above is the DSS parallax angle, and second, as an effect involving the spacecraft parallactic angle,

$$\alpha_S = \alpha(\varepsilon_S - \delta) = \sigma(\varepsilon_S - \delta) + \varepsilon_S - \delta \tag{7}$$

The apparent elevation correction is then merely

$$\sigma_S = \alpha_S - \varepsilon_S = \sigma(\varepsilon_S - \delta) - \delta \tag{8}$$

When δ is known, the apparent elevation and ϖ may be calculated immediately. The station parallax, if indeed that parameter is of direct interest, is seen to satisfy the equation

$$\varpi = \sigma(\varepsilon_S - \delta) - \sigma(\varepsilon_S) - \delta \tag{9}$$

It remains only to determine the spacecraft parallax.

IV. Calculating the Spacecraft Parallax

Since the geocentric elevation of the spacecraft (see Fig. 2) is equal to the ray angle $\varepsilon_S - \delta$ at the spacecraft plus the ray-radius angle ϕ_S,

$$\theta = (\varepsilon_S - \delta) + \phi_S \tag{10}$$

the spacecraft parallax angle δ must be a root of the function

$$f(\delta) = \theta - \varepsilon_S + \delta - \arcsin\left(\frac{an_0}{SC}\cos\left(\varepsilon_S - \delta + \sigma(\varepsilon_S - \delta)\right)\right) \tag{11}$$

This formulation is exact under the assumptions enumerated earlier (spherical Earth, radial refractive index gradient, continuous 1–1 refraction function). All quantities are known except δ, and its value is the root of the function above. Moreover, δ is bounded above by some positive angle, because there is some maximum parallax angle (spacecraft at the atmospheric limit), and is bounded below by zero.

Since δ is expected to be small, it may be estimated using a first-order Taylor series expansion of the function above. Routine grinding or use of a tool such as Mathematica readily finds the solution to be

$$\delta \approx \frac{\varepsilon_S - \theta + \arcsin\left(\frac{a\,n_0}{SC}\cos\left(\varepsilon_S + \sigma(\varepsilon_S)\right)\right)}{1 - \frac{a\,n_0}{SC}\,\frac{\sin\left(\varepsilon_S + \sigma(\varepsilon_S)\right)\left(1 + \sigma'(\varepsilon_S)\right)}{\sqrt{1 - \left(\frac{a\,n_0}{SC}\cos\left(\varepsilon_S + \sigma(\varepsilon_S)\right)\right)^2}}} \tag{12}$$

Values of the refraction derivative may be readily approximated to the accuracy required by the sample derivative. The error in this approximation is proportional to δ^2 and is consequently very small. In a Mathematica study (AtmosphericParallax.nb available via the SPS portal), the author showed that this approximation is good enough for the MPG. Other applications may require more accurate solutions using a root-finder subroutine available in most numeric libraries, such as the Van Wijngaarden–Dekker–Brent method found in [6].

V. Comparison with NSS Parallax

The refraction correction used in the NSS MP (and also the ODP and an early DPODP version) was also developed by Dan Cain in [1] and revised in 1988.[3] The station parallax computed using the NSS MP refractivity is compared with the Cain parallax model in Fig. 3, where it may be noted that there is a serious disagreement at low elevations. In the region above 6-deg elevation, the two models agree within about 1 mdeg, and the disagreement falls within 0.5 mdeg above 9 deg. As mentioned earlier, the NSS MP parallax becomes infinite at −2.71 deg.

The above comparisons were made for a fictitious spacecraft in a 500-km circular orbit about Earth passing directly overhead at the DSS. The situation improves, of course, as the spacecraft range gets

[3] The only extant documentation of this algorithm appears to be the NSS MP function (ARFC) code.

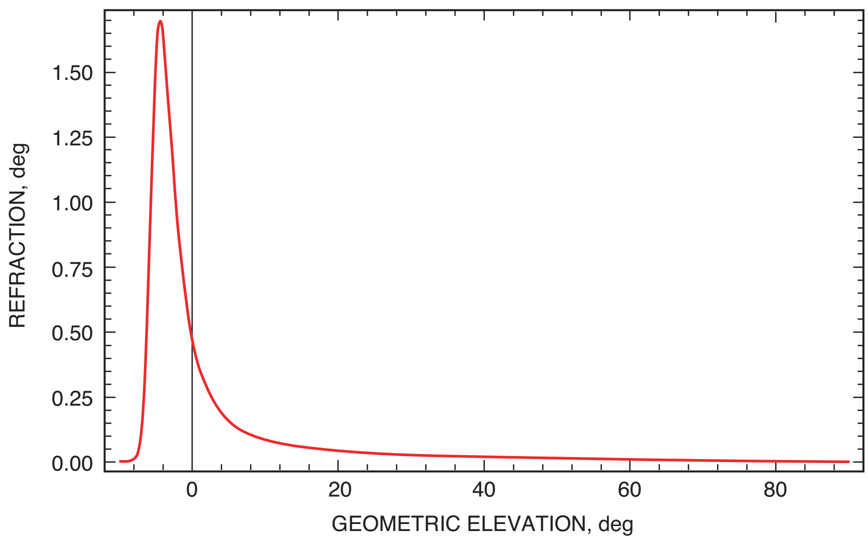

Fig. 3. Hybrid Lanyi/Berman–Rockwell model refraction: The hybrid consists of the Lanyi model above 6 deg, the Berman–Rockwell model below 0 deg, and a nonlinear interpolation between these in between. This refraction is continuous, produces a monotonic apparent elevation characteristic, and is accurate enough for MPG elevation event predictions for all DSN missions.

larger, because the parallax itself becomes smaller. In fact, the difference drops below about 10 mdeg for spacecraft exceeding 1 Earth radius in distance at 1-deg elevation.

A. Parallax for a Hybrid Lanyi/Berman–Rockwell Refraction Model

The Lanyi refraction model,[4,5] reported in [7], is currently in use in DSN antenna pointing controllers. It claims accuracy in the millidegree region at elevations above 6 deg. However, it becomes discontinuous at 0 deg, and thus unusable for MPG purposes at and below this region. The Berman–Rockwell (BR) model [8–10] purports to be accurate within 3 mdeg at 5-deg elevation, which degrades to only 10-mdeg error at −3-deg elevation. Moreover, it is well behaved at all elevations.

The MPG, therefore, has developed a hybrid refraction model combining the two: the Lanyi model above 6 deg, the BR model below 0 deg, and a nonlinear interpolation of the two in the range in between these two limits. Inputs to the hybrid model are surface pressure (P, in mbar), temperature (T, in K), and relative humidity (RH, as a decimal fraction). The hybrid Lanyi/Berman–Rockwell (HLBR) characteristic shown in Fig. 4 was calculated for the default atmospheric parameters at DSS 13, which are

$$\begin{aligned} T &= 295.09 \text{ K} \\ P &= 901.09 \text{ mbar} \\ RH &= 0.3137 \end{aligned}$$

The same fictitious spacecraft in circular orbit as used in the previous example is assumed here.

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[4] G. A. Lanyi, "Atmospheric Refraction Corrections to Antenna Pointing at 1 Millidegree Accuracy," JPL Interoffice Memorandum 335.3-89-026 (internal document), Jet Propulsion Laboratory, Pasadena, California, March 24, 1989.

[5] R. Riggs, "A New Atmospheric Refraction Model," JPL Interoffice Memorandum 3320-90-002 (internal document), Jet Propulsion Laboratory, Pasadena, California, January 5, 1990.

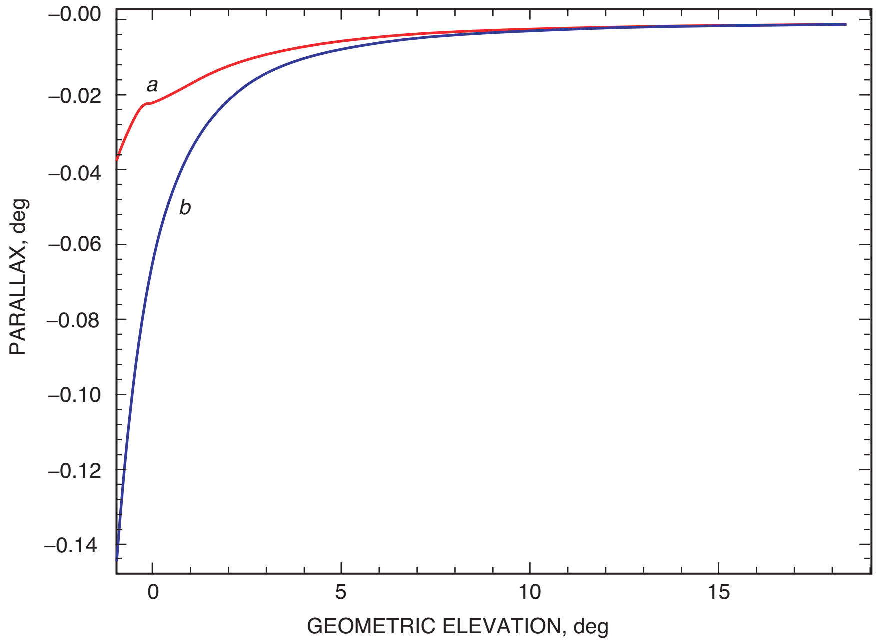

Fig. 4. Computed parallax comparison: Curve *a* is the station parallax found using Eq. (11), while curve *b* is the parallax given by the model used by the NSS MP for a fictitious spacecraft in a 500-km circular orbit passing directly over the DSS. These are seen to be in significant disagreement at geometric angles below –1 deg. The refraction correction model is limited to elevations above –3 deg, the parallax model becomes infinite at –2.71 deg, and both require special treatment in the event detection software.

Figure 5 displays the station parallax along with the total refraction, negated for comparison. It may at first appear odd that the parallax exceeds the total refraction in magnitude at elevations below about 7.5 deg. After all, physical law demands that this cannot be so, and the derivation of the parallax equation was claimed to be correct and rigorous. What's amiss?

This parallax paradox fades when one realizes that the refraction function becomes grossly inaccurate and physically without meaning in this region. Thus, the refraction model no longer combines with the invariance condition in a consistent way. Nor does this pose a problem to the MPG, because all that is required in this region is continuity, reasonable accuracy, and no surprises.

Finally, Fig. 6 shows the difference between the approximate spacecraft parallax and that found as the root of $f(\delta) = 0$. The maximum error in the approximation is 6.4 mdeg at an elevation of -4.89 deg for the fictitious mission. The error is less than 0.17 mdeg above -3 deg elevation. Inasmuch as the BR model fails in accuracy below -3 deg and no current DSN missions have masks in this range, the root-finder solution never really needs to be exercised at all.

VI. Conclusion

This study has demonstrated that parallax adjustments to elevation refraction corrections can be determined directly from the assumed astronomical refraction function and do not require a separate and independent model. Parallax corrections derived from the refraction function are rigorous, exact, and consistent with that model in its region of accuracy. The parallax computation given here has no inherent inaccuracy limitations of its own, but carries through any inaccuracies that might be present in the refraction model on which it is based.

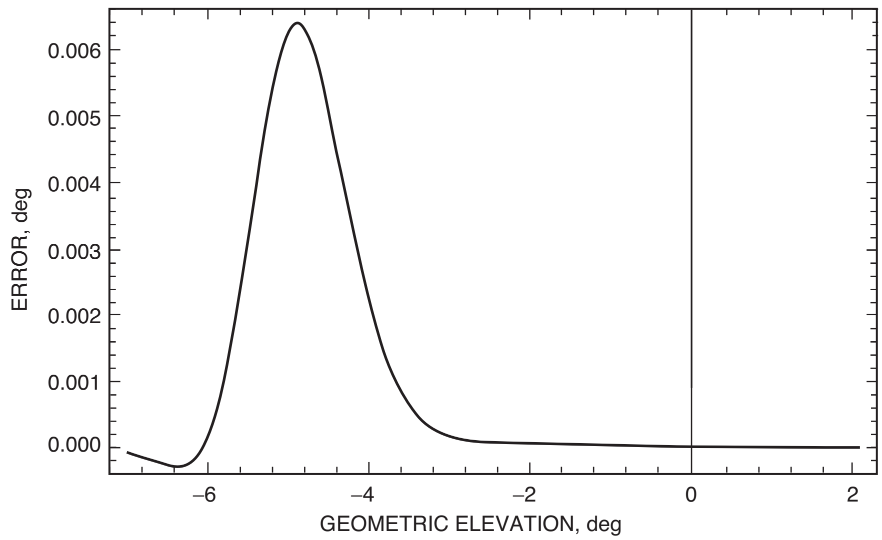

Fig. 5. Spacecraft parallax approximation error: The error in the formula of Eq. (12) for the test orbit is a maximum of about 6.4 mdeg.

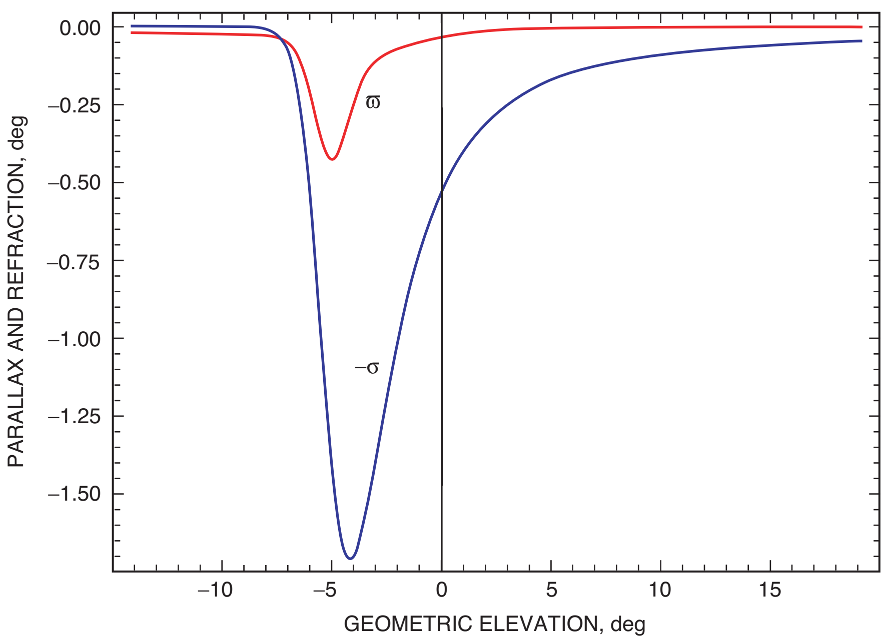

Fig. 6. Comparison of parallax to (negative of) HLBR refraction for test orbit. Note that parallax seemingly exceeds refraction below about –7.5 deg. This paradox is due to breakdown of the refraction model at low elevations.

The effects of refraction and distance can be combined into one apparent elevation computation, without the need to compute the separate effects of refraction at stellar distance and parallax effects at finite distances, as was done in the NSS MP. The approximate spacecraft parallax formula applies within the required accuracy at all elevations down to -3 deg, which is well below all DSN mission horizons.

The use of a separate formula, as was done in the NSS MP, is both redundant and a source of potential error. The NSS MP refraction and parallax models are in significant disagreement at geometric angles below -1 deg. The refraction correction model is limited to elevations above -3 deg, the parallax is infinite at -2.71 deg, and both require special treatment in the event-detection software.

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