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A New Geometric Trilateration Scheme for GPS-Style Localization

Kar-Ming Cheung* and Charles Lee*

ABSTRACT. — In this article, we introduce a new trilateration method for GPS-style localization. We show by simulations that the localization accuracy of the new method is indistinguishable from that of the traditional GPS approach. On the user segment side, the new scheme is computationally more efficient. This has the potential to translate into lower cost, and to enable faster location acquisition in the more challenging and dynamic operation environments.

I. Introduction

The United States' Global Positioning System (GPS) infrastructure consists of 31 satellites [1] in medium Earth orbit (MEO) that provide 24/7 and global location and timing services for users on Earth's surface and in low Earth orbit (LEO). The cost of development, deployment, and operation of GPS is estimated to be about \$33 billion, and the annual operation and maintenance cost is about \$1 billion [1].

GPS provides three-dimensional (3-D) position estimates via trilateration, which refers to the general technique of computing position based on measurement of distances. The standard GPS trilateration scheme is expressed in terms of distance measurements and positions in an Earth-centered Cartesian coordinate system. The set of simultaneous equations is of the form

$$\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}+c\Delta t=d_i \qquad i=1,\ldots,n \tag{1}$$

where (x,y,z) is the position of vehicle V to be estimated, (x_i,y_i,z_i) are known positions of the GPS satellite S_i,[1] and n is the number of satellites; Δt is the clock bias between V and the GPS time standard, which is maintained by the GPS operation segments; and c is the speed of light. In the GPS trilateration computation, (x,y,z) and $c\Delta t$ can be solved uniquely for $n \geq 4$. The standard approach to solve Equation (1) is known as the Newton–Raphson

* Communications Architectures and Research Section.

[1] GPS satellites, and we assume S_i's, are all time-synchronized.

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method, which is a general iterative method that uses linear regression to find the root of a function [2].

In this article, we present a different trilateration framework and a new iterative method that is based on the Pythagorean theorem and solves the set of simultaneous equations for position estimation. The scheme differs from the traditional GPS trilateration scheme in the following manner:

(1) The current GPS approach (Newton–Raphson method) uses the absolute locations (x_i, y_i, z_i)'s of the GPS satellites as input to each step of the localization computation. The new method uses the directional cosines U_i's from Earth's center to the GPS satellite S_i.

(2) Both the Newton–Raphson method and the new method iterate to converge to a localized solution. In each iteration step, multiple computation-intensive matrix operations are performed. The Newton–Raphson method constructs a different matrix in each iterative step, and thus requires performing a new set of matrix operations in each step. The new scheme uses the same matrix in each iteration, and thus requires computing the matrix operations only once for all subsequent iterations.

In a later section, we show by simulations that for the same GPS ephemeris errors and pseudorange measurement errors, the localization performance of the new scheme is indistinguishable from that of the Newton–Raphson method.

On the user segment side, the new scheme provides a clear computation advantage. It requires only one set of matrix calculations for all iterations to converge to a localized solution, whereas the Newton–Raphson method performs a different set of matrix calculations per iteration. This has the potential to translate into lower cost, and to enable faster location acquisition in the more challenging and dynamic operation environments.

Using a similar problem formulation, we show that the new trilateration scheme can be used for relative positioning between aircraft or spacecraft separated by hundreds of kilometers, and still deliver meter-level accuracy. This enables precision formation flying. The results can be found in [3]. Surprisingly, the trilateration scheme for GPS-style absolute positioning executes the same computation procedures as those for relative positioning. Only the inputs to the algorithm are different. Thus, the same software or hardware implementation can be used for both applications.

The rest of the article is organized as follows: Section II outlines the traditional Newton–Raphson method for GPS trilateration position determination. Section III takes into account the clock bias between a vehicle and the GPS satellite constellation, and derives the localization method using pseudorange measurements from four or more GPS satellites. Section III also discusses the computation advantage of the new method, and compares the root-mean-square error (RMSE) performances with the Newton–Raphson method under different combinations of GPS ephemeris errors and pseudorange measurement errors. Section IV provides concluding remarks.

II. A Brief Outline of the Newton–Raphson Method for GPS Localization

The Newton–Raphson iterative method and its convergence are based on the approach of linear regression. Let $\vec{P} = (\tilde{x}, \tilde{y}, \tilde{z})$ be the estimated location for a given iteration. A residual location $\Delta\vec{P} = (\Delta x, \Delta y, \Delta z)$, and an estimated clock offset $\Delta = c\Delta t$ are computed by solving the following equation:

$$\Delta\vec{P}' = \left(G^T G\right)^{-1} GT\, \vec{d}$$

where

$$\Delta\vec{P}' = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta \end{bmatrix},$$

and

$$\vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}_{nx1}$$

for $n \geq 4$.

The matrix G is of the form

$$G = \begin{bmatrix} \frac{x_1 - \tilde{x}}{d_1} & \frac{y_1 - \tilde{y}}{d_1} & \frac{z_1 - \tilde{z}}{d_1} & -1 \\ \frac{x_1 - \tilde{x}}{d_2} & \frac{y_2 - \tilde{y}}{d_2} & \frac{z_2 - \tilde{z}}{d_2} & -1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x_n - \tilde{x}}{d_n} & \frac{y_n - \tilde{y}}{d_n} & \frac{z_n - \tilde{z}}{d_n} & -1 \end{bmatrix}_{nx4}.$$

The estimated location $(\tilde{x}, \tilde{y}, \tilde{z})$ is then updated as $(\tilde{x}, \tilde{y}, \tilde{z}) + (\Delta x, \Delta y, \Delta z) \rightarrow (\tilde{x}, \tilde{y}, \tilde{z})$. Note that

(1) The matrix G is constructed using the GPS satellite locations (x_i, y_i, z_i) as well as the estimated location $(\tilde{x}, \tilde{y}, \tilde{z})$ of P for a given iteration.

(2) The first three entries of row i in G correspond to the unit vector from the intermediate location $(\tilde{x}, \tilde{y}, \tilde{z})$ of each iterative step to the GPS satellite S_i.

(3) The estimated location $(\tilde{x}, \tilde{y}, \tilde{z})$ is different in each iterative step, thus the matrix G is different, and the complicated computation of $(G^T G)^{-1} G^T$ has to be performed in each step.

The details of this method can be found in many GPS books, e.g., [1].

III. The New Geometric Trilateration Scheme for Localization

In this section, we derive the new trilateration method and compare its performance with the Newton–Raphson scheme.

A. Derivation of Iterative Algorithm

Let E denote the center of Earth with coordinates $(0, 0, 0)$. Consider three points V, E, and S_1 that form a triangle $\wedge_1$ in the Euclidean space, as shown in Figure 1. Let r_1 be the range between E and S_1, and $r_1{}'$ be the pseudorange measurements between V and S_1. We consider the presence of the clock bias Δt between the vehicle V and the GPS satellites S_i's, $1 \leq i \leq n$. We assume that the clocks of the GPS satellites are perfectly synchronized. We express the unknown clock bias of the vehicle V with respect to S_1 as an unknown correction factor $\Delta = c\Delta t$ in the pseudorange measurements $r_1{}'$. The same correction factor Δ occurs in all other pseudorange measurements $r_i{}'$, $1 \leq i \leq n$. The problem formulation of this case is illustrated in Figure 1.

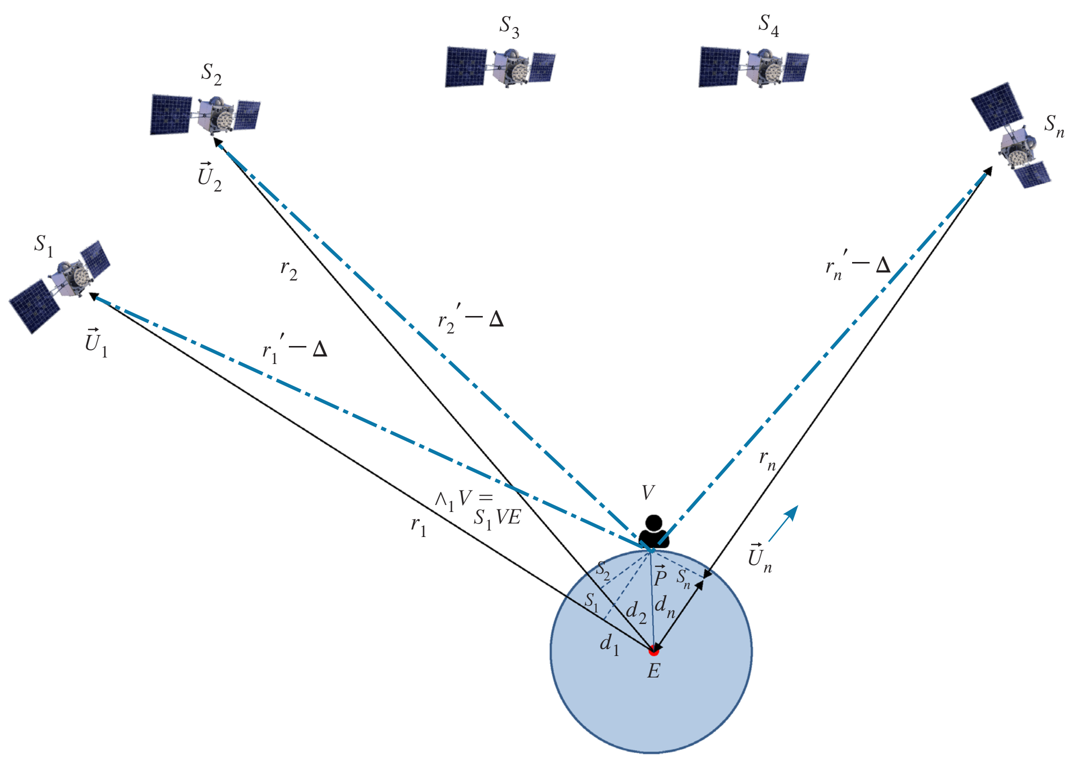

Figure 1. Geometry of the problem formulation.

Let

$$\vec{U}_1 = \begin{pmatrix} u_{x1} \\ u_{y1} \\ u_{z1} \end{pmatrix}$$

be the directional cosine of $\vec{S}_1$, i.e., the unit vector from E to S_1. Let s_1 be the altitude of $\wedge_1$ through V. Denote the projection of $\vec{P}$ onto $\vec{S}_1$ to be d_1, which can be expressed as the dot product between $\vec{U}_1$ and $\vec{P}$ (denoted by $\vec{U}_1 \circ \vec{P}$). Note that $\wedge_1$ is made up of two right-angled triangles that share a common side s_1, where $s_1 \leq \| \vec{P} \|$. Similarly, for each GPS satellite S_i, we form a triangle $\wedge_i$.

We construct the following relationships by applying the Pythagorean theorem on the two right-angled triangles of $\wedge_i$ in Figure 1:

$$s_i^2 = \| \vec{P} \|^2 - \left\| \vec{U}_i \circ \vec{P} \right\|^2, \tag{2a}$$

$$d_i = r_i - \left(\left(r_i' - \Delta \right)^2 - s^2 \right)^{\frac{1}{2}}. \tag{2b}$$

For $n = 4$, we define the vector

$$\vec{P}' = \begin{bmatrix} p_x \\ p_y \\ p_z \\ \Delta \end{bmatrix},$$

the matrix

$$A' = \begin{bmatrix} \vec{U}_1^T - 1 \\ \vec{U}_2^T - 1 \\ \vec{U}_3^T - 1 \\ \vec{U}_4^T - 1 \end{bmatrix},$$

and

$$\vec{d}' = \begin{bmatrix} d_1 - \Delta \\ d_2 - \Delta \\ d_3 - \Delta \\ d_4 - \Delta \end{bmatrix},$$

such that

$$\vec{P}' = \left(A' \right)^{-1} \vec{d}'. \tag{3}$$

Equation (3) forms an iterative relationship with Equations (2a) and (2b). In general, when there are n anchors, where $n \geq 4$, one can form additional Pythagorean relationships, as shown in the above equations, and compute the least-mean-square solution of an overdetermined system as follows:

$$A' = \begin{bmatrix} \vec{U}_1^T - 1 \\ \vec{U}_1^T - 1 \\ \vdots \\ \vec{U}_n^T - 1 \end{bmatrix},$$

and

$$\vec{d}' = \begin{bmatrix} d_1 - \Delta \\ d_2 - \Delta \\ \vdots \\ d_n - \Delta \end{bmatrix} = \begin{bmatrix} r_1 \sqrt{\left(r_1' - \Delta\right)^2 - s_1^2} - \Delta \\ r_2 \sqrt{\left(r_2' - \Delta\right)^2 - s_2^2} - \Delta \\ \vdots \\ r_n \sqrt{\left(r_n' - \Delta\right)^2 - s_n^2} - \Delta \end{bmatrix},$$

$$\vec{P}' = \left(A'^T A'\right)^{-1} A'^T \vec{d}'. \tag{4}$$

Based on the above formulation, we construct an iterative method that guarantees convergence to the vector $\vec{P}'$ using the GPS satellite ranges of $r_1, r_2 \cdots, r_n$ and pseudorange measurements of $r_1', r_2' \cdots, r_n'$, where $n \geq 4$. We outline the method for the case $n = 4$:

Iterative Procedure

(1) Initialization:

(a) Compute the directional cosines $\vec{U}_1$, $\vec{U}_2$, $\vec{U}_3$, and $\vec{U}_4$, and construct the matrix A'.

(b) Set $s_1 = s_2 = s_3 = s_4 = 0$.

(c) Set $\Delta = 0$.

(d) Compute $M' = A'^{-1}$ (or $M' = (A'^T A')^{-1} A'^T$ for $n \geq 5$).

(2) Compute d_1, d_2, d_3, and d_4 according to Equation (2b).

(3) Compute $\vec{P}' = M' \vec{d}'$ according to Equation (3).

(4) Compute s_1^2, s_2^2, s_3^2, and s_4^2 according to Equation (2a).

(5) Go to 2, and compute $\vec{P}'$ until $\vec{P}'$ converges.

Note that in the above iterative procedure, the matrix M' only needs to be computed once. We found that $\vec{P}'$ typically converges within five or six iterations in our simulation.

In North America, over 99.9 percent of the time vehicles see 6 to 12 GPS satellites [1], and therefore the matrix A' is of the order of 6 to 12. In every iteration step of trilateration, the calculation of $(A'^T A')^{-1} A'^T$ can be computationally intensive, and the convergence is a serial process that cannot be parallelized. Unlike the Newton–Raphson method that forms a different matrix G in each iterative step (see Section II), the proposed scheme forms a constant matrix A' that consists of directional cosines of the GPS satellites in view. Thus, the term $(A'^T A')^{-1} A'^T$ only needs to be computed once, and remains the same in each subsequent iteration.

B. Positioning Performance Simulations and Comparison

We consider a scenario of a vehicle V on Earth's surface, which is tracked by four GPS satellites at MEO. The positions of the vehicle and the satellites are specified in Table 1.

Table 1. Positions of user vehicle on Earth's surface and the GPS satellites.

Vehicle/ Satellite	Longitude, deg due east	Latitude, deg	Altitude, km
Surface Vehicle	280.3433	0	0
GPS Satellite 1	243.1105	35.4259	20200
GPS Satellite 2	253.6700	32.6300	20200
GPS Satellite 3	290.6021	–35.7773	20200
GPS Satellite 4	307.1953	5.2514	20200

We compute the RMSE performance of the new trilateration method derived in Section III.A, and compare with that of the Newton–Raphson scheme under the following simulation setup:

(1) The GPS is a highly calibrated system. The current GPS ephemeris errors are bounded to within 2 to 3 m (RMS).[2] We considered the cases with 3-D GPS ephemeris errors (RMS) of 0.0 m, 0.5 m, 1.0 m, 2.0 m, 5.0 m, 10.0 m, 30.0 m, and 35.0 m.

(2) For the GPS range measurement r_i and the pseudorange measurement $r_1{}'$,[3] $1 \le i \le n$, we add independent random errors of Gaussian distribution with zero mean and standard deviation of 0.0 mm, 1.0 mm, 2.5 mm, 5.0 mm, 1.0 cm, 2.0 cm, and 5.0 cm.

(3) We consider the cases of no media error (atmospheric delay).

(4) We perform 10,000 simulations for each combination of GPS ephemeris errors and pseudorange measurement errors, and compute the RMSE performance in each case.

The RMS localization error performance (in centimeters) of the new method and the Newton–Raphson scheme are shown in Tables 2 and 3, respectively. They are almost indistinguishable.

IV. Conclusion

In this article, we introduce a new trilateration scheme for GPS-style 3-D positioning that is based on the Pythagorean theorem rather than the Newton–Raphson linear regression method. We show by simulations that the localization accuracy of the new approach is almost indistinguishable from that of the Newton–Raphson scheme for the same combination of GPS ephemeris errors and pseudorange measurement errors.

On the user segment side, the new scheme provides a clear computation advantage. It requires only one set of matrix calculations for all iterations to provide a localized solution, whereas the Newton–Raphson method performs a different set of matrix calculations per

[2] See https://en.wikipedia.org/wiki/Error_analysis_for_the_Global_Positioning_System.

[3] In practice, the GPS satellites are well-calibrated; the GPS satellite S_i's range measurement r_i has much fewer random measurement errors compared to those of the vehicle's pseudorange measurements r_i'. However, in this article, for the sake of comparing the new scheme with the Newton–Raphson method, we use the same random error statistics for both r_i and r_i'.

Table 2. RMS localization error, in centimeters, of our proposed scheme.

Pseudorange Error, cm	GPS Satellite Position Error							
	0 m	0.5 m	1 m	2 m	5 m	10 m	30 m	35 m
0	0.00	115.45	231.59	459.94	1151.88	2306.75	6905.59	8030.99
0.10	1.88	116.01	228.51	463.12	1151.62	2293.96	6963.31	8053.13
0.25	4.63	115.35	230.68	461.79	1144.89	2303.23	6857.16	8079.36
0.50	9.28	115.90	231.00	460.80	1155.96	2299.06	6884.14	8109.05
1.00	18.64	116.39	231.14	457.64	1151.45	2292.72	6908.46	8052.74
2.00	37.11	120.61	233.60	465.27	1156.76	2318.15	6939.10	8086.46
5.00	93.41	147.56	247.40	470.02	1167.13	2320.61	6920.64	8106.20

Table 3. RMS localization error, in centimeters, of the Newton–Raphson scheme (traditional GPS scheme).

Pseudorange Error, cm	GPS Satellite Position Error							
	0 m	0.5 m	1 m	2 m	5 m	10 m	30 m	35 m
0	0.00	115.38	231.21	458.45	1148.60	2298.68	6911.34	8050.98
0.10	1.88	115.59	229.82	461.27	1147.69	2301.32	6904.79	8084.02
0.25	4.67	115.32	228.56	458.70	1152.21	2287.62	6908.94	8056.46
0.50	9.34	115.73	231.56	461.98	1150.70	2312.89	6979.49	8135.06
1.00	18.55	114.91	231.51	458.36	1157.32	2295.37	6916.55	8066.98
2.00	37.56	121.62	233.89	458.97	1143.87	2310.88	6919.93	8066.90
5.00	92.50	148.73	249.23	470.51	1161.83	2305.15	6882.36	8105.19

iteration. This opens up the opportunities for lower implementation cost, and enables new applications that require faster location acquisition in the more challenging and dynamic operation environments.

Another advantage of this new GPS-style trilateration scheme discussed in this article is that it executes the same computation procedures as the trilateration scheme for relative positioning [3]; only the inputs to the algorithm are different. Thus, the same software or hardware implementation can be used for both absolute positioning and relative positioning applications.

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References

[1] P. Misra and P. Enge, *Global Positioning System: Signals, Measurements, and Performance*, 2nd Edition, Ganga-Jamuna Press, 2012.

[2] B. Kumar, “Determination of GPS Receiver Position Using Multivariate Newton–Raphson Technique for Over Specified Cases,” *International Journal of Applied Engineering Research*, vol. 3, no. 11, pp. 1457–1460, 2008.

[3] K. Cheung and C. Lee, “A Trilateration Scheme for Relative Positioning,” presented at IEEE Aerospace Conference, Big Sky, Montana, March 4–11, 2017.