The Ultimate Limits of Binary Coding for a Wideband Gaussian Channel

S. A. Butman and R. J. McEliece
Communications Systems Research Section

This article presents, in graphical form, the theoretical relationship between bit signal-to-noise ratio, bit error probability, and bandwidth expansion factor for binary coded telemetry on a wideband Gaussian channel.

I. Introduction

In this article we calculate the theoretical limits of the performance of binary coding on a wideband Gaussian channel, such as the deep-space downlink on the National Aeronautics and Space Administration's (NASA's) planetary probes. Our results show the inescapable tradeoffs which must be made between signal power, data rate, bit error probability, and bandwidth occupancy.

Our motive, primarily, is to provide a convenient reference for system designers and coders. For example, it is known (Ref. 1) that a rate ½ (rate R of a binary-coded system is the number of information symbols per transmitted encoded symbol), constraint length 7 convolutional code, which is decoded with the Viterbi algorithm, requires a bit signal-to-noise ratio (E_b/N_0) of 2.6 dB in order to achieve a bit error probability (P_E) of 0.005. Figure 1 shows that it is theoretically possible to achieve the same P_E at the same rate with binary coding at a value of E_b/N_0 of only -0.1 dB. The minimum conceivable E_b/N_0 required to achieve $P_E = 0.005$ with no restriction on bandwidth expansion is seen to be about -1.8 dB. (Although current *Mariner* bit rates of the order 10^5 bits per second make binary codes of rates much less than ½ impractical, probes to the very distant outer planets will have much smaller bit rates and will allow the use of codes with larger bandwidth expansion, i.e., larger symbol-to-bit ratios. Thus it is important to know the theoretical limits of binary codes for arbitrary small values of R.)

The above example illustrates the usefulness of our results. They provide yardsticks against which the performance of an actual system can be measured.

II. Calculations

Let us consider a communications channel which accepts one of two real numbers, $\pm\alpha$, and adds to the transmitted number a normally distributed random variable z which has mean 0 and variance 1. This channel is an accurate model of a wideband Gaussian channel in which binary modulation is used and for which the output is sampled at the Nyquist rate, i.e., at a rate of $2W$, where W is the bandwidth of the signal. If a code of rate R is being used, the value of α turns out to be $(2RE_b/N_0)^{1/2}$. The formula for the capacity of the above two-input Gaussian channel is (Ref. 2)

$$C_2(\alpha) = \alpha^2 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} \log \cosh (\alpha^2 + \alpha y)\, dy \qquad \text{(nats/symbol)} \qquad (1)$$

Hence, according to well-known results of information theory, a bit error probability of p is achievable as long as

$$R(1 - H(p)) \leq C(\alpha), \qquad \alpha = (2RE_b/N_0)^{1/2} \qquad (2)$$

where $H(p) = -p \log p - (1-p) \log (1-p)$ is the natural entropy function. Formula (2), with equality substituted for the inequality, gives the relationship between the rate R and the error probability P_E, which we have plotted in Fig. 1.

We have restricted our attention to binary input schemes in this note because of their widespread, almost universal, use in practical systems. However, it is interesting to compare binary codes to codes which are allowed to use an arbitrary number of input levels. Here the formula corresponding to (2) is

$$R(1 - H_2(p)) \leq \frac{1}{2} \log_2 (1 + \alpha^2) \qquad (3)$$

where $H_2(p) = H(p)/\log 2$ is the binary entropy function. Rather than give another figure like Fig. 1 for nonbinary codes, we present Fig. 2, which compares the minimum possible value of (E_b/N_0) required to achieve $P_E = 0$ for binary and nonbinary codes as a function of the rate R. Notice that the gain possible by going to nonbinary codes is negligible until R reaches a value of 0.5 or more.

References

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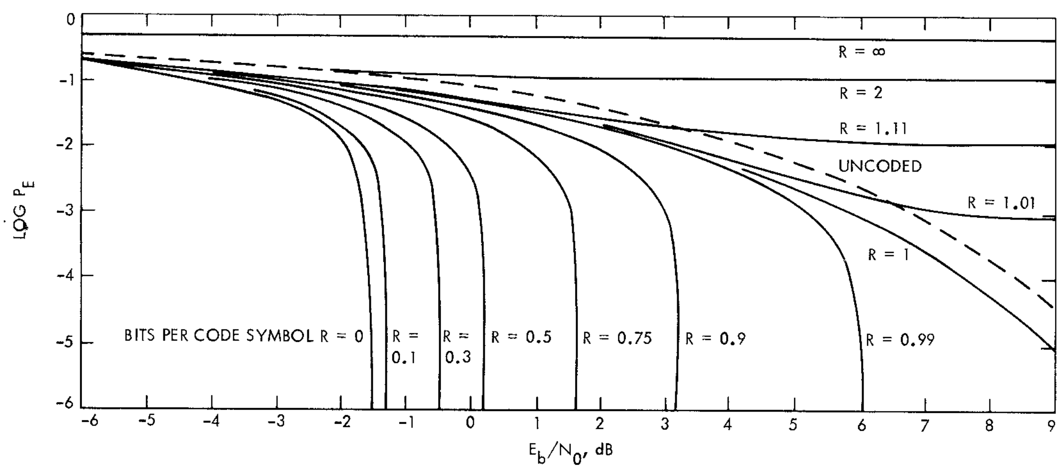

Fig. 1. Binary coding limits

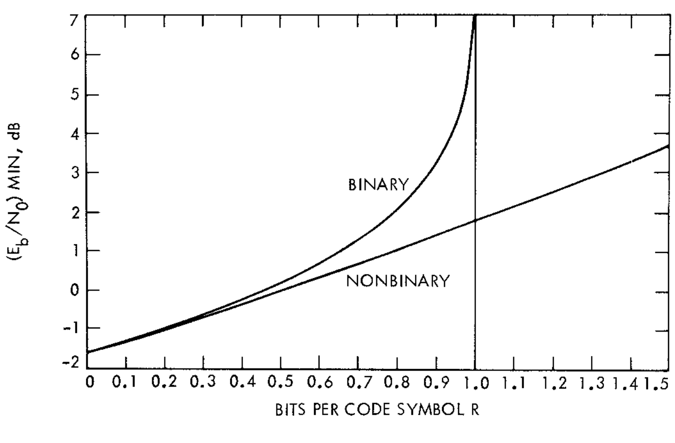

Fig. 2. Comparison of binary and nonbinary coding limits