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Statistical Link Analysis — A Risk Analysis Perspective

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This article discusses the advantages and caveats of a statistical link analysis method established by J. Yuen in the 1970s. Since then, the Jet Propulsion Laboratory has adopted this approach as flight principles in order to conduct link analysis for its deep-space missions. This article presents three new results: 1) we show that by invoking Lyapunov's condition of the central limit theorem, the Gaussian approximation of statistical link analysis is indeed mathematically valid if there is no single link parameter with a variance that is much larger than the others; 2) we discuss a system engineering approach of incorporating expert opinions as a mean to mitigate the lack of statistical knowledge of the link parameters; and 3) we also introduce the use of saddle-point approximation to estimate the tail probability in statistical link analysis in situations when the Gaussian approximation does not apply.

I. Introduction

A link budget is a system engineering tool that is used to evaluate mission data return and aid in communication system design. It consists of the calculation and tabulation of the useful signal power and the undesirable noise power available at the receiver. The signal and noise terms in the link equation are mathematical abstractions of the performance behavior expressed in decibel[1] (dB) units, and by summing up these terms, one can come up with an overall signal-to-noise ratio (SNR) estimate that can be used to characterize communication system performance, to support system design trade-offs, and to manage the operational risks associated with the usage of a link.

The main purpose of carrying out communication link analysis is to maximize the data throughput over a noisy channel, and at the same time to maintain the integrity of the data. Communication link characterizations are inherently statistical due to the uncertainties associated with the signal (distortion) and the channel (noise). The link parameters in a link equation are also statistical in nature as they are typically obtained from measurements of the hardware components, and are subject to measurement uncertainties.

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[1] The decibel unit (dB) is historically used as a measure of sound pressure level, but is also used for a wide variety of other measurements in science and engineering.

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In the context of maximizing mission data return for a communications pass, the link margin policy associated with the data rate is analogous to booking passengers on a jet flight: when the aircraft takes off, all the empty seats are useless.[2] Thus, it is highly desirable that one can accurately characterize the link performance in a statistical manner, such that one can design a link and establish the link margin policy based on quantifiable statistical confidence levels that meet the link quality requirements.[3]

J. Yuen formulated the analysis framework of statistical link analysis in the 1970s [1,2,3]. Since then, the Jet Propulsion Laboratory (JPL) has adopted this approach in the form of flight principles for conducting link analysis for its deep-space missions. The JPL projects and the Deep Space Network (DSN) have been measuring the performance statistics of hardware components and conducting experiments to characterize the statistics of weather effects on the link. These statistical data are thus folded into the statistical link analysis process.

Statistical link analysis has not been popular outside JPL. Part of the reason for this is not just the lack of understanding of the methodology, but also the main challenge with the unavailability of the link parameter statistics and the effort required to characterize it. This is particularly true for missions in the proposal phase, where information about flight or ground communications hardware and systems can be sketchy and sometimes might not exist. Therefore, the communication system performance is reliant upon educated guesses and speculation. This kind of link analysis typically assumes a single value for each link parameter, and assumes an arbitrary number in dB as a link margin requirement[4] [4]. When the link calculation yields a margin higher than or equal to the requirement, one would declare that the link is "closed." The problems of this approach are: 1) what are the values to be used as link parameters: best case, nominal case, or worst case? 2) What is the appropriate link margin policy? If 3 dB is chosen as the link margin policy, why is 3 dB enough? In other words, the typical link analysis approach is a "rule-of-thumb" method at best, and has no mathematical and statistical justification and cannot fulfill the fundamental objective of link analysis, which is to quantify the likelihood of whether a communication session with a given link configuration would be successful in transporting the data from point A to point B. In the absence of detailed statistical knowledge of link parameters, some telecommunication analysts resort to using numbers for a worst-case scenario. This approach does not take advantage of the statistical and random nature of the link, and results in an overly conservative link with a huge hidden link margin — analogous to the empty seats of an airplane, which are useless when the flight takes off.

This article addresses some of the fundamental principles of statistical link analysis. We also try to narrow the gap between a mathematically vigorous approach that demands the complete statistical characterization of the link, which sometimes can be hard to achieve, and the rule-of-thumb approach that uses single link parameters that are easier to realize, yet the approach lags mathematical justification.

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[2] In today's cutthroat competitive environment of the airline industry, airline companies prefer overbooking the flights to ensure full utilization of available seats, and to provide compensation to the frustrated customers who have to catch the next flight. This approach is analogous to the retransmission strategy used in data communication networks.

[3] Typically the link quality requirements are expressed in some form of error rate such as bit-error rate (BER), frame-error rate (FER), packet-loss rate, etc.

[4] The value 3 dB is a popular link margin requirement number.

We discuss the following topics in this article. In Section II, we provide an overview of statistical link analysis and discuss the strengths and caveats of its mathematical basis and assumptions. We also prove that the Gaussian approximation of statistical link analysis is not just a "hand-waving" approximation, but is mathematically valid under certain conditions. In Section III, we instigate a new mindset[5] of link analysis that incorporates expert opinions as a means to overcome the lack of statistical knowledge of link parameters. This approach allows one to use sound statistical instruments to integrate expert opinions in the form of an educated guess of the probability distribution function (pdf) of a link parameter, thus enabling one to quantify the link performance statistically with the best available information. In Section IV, we introduce the use of saddle-point approximation to estimate the tail probability in statistical link analyses in situations when the Gaussian approximation does not apply.

II. Mathematical Validity of Gaussian Approximation of Statistical Link Analysis

Link analysis starts with the following link equation:

$$\frac{P_T}{N_0} = \frac{EIRP \cdot G/T}{k \cdot L_s \cdot L_o} \tag{1}$$

where $EIRP$ is the effective isotropic radiator power of the transmitter, G/T is the gain over system noise temperature, which is a measure of the receiver sensitivity, k is Boltzmann's constant (1.38×10^{-23} J/K), $L_s = \left(\frac{4\pi d}{\lambda}\right)^2$ is the space-loss where d is the distance between transmitter and receiver, λ is the wavelength, and L_o denotes all other losses and degradation factors not specifically addressed in Equation (1).

The $EIRP$ term includes all the gain and loss terms on the transmission side, including pointing loss, the term G/T includes all the gain and loss terms on the receiver side, and the L_o term includes contributions of the intervening transmission media. Note that the link Equation (1) is multiplicative in nature. By taking the base-10 logarithm and multiplying by 10 on both sides of Equation (1), we convert the multiplicative relationship of the gain and loss terms to become an additive relationship. The additive terms are expressed in units of dB. Equation (1) can therefore be rewritten as

$$\frac{P_T}{N_0}(in\ dB) = EIRP(in\ dB) + G/T(in\ dB) - k(in\ dB) - L_s(in\ dB) - L_o(in\ dB) \tag{2}$$

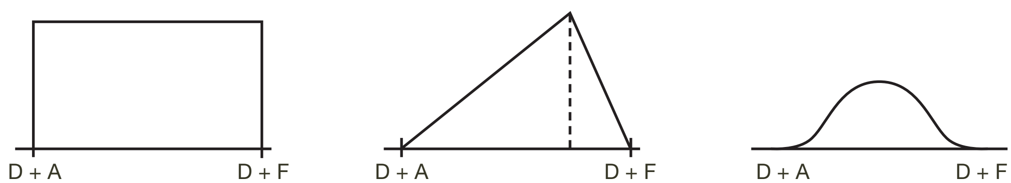
The concept of statistical link analysis relies on the additive nature of the link equation as given in Equation (2). Without loss of generality, we denote the link parameters (with units of dB) in terms of x_i. Each of the statistical link parameters x_i can be described in terms of a design value $x_{design,i}$, a minimum value $x_{\min,i}$, a maximum value $x_{\max,i}$, and a pdf $f_i(x_i)$ such that $f_i(x_i) \neq 0$ for $x_{\min,i} \leq x \leq x_{\max,i}$ and $f_i(x_i) = 0$ for $x_i < x_{\min,i}$ and $x_i > x_{\max,i}$. Some common forms of $f(x)$ are the rectangular (or uniform), triangular, and Gaussian distributions.[6] From this setup, one can deduce the mean of x (denoted by m_x) and the variance of x (denoted by σ_x^2). Let's denote the design value $D_x = x_{design}$, and define the favorable tolerance $F_x = x_{\max} - x_{design}$ and the advance tolerance $A_x = x_{\min} - x_{design}$. The computations of

[5] In the context of this article, the term "mindset" refers to the philosophical view or belief that includes a set of assumptions and methods to approach a problem or situation.

[6] Strictly speaking, the Gaussian distribution is unbounded. In link analysis, it is typically used to model weather effects or to model the combined effect of a number of link parameters (derived parameters).

the mean and variance of the uniform, triangular, and Gaussian distribution are given in Figure 1.

Computation of mean and variance of common probability density functions



Uniform
M = D + (F+A)/2
V = (F–A)^2/12

Triangular
M = D + (F+A)/3
V = (F^2+A^2–AF)/18

Gaussian
M = D + (F+A)/2
V = (F–A)^2/36

D = Design value
M = Mean
V = Variance
A = Adverse tolerance, which has a negative value for all link parameters except noise spectrum density and noise bandwidth
F = Favorable tolerance, which has a positive value for all link parameters except noise spectrum density and noise bandwidth

For noise spectral density and noise bandwidth, the favorable tolerance is a negative number, while the adverse tolerance is a positive number.

Figure 1. Conversions between design, favorable tolerance, and adverse tolerance and mean and variance of some popular probability density functions (from [1]).

Assume that there are n link parameters, x_i's that are independent. The ensemble of these link parameters $z = \sum_i x_i$ has mean $m_z = \sum_i m_{x_i}$ and variance $\sigma_z^2 = \sum_i \sigma_{x_i}^2$. The pdf of z, which we denote as $f(z)$, can be computed by convolving $f_{x_1}(x_1), f_{x_2}(x_2), \ldots, f_{x_n}(x_n)$. This is in general a computationally intensive process, as this involves levels of integration.

To simplify the computation, when a large number of independent link parameters are added together (in dB), Yuen proposed to approximate the resulting received SNR term with a Gaussian distribution $N(m_z, \sigma_z^2)$, where m_z is the mean and σ_z^2 is the variance as defined above. From this, one can design a link and establish link margin policy based on a statistical confidence level measured in terms of the σ of a Gaussian distribution function (e.g., 2σ event, 3σ event, etc.).

Note that in general, link parameters have different means, variances, and pdfs; thus, the above Gaussian approximation approach does not conform to the sufficient conditions of the classical central limit theorem, which requires all the link parameters to be independent and identically distributed. The procedure outlined in [1] did not justify this Gaussian approximation in a mathematically rigorous manner, and it did not explicitly state the conditions under which this Gaussian approximation is valid. However, decades of experience show that for links where there are many link parameters, this approach works well in most cases and closely approximates the Gaussian distribution.

In this article, we fill this gap by invoking a variant of the central limit theorem called Lyapunov's condition to prove that the aforementioned Gaussian approximation is indeed mathematically applicable under certain assumptions.

Lyapunov's condition states that for a sequence of independent variables x_i's such that each x_i has a finite mean m_{x_i} (m_{x_i} can be different from m_{x_j}) and a finite variance $\sigma_{x_i}^2$ ($\sigma_{x_i}^2$ can be different from $\sigma_{x_j}^2$), then $z = \sum_i x_i$ converges in distribution to a Gaussian distribution $N(m_z, \sigma_z^2)$ as long as the following condition is satisfied:

$$\text{For some } \delta = 0, \lim_{N \to \infty} \frac{1}{\sigma_z^{2+\delta}} \sum_{i=1}^{N} E\left(|x_i - m_{x_i}|^{2+\delta}\right) = 0 \tag{3}$$

The physical meaning of Lyapunov's condition is that the random variable x_i's need to have finite means and variances, and there is not a dominant term with a variance that greatly outweighs or exceeds the variances of the other terms.

To illustrate this point, we consider a toy example of the sum of four "well-behaved"[7] random variables with their distributions given in Table 1. Figure 2 shows the different shapes of these distributions.

Table 1. List of probability density functions.

Variables x_i	pdf $f_i(x_i)$		Mean and Variance
x_1: Uniform	$f_1(x_1) = 1$ $= 0$	for $1 \le x_1 \le 2$ otherwise	Mean = 1.5 Variance = 1/12
x_2: Uniform	$f_2(x_2) = 1$ $= 0$	for $2 \le x_2 \le 3$ otherwise	Mean = 2.5 Variance = 1/12
x_3: Triangular	$f_3(x_3) = 4x_3$ $= 4(1 - x_3)$ $= 0$	for $0 \le x_3 \le 0.5$ for $0.5 \le x_3 \le 1.0$ otherwise	Mean = 0.5 Variance = 1/24
x_4: Gaussian	$f_4(x_4) = \frac{4}{\sqrt{2\pi}} e^{\frac{4(x_4-1)^2}{2}}$	for $-\infty \le x_4 \le \infty$	Mean = 1.0 Variance = 1/4

The random variable $z = x_1 + x_2 + x_3 + x_4$ has a mean of 5.5 and a variance of 11/24. Let $f_z^G(z)$ be the Gaussian approximation for z with the same mean and variance. The plots of $f_z(z)$ and $f_z^G(z)$ are compared in Figure 3. One can see from the example that the exact pdf $f_z(z)$, which is very complicated, and its Gaussian approximation $f_z^G(z)$, are almost indistinguishable except in the vicinity about the mean. This example illustrates that for the purpose of finding the tail probability, $q_+(\alpha) = \int_\alpha^\infty f_z(z)dz$, where α of interest is at least 2σ away from the mean, the Gaussian approximation $f_z^G(z)$ is a good approximation to use to evaluate the risk associated with the link usage.

[7] "Well-behaved" means that the pdfs of random variables have variances that are comparable.

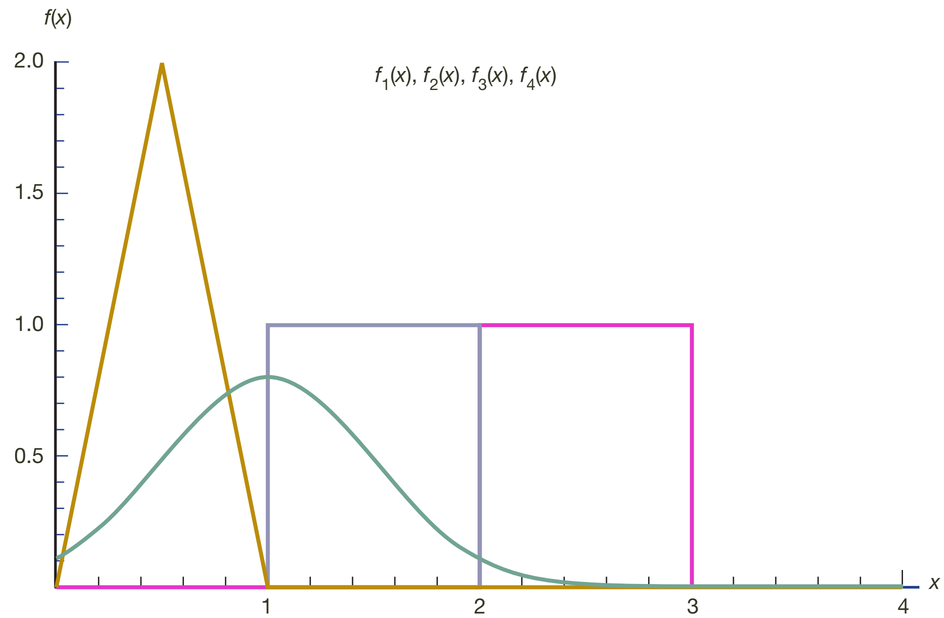


Figure 2. Diagram of pdfs.

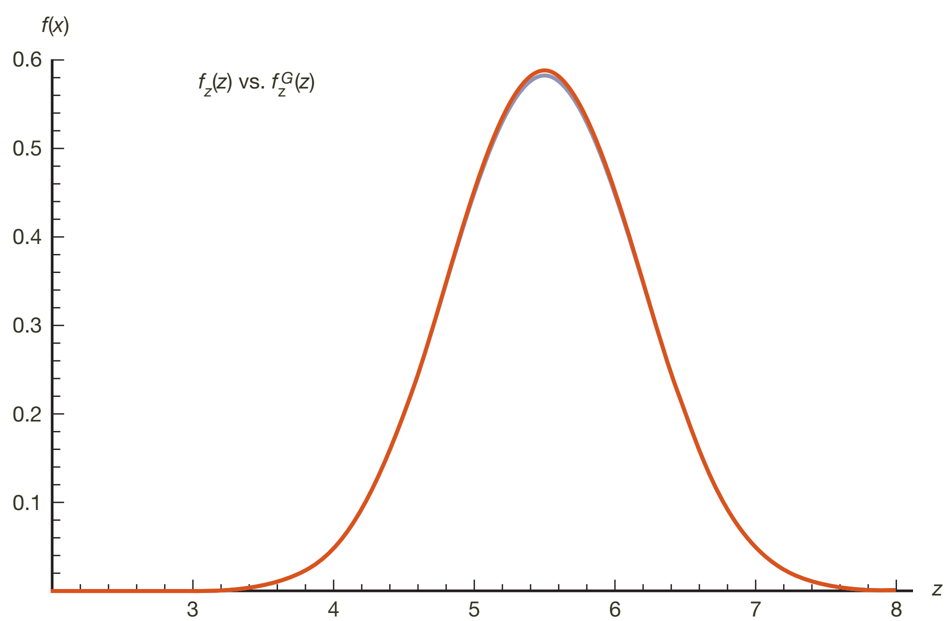


Figure 3. Comparison of $f_z(z)$ and $f_z^G(z)$.

In Section IV, we will describe a method that uses another variant of the central limit theorem called the saddle-point approximation, which is used to estimate the tail probability of the sum of independent variables where there are one or more terms with dominant uncertainties.

III. Notion of Incorporating Expert Opinions as a Means to Mitigate the Lack of Statistical Knowledge of Link Parameters

The main challenge of statistical link analysis is in addressing the difficulties in obtaining the statistical knowledge of the link parameters. For a mission in development or operational phases, the issue primarily lies in the effort required to statistically characterize the link parameters of the spacecraft, intervening media, and the network. For a mission in proposal or formulation phases, the problem is that sometimes the flight or ground communication hardware and systems might not exist, and the communication system performance is reliant upon educated guesses and speculations. Due to the difficulties associated with the lack of statistical knowledge of link parameters, heuristic approaches are typically used. One typical heuristic approach is to consider each link parameter as a single number, and to compute the overall SNR by just adding the gain and loss numbers together. A tendency of this approach is that people typically pick the worst-case numbers for use in the link equation, and the resulting link configuration or communication system design would be highly conservative with a lot of hidden and unquantifiable margin.

To alleviate the problem of the lack of statistical knowledge of link parameters, we propose to use expert opinions in the form of an educated guess of the pdf of a link parameter whose statistical knowledge is sketchy or lacking. By aggregating all the pdfs, both the well-established pdfs generated from historical data and measurements and the expert-opinion-derived pdfs in the statistical link analysis framework as outlined in Section II, the full statistical characterization of the resulting SNR in the form of the Gaussian approximation of the pdf can be computed based on the best available information at a given time. When more information is available and new measurements are made to improve the statistical knowledge of the link parameters, the expert-opinion-driven pdfs can be refined and incorporated in the statistical link analysis framework. This approach helps to bound the trade-space and to avoid the link design and development from marching into a solution that can be either too conservative or too risky in the time-evolving design process, as redesign in the late phase is always an expensive alternative.

Similar approaches of eliciting expert opinions to fill the gap of the lack of knowledge in the design process can be found in the recent literature in the area of system design and cost analysis [5,6], and are being studied from the viewpoint of merging well-established old information with sketchy new information in large-scale system design. The expert opinion-driven risk analysis methodology is also being explored in the general areas of spacecraft communications system design and spacecraft system design [7]. The following are the guidelines of formulating the expert opinions and performing statistical link analysis:

(1) For links in which all signal and noise parameters with uncertainties that are approximately the same order of magnitude, the Gaussian approximation as described in Section II can be used to perform statistical link analysis. For example, such cases would include space-to-space links and space-to-Earth links.

(2) There are links that have a gain or loss term with a dominant uncertainty. For example, fading loss as a result of ionization due to vehicle hypersonic entry into a planetary atmosphere, where vehicle movement causes large excursions in off-boresight angle of an antenna, and large features on surface terrain can be of the order of tens of dB. This fading term should be added to the rest of the parameters without invoking the Gaussian assumption. Mathematically, the resulting distribution of the sum is evaluated as the convolution of the pdf of this term with the Gaussian distribution of the ensemble of the rest of the terms. The link margin can then be computed from this resulting distribution, which can be complicated. Another approach is to use a variant of the saddle-point approximation to compute the tail probability as described in Section IV.

(3) The probability distribution functions of flight components will be either rectangular or triangular. For builds of six or fewer units, a rectangular pdf is the most likely. If more than six items of a design have been built, the actual values may suggest a triangular pdf.[8]

(4) For circumstances where the spacecraft is rolling about an axis, the worst-case antenna gain pattern cut should be used as the design value. Uncertainties about that value will include measurement error. An exception could be made for an "outlier" cut representing less than 1 percent of the pattern. For such a 99 percent performance consideration, that cut could be discarded.

(5) A list of common link parameters and their associated pdfs is given in Table 2(a) for uplink parameters, Table 2(b) for downlink parameters, and Table 2(c) for ranging parameters. These models were implemented in the JPL operational link analysis tool suites: the Telecommunications Forecaster Predictor [8,9]. Note that all derived parameters, which are functions of other parameters, are assigned Gaussian distributions.

IV. Estimating Link Margin in the Presence of One or More Dominant Terms

As shown in Section II, the Gaussian approximation can be used to greatly simplify the calculations of statistical link analysis as long as there is no dominant term with a large variance that outweighs that of all the other terms. In situations when one or more dominant terms exist, direct convolution or another approximation approach is needed.

In this section, we introduce the use of saddle-point approximation to estimate the tail probability in statistical link analysis. The details of this technique are described in [10], which uses a variant of the saddle-point approximation to estimate the detection probabilities of radar and optical communication systems.

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[8] Richard Horttor, personal communication, circa 2008.

Table 2(a). Uplink parameters.

Parameter Name	PDF Type
DSN transmitter power	Triangular
DSN transmitter waveguide loss	Triangular
DSN transmit gain	Triangular
DSN pointing control loss	Triangular
DSN EIRP	Gaussian (derived)
UL space loss	Deterministic
UL atmospheric attenuation	Deterministic
S/C receive/gain	Triangular
S/C pointing control loss	Uniform
S/C offpoint loss	Uniform
UL polarization loss	Uniform
S/C receive circuit loss	Uniform
UL Ptotal	Gaussian (derived)
UL SNT	Triangular
UL No	Triangular
UL Pt/No	Gaussian (derived)
UL command carrier/range suppression	Triangular
UL ranging carrier/data suppression	Triangular
UL Pcarrier	Gaussian (derived)
UL Pc/No	Gaussian (derived)
UL carrier loop noise bandwidth	Uniform
UL carrier loop SNR	Gaussian (derived)
UL command data suppression	Triangular
UL Pd/No	Gaussian (derived)
UL data rate	Deterministic
UL implementation loss or UL radio loss	Triangular
UL Eb/No	Gaussian (derived)
UL data rate capability	Deterministic (using margined Pt/No or Pd/No to calculate)

Table 2(b). Downlink parameters.

Parameter Name	PDF Type
S/C transmitter power	Triangular
S/C circuit loss	Uniform
S/C transmit gain	Triangular
S/C offpoint loss	Uniform
S/C pointing control loss	Uniform
S/C EIRP	Gaussian (derived)
DL space loss	Deterministic
DL atmospheric attenuation	Deterministic
DSN receive gain	Triangular
DSN pointing control loss	Uniform
DL polarization loss	Uniform
(For array) DSN array combining loss	Triangular
DL Ptotal	Gaussian (derived)
DL SNT	Gaussian
DL No	Gaussian
DL Pt/No	Gaussian (derived)
DL telemetry carrier/range suppression	Triangular
DL ranging carrier/data suppression	Triangular
(For DOR) DL DOR carrier/data suppression	Triangular
DL Pcarrier	Gaussian (derived)
DL Pc/No	Gaussian (derived)
(For DOR) DL DOR tone suppression	Triangular
(For DOR) DL Ptone/No	Gaussian (derived)
DL carrier loop noise bandwidth	Deterministic
DL carrier loop SNR	Gaussian (derived)
DL telemetry data suppression	Triangular
DL Pd/No	Gaussian (derived)
DL data rate	Deterministic
DL symbol per bit	Deterministic
(Non-BVR receiver, or measured system loss) DL implementation loss	Triangular
(BVR) DL radio loss	Triangular: if strong signal, deterministic (fixed at 0.3 dB)
(BVR) DL subcarrier demod loss	Triangular
(BVR) DL symbol sync loss	Triangular
(Optional) DL decoder loss	Triangular
(Optional) DL waveform distortion loss	Triangular
DL Eb/No	Gaussian (derived)
DL data rate capability	Deterministic (using margined Pt/No or Pd/No to calculate

Table 2(c). Ranging parameters.

Parameter Name	PDF Type
UL Pt/No	Gaussian (derived)
UL ranging suppression due to ranging	Triangular
UL ranging filtering loss	Triangular
UL Pr/No	Gaussian (derived)
UL ranging channel BW	Triangular
UL ranging channel SNR	Gaussian (derived)
DL Pt/No	Gaussian (derived)
DL ranging suppression due to ranging	Triangular
DL Pr/No	Gaussian (derived)
DL noisy reference loss	Triangular
DL output Pr/No	Gaussian (derived)

Let z denote the sum n of independent random variables $x_1, x_2, \ldots, x_n$, where x_i has a pdf $f_{x_i}(x_i)$ for $1 \le i \le n$. That is, $z = \sum_{i=1}^{n} x_i$. Let $f_z(z)$ denote the pdf of z, and $\Psi_z(s) = \int_{-\infty}^{\infty} e^{sz} f_z(z)dz$ denote the characteristic function of $f_z(z)$. The straightforward approach to evaluate $f_z(z)$ as the convolution of $f_{x_1}(x_1), f_{x_2}(x_2), \ldots f_{x_n}(x_n)$ is usually impractical as this involves $n-1$ levels of integration. Another approach is to evaluate the characteristic function $\Psi_z(s)$ of $f_z(z)$, which is the product of the characteristic functions $\Psi_{x_1}(s), \Psi_{x_2}(s), \ldots \Psi_{x_n}(s)$ of $f_{x_1}(x_1), f_{x_2}(x_2), \ldots f_{x_n}(x_n)$. The problem with this approach is that it is difficult to invert $\Psi_z(s)$, which can be a very complicated expression, back to $f_z(z)$. Helstrom [10] introduced a variant of the saddle-point approximation that estimates the tail probability $q_+(\alpha) = \int_{\alpha}^{\infty} f_z(z)dz$. This method is useful in the case where the pdf $f_z(z)$ can be arbitrarily complicated but its characteristic function $\Psi_z(s)$ is known. This approximation is particularly useful for small $q_+(\alpha)$. The key result is that the approximation of $q_+(\alpha)$ can be expressed as a function of the characteristic function $\Psi_z(s_o)$, its first derivative $\Psi'_z(s_o)$, and its second derivative $\Psi''_z(s_o)$, where (s_o) is a positive root of some function $\psi(s)$. It is shown in [10] that the root exists and there is no need to invert $\Psi_z(s)$. The main result is described below.

Define the function $\psi(s)$ as follows:

$$e^{\psi(s)} = \frac{e^{-s\alpha}\Psi_z(s)}{s} \tag{4}$$

It was shown in [10] that the solution of $\psi(s)$'s first derivative $\psi'(s) = 0$ exists, and is denoted by (s_o). By applying the Taylor series expansion of $\psi(s)$ and truncating at the second term, it can be shown that the tail probability $q_+(\alpha)$ can be approximated by

$$q_+(\alpha) \approx \frac{e^{\psi(s_0)}}{\sqrt{2\pi\psi''(s_0)}} \tag{5}$$

where $\psi''(s)$denotes the second derivative of $\psi(s)$. It was shown in [10] that Equation (5) suffices for most engineering applications when α is at least one standard deviation from the mean $E(z)$. Typical link analysis usually requires a link margin of 2 or 3 σ (sigma);

i.e., a tail probability $q_+(\alpha)$ of 2 or 3 standard deviations from the mean $E(z)$. Thus, this method is useful for statistical link analysis. Table 3 shows the characteristic functions of some popular pdfs that can be useful in the saddle-point approximation. A detailed description of the computation of $q_+(\alpha)$ can be found in [11].

Table 3. Characteristic functions of some popular probability density functions.

Name	Probability Density Function	Characteristic Function
Uniform distribution (continuous)	$\frac{1}{b-a}$ for $\alpha \le x \le b$ 0 for $x < a$ or $x > b$	$\frac{e^{sb} - e^{sa}}{s(b-a)}$
Triangular distribution	$\frac{2(x-a)}{(b-a)(c-a)}$ for $\alpha \le x \le c$ $\frac{2(b-x)}{(b-a)(b-c)}$ for $c \le x \le b$	$2\frac{(b-c)e^{as} - (b-a)e^{as} + (c-a)e^{bs}}{(b-\alpha)(c-a)(b-c)s^2}$
Normal distribution	$\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$e^{\left(\mu s + \frac{\sigma^2 s^2}{2}\right)}$
Exponential distribution	$\lambda e^{-\lambda x}$	$\left(1 - \frac{s}{\lambda}\right)^{-1}$
Gamma distribution	$x^{k-1}\frac{e^{(-x/\theta)}}{\Gamma(k)\theta^k}$	$(1-\theta s)^{-k}$ for $s < 1/\theta$
Beta distribution	$\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$	$1 + \sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{s^k}{k!}$

We illustrate the saddle-point approximation technique with the following example. Consider the set of three distributions as shown in Table 4. These distributions have a wide range of variances. Figure 4 shows the sizes and shapes of these distributions.

Table 4. List of probability density functions for illustrative example.

Variables x_i	pdf $f_i(x_i)$		Mean and Variance
x_1: Uniform	$f_1(x_1) = 1/4$ $= 0$	for $2 \le x_1 \le 6$ otherwise	Mean = 4.0 Variance = 4/3
x_2: Triangular	$f_2(x_2) = 4x_2$ $= 4(1 - x_2)$ $= 0$	for $0 \le x_2 \le 1/2$ for $1/2 \le x_2 \le 1$ otherwise	Mean = 1/2 Variance = 1/24
x_3: Triangular	$f_3(x_3) = 4/3\ (x_3 - 4)$ $= 28/15 - 4/15\ x_3$ $= 0$	for $4 \le x_3 \le 4\,1/2$ for $4\,1/2 \le x_3 \le 7$ otherwise	Mean = 5 1/6 Variance = 31/72

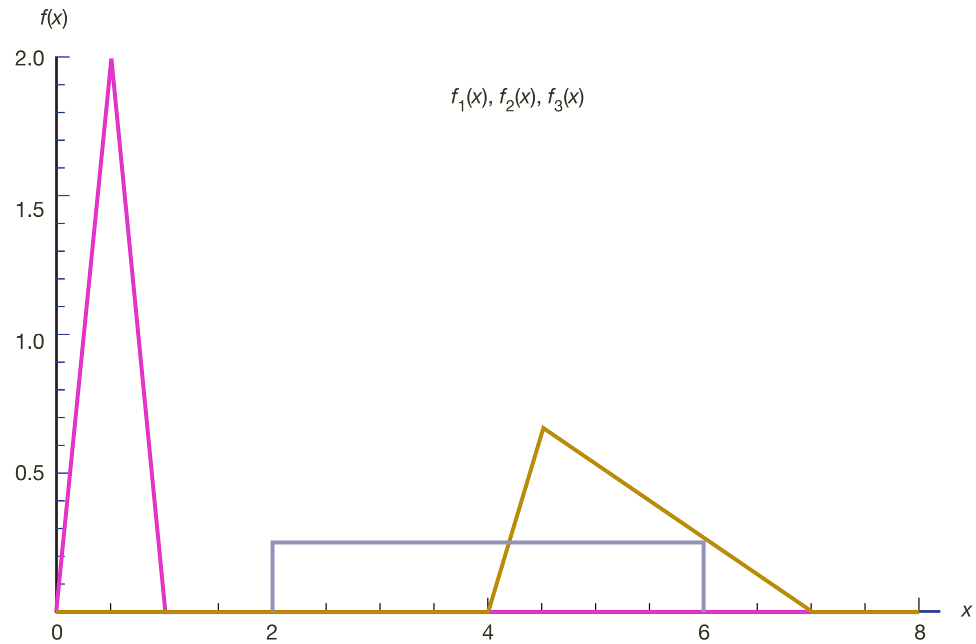

Figure 4. Diagram of pdfs used in example.

The random variable $z = x_1 + x_2 + x_3$ has a distribution $f(z)$ that can be computed by direct convolution of $f_1(x_1)$, $f_1(x_1)$, and $f_3(x_3)$, and is given by the following expression:

$$
f[z] = \begin{cases}
\frac{1}{90}(-14+z)^4 & \frac{27}{2} < z < 14 \\
\frac{1}{18}(-6+z)^4 & 6 < z \leq \frac{13}{2} \\
\frac{1}{720}(-1987 + 456\,z - 24\,z^2) & \frac{15}{2} < z \leq 9 \\
\frac{1}{720}(4375 - 648\,z + 24\,z^2) & \frac{23}{2} < z \leq 13 \\
\frac{1}{180}(-208787 + 72840\,z - 9516\,z^2 + 552\,z^3 - 12\,z^4) & 11 < z \leq \frac{23}{2} \\
\frac{1}{90}(-22081 + 13256\,z - 2976\,z^2 + 296\,z^3 - 11\,z^4) & \frac{13}{2} < z \leq 7 \\
\frac{1}{720}(-224113 + 69656\,z - 8088\,z^2 + 416\,z^3 - 8\,z^4) & 13 < z \leq \frac{27}{2} \\
\frac{1}{36}(-19991 + 8000\,z - 1200\,z^2 + 80\,z^3 - 2\,z^4) & 10 < z \leq \frac{21}{2} \\
\frac{1}{180}(-19955 + 8000\,z - 1200\,z^2 + 80\,z^3 - 2\,z^4) & \frac{19}{2} < z \leq 10 \\
\frac{1}{45}(9368 - 5034\,z + 1011\,z^2 - 90\,z^3 + 3\,z^4) & 7 < z \leq \frac{15}{2} \\
\frac{1}{720}(50501 - 22872\,z + 3864\,z^2 - 288\,z^3 + 8\,z^4) & 9 < z \leq \frac{19}{2} \\
\frac{1}{180}(289007 - 108176\,z + 15168\,z^2 - 944\,z^3 + 22\,z^4) & \frac{21}{2} < z \leq 11 \\
0 & \text{Otherwise}
\end{cases}
$$

The term $f(z)$ and its Gaussian approximation $f_z^G(z)$ are compared in Figure 5, which shows that the Gaussian approximation is not a good approximation in this case when the random variables have a wide range of variance values.

Using the procedure as outlined above, we compute the saddle-point approximation of the tail probability $q_+(\alpha)$ for $9 \leq \alpha \leq 14$. Figure 6 shows the approximated values (dotted line) compared to the exact values (solid line). This example shows that the saddle-point approximation matches well with the exact function, especially when α is small.

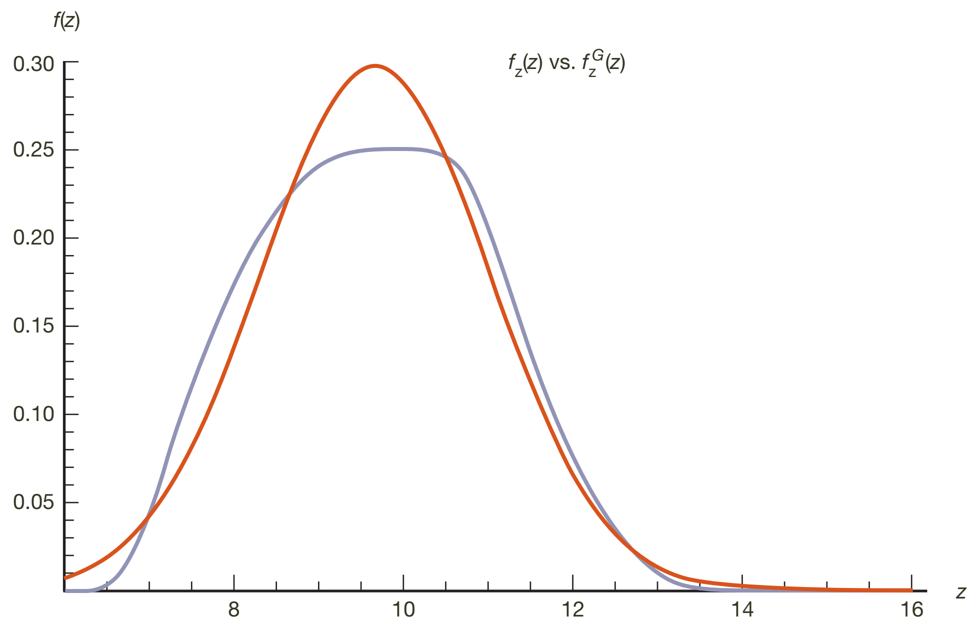

Figure 5. $f_z(z)$ and its Gaussian approximation.

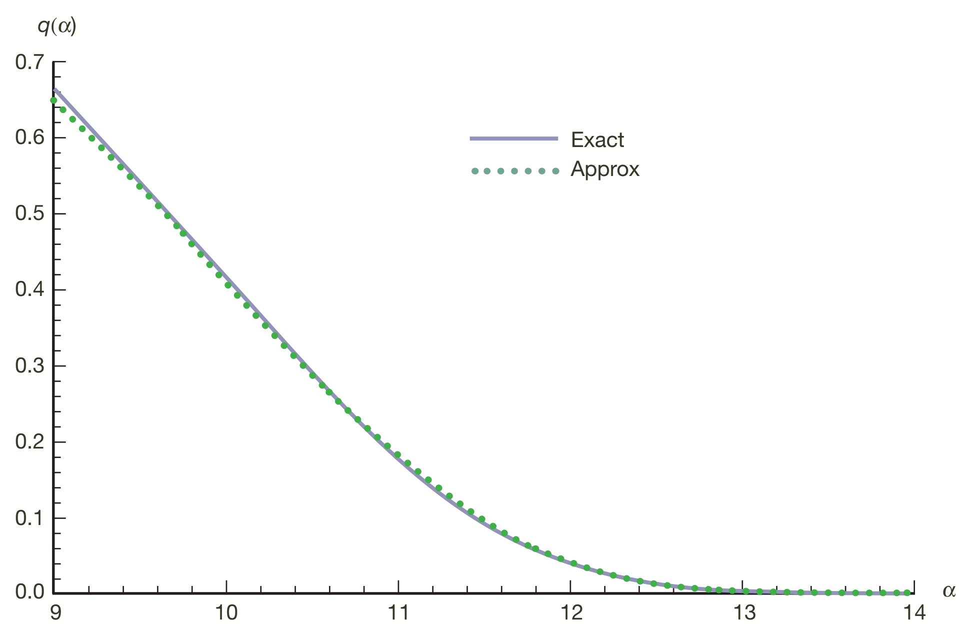

Figure 6. Tail probability $q_+(\alpha)$ and its saddle-point approximation.

IV. Conclusion

This article presents three new results in statistical link analyses:

(1) A theoretical result that shows that the Gaussian approximation of statistical link analysis is indeed mathematically valid if there is no link parameter with a variance that is much larger than the others.

(2) A system engineering result that introduces the concept of incorporating expert opinions as a means to mitigate the lack of statistical knowledge of the link parameters.

(3) An implementation result that bypasses the tedious convolution and integration procedures by using a variant of saddle-point approximation to compute the tail probability of a statistical link analysis.

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