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Combining Loss of a Transmitting Array due to Phase Errors

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This article reviews the theoretical model for the combining loss of a transmitting antenna array in the presence of phase errors, and gives some examples of its application. By reciprocity, similar results apply to a receiving array. The phase errors are treated statistically. For an array on Earth, one cause of such errors is turbulence in the atmosphere such that the mean index of refraction differs among the paths from the separate antennas. The theory is applied to that case via the spatial structure function of the turbulence.

I. Introduction

For a transmitting phased array, a simple theory determines the combined power density at a far-field receiver. For narrow-bandwidth signals represented as modulated sinusoids, maximum power is obtained when the phases are aligned at the target, but for various reasons the phase alignment may not be perfect. This article considers the received power as a function of the amount of misalignment and of the number of antennas in the array. The treatment here is based on purely sinusoidal signals, but the results are the same for modulated signals provided that the modulation is identical for all elements of the array and is aligned in time when the signals arrive at the target.

The theory applies regardless of the cause of the misalignment, but a case of particular practical interest occurs when the transmitting antennas are on Earth and the signals pass through the atmosphere. Turbulence can then cause the mean refractive index along the path from an antenna to the target to be different among the antennas. Examples of such situations are analyzed in Section III.

II. Analysis

Consider a two-element phased array transmitting co-polarized sinusoidal signals at angular frequency ω. In the far field, the combined field strength will be

$$E(t) = e_1 \cos(\omega t + \phi_1) + e_2 \cos(\omega t + \phi_2)$$

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where e_1, e_2 are the electric field strengths of the individual signals and ϕ_1, ϕ_2 are their phases. We are concerned about the effects of phase errors, so for simplicity let $e_1 = e_2 = e$. The power density at the same far-field location is then

$$\begin{aligned} P &= \overline{E(t)^2}/Z_0 \\ &= (e^2/Z_0)\,[2 + 2\,\overline{\cos(\omega t + \phi_1)\cos(\omega t + \phi_2)}] \\ &= 2\,(e^2/Z_0)\,[1 + \cos(\phi_1 - \phi_2)] \end{aligned} \tag{1}$$

where Z_0 is the impedance of free space and the overbar represents time averaging.

Now assume that the phase error $\delta\phi = \phi_1 - \phi_2$ is a normally distributed, zero mean random variable with variance $\sigma^2 = \langle \delta\phi^2 \rangle$. Then the expected value of the power density is

$$\langle P \rangle = 2\,(e^2/Z_0)\,(1 + \langle \cos\delta\phi \rangle) = 2\,(e^2/Z_0)\,(1 + e^{-\sigma^2/2}) \tag{2}$$

where we have used the identity $\langle \cos x \rangle = e^{-\sigma_x{}^2/2}$ where σ_x is the standard deviation of x. Since the maximum value of $\langle P \rangle$ is $P_{\max} = 4e^2/Z_0$,

$$\langle P \rangle / P_{\max} = \frac{1}{2}(1 + e^{-\sigma^2/2}) \tag{3}$$

More generally, the phase combining loss factor for an array of N identical elements is given [1] by

$$\langle P \rangle / P_{\max} = \frac{1}{N^2} \sum_k \sum_m e^{-\sigma_{km}^2/2} \tag{4}$$

where k and m range over the N elements and σ_{km}^2 is the variance of the phase difference at the receiver of the signals from elements k and m. It can easily be shown that for $N = 2$, Equation (4) reduces to Equation (3). Note that $\sigma_{km} = 0$ when $k = m$.

In the special case that each antenna has a phase error independent of the others and all errors are identically distributed with variance σ_0^2, the variances of the phase differences (by baseline) are all $\sigma_{km}^2 = 2\sigma_0^2$ when $k \neq m$. Then

$$\begin{aligned} \langle P \rangle / P_{\max} &= \frac{1}{N^2}[N + \sum_k \sum_{m \neq k} e^{-\sigma_0^2}] \\ &= \frac{1}{N^2}\left[N + N(N-1)e^{-\sigma_0^2}\right] \\ &= \frac{1 - e^{-\sigma_0^2}}{N} + e^{-\sigma_0^2} \end{aligned} \tag{5}$$

In the limit $N \to \infty$, this reduces to

$$\langle P \rangle / P_{\max} = e^{-\sigma_0^2} \tag{6}$$

which is sometimes called the “Ruze formula.” [2]

Figure 1 is a plot of the loss for the 2-element and large-N idealized cases (identical antennas with independent, equal-variance phase errors) as a function of the rms phase error of each antenna.

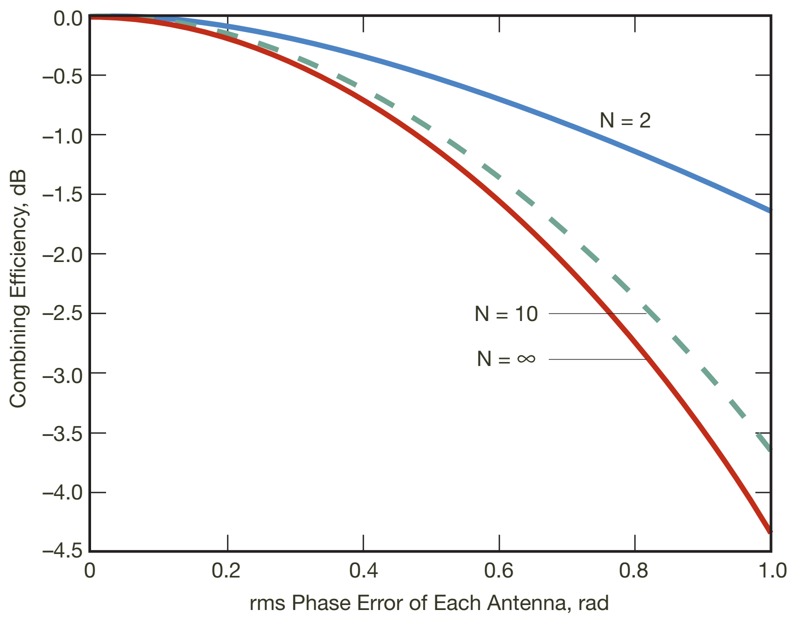

Figure 1. Combining efficiency from Equation (5) vs. rms phase error σ_0 for various numbers of antennas *N*. All antennas are assumed to be transmitting the same EIRP toward the target, and all have independent, zero mean, normally distributed phase errors.

Now consider the situation where the phase errors are partially correlated among the antennas, as is the case for atmospheric turbulence. Paths from antennas near each other are likely to suffer similar excess delay, while paths from antennas more widely separated are likely to be affected differently. This is modeled by the spatial structure function of delay, defined as

$$D_\tau(\vec{r}) = \left\langle \left[\tau(\vec{x}) - \tau(\vec{x} - \vec{r})\right]^2 \right\rangle \tag{7}$$

where $\tau(\vec{x})$ is the excess delay along the path from antenna position $\vec{x}$. The antenna positions are taken to be in a plane, so that $\vec{x}$ is two-dimensional. It is assumed that the mean-square delay difference is shift-independent, as is expected in the frozen flow model, where a fixed pattern of density fluctuations is blown across the array by the wind. The structure function will generally be different for different target positions, typically being larger at lower elevation angles. Sometimes horizontal asymmetries cause $D_\tau(\vec{r})$ to vary with direction, but often it is reasonable to assume that it depends only on distance $r = |\vec{r}|$, so we write $D_\tau(r)$. The structure function of delay can be scaled to obtain the structure function of phase at frequency f by

$$D_\phi(r) = (2\pi f)^2 D_\tau(r) \tag{8}$$

Inspection of Equations (7) and (8) shows that the structure function of phase at the spacing of antennas k and m is exactly σ_{km}^2 in Equation (4), hence

$$\langle P \rangle / P_{\max} = \frac{1}{N^2} \sum_k \sum_m \exp\left[-D_\phi \left(\left| \vec{x}_k - \vec{x}_m \right| \right) / 2 \right] \tag{9}$$

where $\vec{x}_k$ and $\vec{x}_m$ are the positions of antennas k and m, respectively.

III. Examples

The result in Equation (9) was given in [1],[1] where it was evaluated for a 400-antenna array in a specific pseudo-random configuration about 1.6 km in extent. It was also used in [3] to predict combining losses at Goldstone for a close-packed 19-element array of 12-m-diameter antennas 288 m in extent. The latter results are repeated here in Figure 2. The loss vs. frequency is computed for transmission at elevation angle 20 deg, over a range of values of the zenith rms delay fluctuation at 250 m separation. The delay fluctuation at each of the various separations in the array was scaled from the value at 250 m by assuming that the structure function follows a power law with exponent 5/3.

Another example, computed in the same way, is shown in Figure 3. Here the array consists of the three existing 34-m-diameter antennas at Goldstone known as DSS24, DSS25, and DSS26.

It can be seen that the performance of the three-element array is somewhat worse than that of the 19-element array. This is entirely attributable to the fact that most of the antenna separations of the latter are much smaller, so the delay fluctuations have greater correlation and the pair-wise rms phase differences are smaller. The three-element array has separations of 258 m to 494 m, while the 19-element array has many separations of 69.1 m and a maximum of 276 m.

Measurements over 12 months using a site test interferometer at Goldstone [3] gave a 90th percentile rms zenith delay at 250 m separation of 3.03 ps in the worst month (August) and 1.23 ps in the best month (November). Analysis of the same data[2] shows that the phase difference fluctuations on this baseline are normally distributed, as was assumed in the analysis.

IV. Conclusions

The theory leading to a model for the combining efficiency of a transmitting array has been reviewed, and the results have been applied to phase errors caused by atmospheric turbulence. Examples were presented for one hypothetical and one actual array at Goldstone. The relationship of these results to the well-known Ruze formula [2] for reflector antenna efficiency has been explained.

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[1] Equation (6) of reference [1] contains an error: The exponent should contain an additional factor of 1/2. Numerical results in that paper are nevertheless correct.

[2] D. Morabito, personal communication, Jet Propulsion Laboratory, Pasadena, California.

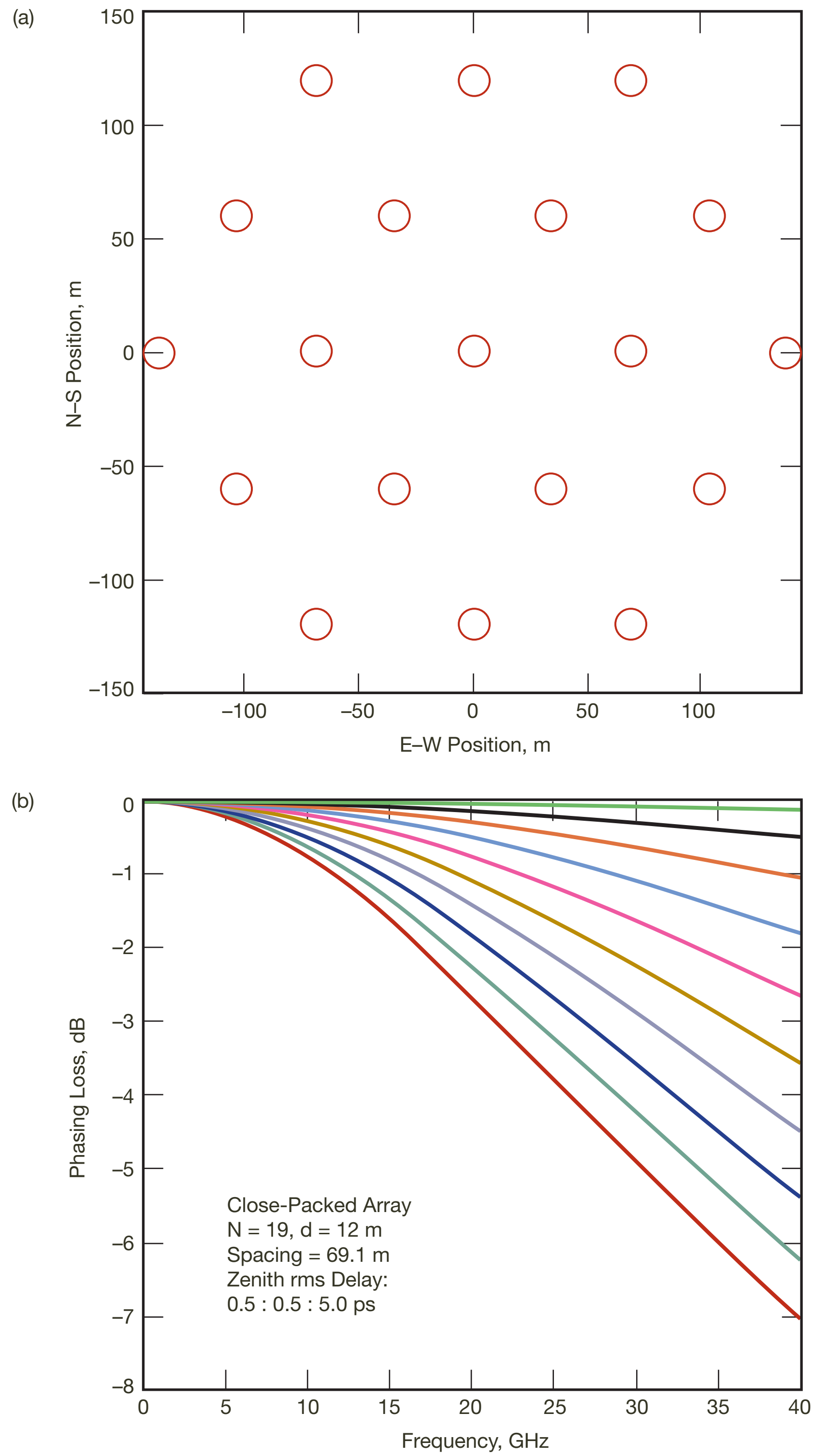

Figure 2. (a) Configuration on the ground of a close-packed array of nineteen 12-m-diameter antennas, where the spacing was chosen to avoid shadowing for elevations above 10 deg. (b) Calculated combining efficiency as a function of frequency for values of zenith rms delay fluctuation from 0.5 ps to 5.0 ps in steps of 0.5 ps. See text for further explanation.

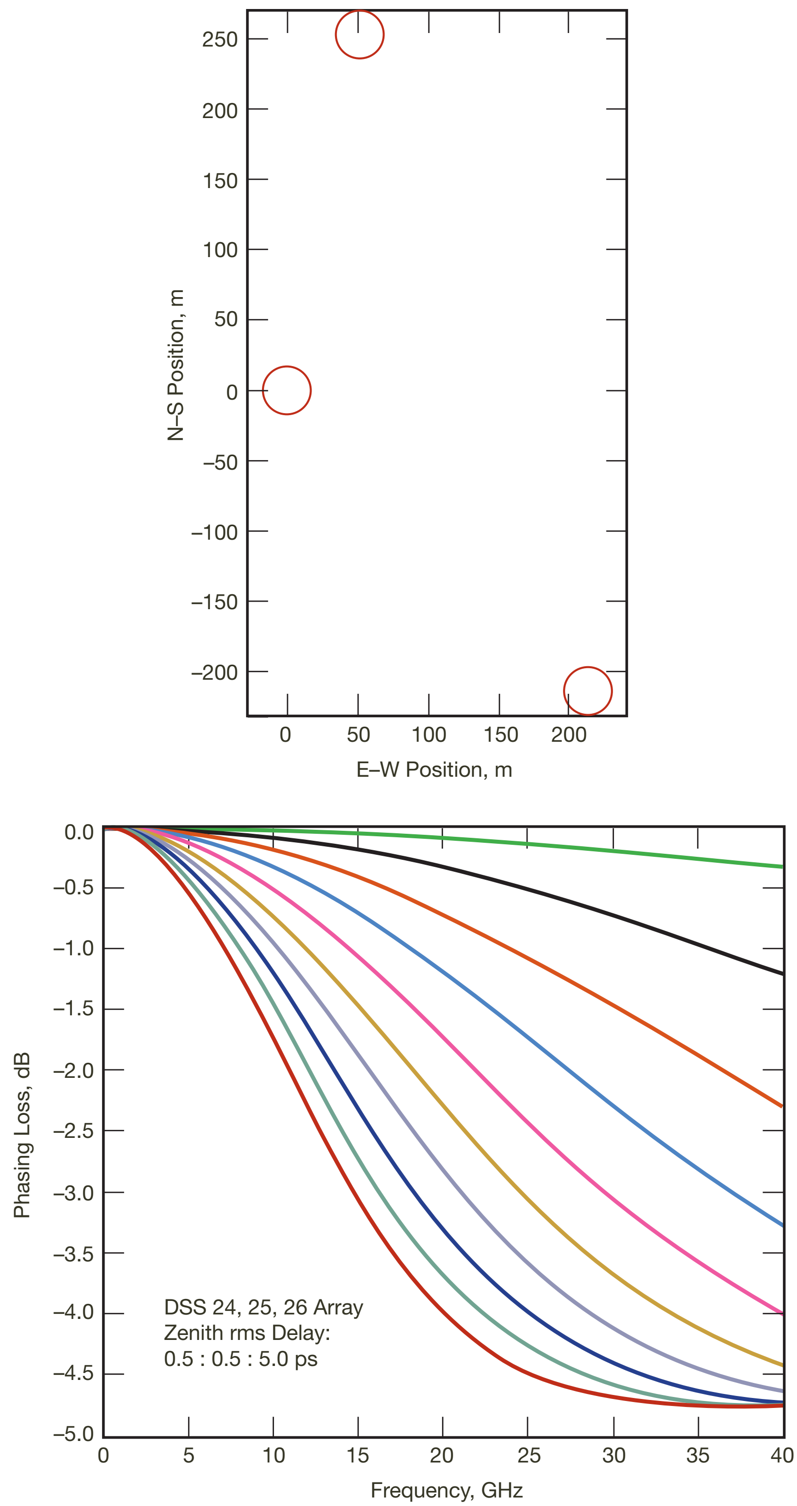

Figure 3. Similar to Figure 2, but for an array consisting of the three 34-m antennas known as DSS24, DSS25, and DSS26 at Goldstone's Apollo Complex.

Acknowledgments

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