Performance Degradation of Uncoded and Sequentially Decoded PSK Systems Due to Log-Normal Fading

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The 1978 Pioneer Venus (PV'78) orbiter will dispatch several probes into the planet atmosphere. The telemetry from these probes will be transmitted directly to Earth as coherent, binary phase-shift-keyed (PSK) uncoded or convolutionally encoded/sequentially decoded data. These communication links will be subjected to log-normal fading due to turbulence in the atmosphere of Venus. This paper offers a theoretical model for predicting the effects of the channel fading on PV'78 telemetry performance. Because this model considers the effects of a noisy carrier reference on the telemetry performance, it permits the determination of the optimum modulation angle which minimizes the link error rate for a given system. The model predicts that the fading will cause a 1.1 to 1.3 dB increase in the signal-to-noise levels required to achieve a frame deletion rate of 10^{-3} for the PV'78 coded telemetry modes.

I. Introduction

The 1978 Pioneer Venus mission (PV'78) will dispatch several probes into the planet atmosphere. The telemetry from these probes will be transmitted directly to Earth as coherent, binary phase-shift-keyed (PSK) uncoded or convolutionally encoded/sequentially decoded data. Data from the Russian spacecraft Venera 4 indicate that these communication links will be subjected to log-normal fading due to turbulence in the atmosphere of Venus (Ref. 1). This paper offers a theoretical model for predicting the effects of the channel fading on PV'78 telemetry performance.

An earlier article (Ref. 2) analyzed the degradation of PV'78 uncoded telemetry due to log-normal fading; however, this analysis was based on the idealization that the receiver was operating with a strong (noise-free) carrier reference (implying a small modulation angle). Layland has examined the problems of uncoded (Ref. 3) and sequentially decoded (Refs. 4, 5) detection with a noisy carrier reference. This paper extends his noisy reference models to include the effects of log-normal fading. As in the non-fading case, the noisy reference model illustrates that a given telemetry link has an optimum modulation angle which minimizes the link error rate.

II. Analysis

Signaling errors occur as bit crossovers in the uncoded case, and as frame deletions (buffer overflows) in the sequential decoding case. Conditioned on the effective received signal-to-noise ratio R in the data channel, the error rate is given by

$$P(\epsilon \mid R) = \begin{cases} Q(\sqrt{2R}); & \text{uncoded case} \\ D(R, N); & \text{sequential decoding case} \end{cases} \tag{1}$$

$Q(\cdot)$ is the gaussian error function defined by

$$Q(x) \equiv \frac{1}{\sqrt{2\pi}} \int_x^{\infty} dy \exp\left(-\frac{y^2}{2}\right) \tag{2}$$

Layland (Ref. 4, Eq. 3) has determined an approximation for the deletion rate of the form

$$D(R, N) \cong \min\left[1, \exp\left(\sum_{n,r} A_{n,r} R^r (\ln N)^n\right)\right] \tag{3}$$

where N is the computational capacity of the sequential decoder in computation per bit. The coefficients $\{A_{n,r}\}$ should be determined empirically. Pioneer Venus '78 will use a constraint length 32, rate 1/2 convolutional code, with a 512-bit frame. At the present time, an insufficient amount of PV'78 simulated sequential decoding data exists to determine the $A_{n,r}$'s. Consequently, the analysis below uses the $A_{n,r}$'s of Table 1, determined by Layland (Ref. 5, Table 1) for experimental Helios data. Although Helios uses an 1152 bit frame, its sequential decoding computation distribution should be similar to that for PV'78.

There is, in general, an effective memory duration T_m over which the decoder reaches certain intermediate decisions in its detection procedure. In the uncoded case, the detector makes a single decision based on an integration over a bit time T_B to decode each received bit:

$$T_m = T_B; \qquad \text{uncoded case} \tag{4}$$

For sequential decoding, the value of T_m is more obscure: a hard decision on a received bit may involve many individual searches of different lengths into the code tree. Layland (Ref. 6) has considered this problem at length, and has used a simplified analysis to derive an approximate formula for T_m (Ref. 6, Eq. 6):

$$T_m \cong 2T_B\left[1 - \frac{1}{N}\log_2\left(1 + \frac{N}{2}\right)\right]; \qquad \text{sequential decoding case} \tag{5}$$

Because of the log-normal fading and the noisy carrier reference, the received signal-to-noise ratio in the data channel is a random process of the form

$$R(t) = \rho[e^{\chi(t)} \cos\phi(t)]^2 \tag{6}$$

where

$$\rho \equiv \frac{P_T}{N_0} T_B \sin^2\theta \tag{7}$$

In the equations above, $e^{\chi(t)}$ is the log-normal fading process (Ref. 2, Eq. 2), $\phi(t)$ is the carrier reference phase error (e.g., Ref. 3, Eq. 1b), P_T/N_0 is the total received signal-to-noise ratio, and θ is the modulation angle. At very high data rates $R_B = 1/T_B$, $R(t)$ is essentially constant over T_m, and $\chi(t)$ and $\phi(t)$ can be represented by the random variables χ and ϕ. The expected link performance is then characterized by

$$P(\epsilon) = \overline{P(\epsilon \mid R = \rho\, e^{2\chi} \cos^2\phi)}^{\chi,\phi} \tag{8}$$

For small values of R_B, such that $R(t)$ contains many degrees of freedom over T_m,

$$P(\epsilon) = P\left[\epsilon \mid R = \rho\left(\overline{e^{\chi}\cos\phi}^{\chi,\phi}\right)^2\right] \tag{9}$$

Note in Eq. (9) that the expectation is not over the received signal-to-noise ratio $R = \rho\, e^{2\chi}\cos^2\phi$, but rather over $\sqrt{R}$, since it is, in fact, this parameter that characterizes the decoder performance.

The medium rate model is intended to predict the expected error rate for data rates between the two extremes above. A generalization of the approaches used separately for the fading and noisy reference problems (Refs. 2, 3, 5), it is based on the hypothesis that the conditional error rate is given by Eq. (1) with

$$R = \rho\,\alpha^2 \tag{10}$$

where

$$\alpha \equiv \frac{1}{T_m} \int_0^{T_m} \mathrm{d}t \, e^{\chi(t)} \cos \phi(t) \tag{11}$$

The determination of the statistical behavior of the random variable α is complicated by the dependence of $\phi(t)$ on $\chi(t)$. This problem is examined in Appendices A and B, where it is shown that α has the approximate form

$$\alpha \cong e^{\gamma}\left(1 - \frac{\sigma_\phi^2 \mu}{2}\right) \tag{12}$$

and expressions are derived for σ_ϕ^2, $p(\gamma)$, and $p(\mu|\gamma)$. Then, numerical integration techniques can be used to compute

$$P(\epsilon) = \int_{-\infty}^{\infty} d\gamma \, p(\gamma) \int_0^{\infty} d\mu \, p(\mu|\gamma) \, P\left[\epsilon \,|\, R = \rho e^{2\gamma}\left(1 - \frac{\sigma_\phi^2 \mu}{2}\right)^2\right] \tag{13}$$

Error rates computed using this model merge smoothly with those of Eqs. (8) and (9) as R_B approaches its extremes.

III. Results

Equation (13) was used to plot $P(\epsilon)$ versus θ, with parameter P_T/N_0, for the PV'78 telemetry modes under consideration. For example, Figs. 1 to 4 compare the non-fading ($\gamma = 0$) and fading cases for uncoded and sequentially decoded systems, for $R_B = 256$ bps. For a given P_T/N_0; it is seen that $P(\epsilon)$ is convex $\cup$ over θ, implying an optimum modulation angle θ_{opt}; because these curves have broad minima, deviations of several degrees from θ_{opt} do not significantly alter system performance. Figures 3 and 4 introduce the computational capacity

$$C \equiv NR_B \text{ computations per sec (cps)} \tag{14}$$

the present DSS capability is $C \sim 25{,}000$ cps.

Current PV'78 telemetry design objectives are a maximum bit error rate of 10^{-3} in the uncoded case, and a maximum frame deletion rate of 10^{-3} in the coded case. Table 2 compares the minimum values of P_T/N_0 required to achieve these error rates at the corresponding optimum modulation angles θ_{opt} for the four PV'78 data rates R_B. It is shown that θ_{opt} is typically about 10° lower for the sequential decoding modes as compared with the uncoded modes. Also, the model predicts that atmospheric fading will cause a 0.6 to 0.7 dB loss in P_T/N_0 in the uncoded case, and a 1.1 to 1.3 dB loss in the coded case.

IV. Commentary

Although the results above, and Table 2 in particular, have an air of finality, it should be remembered that they are based on a theoretical model, the derivation of which required several approximations. For the moment, these results should be accepted only as rough performance predictions, with further refinements required from telemetry simulation tests.

One particularly weak link in the medium rate model that applies only to the sequential decoding case should be identified. It is generally valid that the instantaneous received signal-to-noise ratio $R(t)$ has the form of Eq. (6) in accounting for the log-normal fading and noisy carrier reference. What may be questionable is the hypothesis that the conditional probability of a frame deletion is characterized by the time average of $\sqrt{R(t)}$ over a *single* interval T_m, as denoted by the random variable α in Eq. (11), despite the fact that many individual decisions over widely varying time intervals are made in decoding a frame of received data. This argument is based on the precedent of Layland's medium rate model for the effects of a noisy carrier reference on sequential decoding performance (Refs. 5 and 6). If we accept the premise that a unique value of T_m exists for which the single time average model yields accurate results, we may still question whether T_m is correctly specified by the theoretical approximation of Eq. (5). When some telemetry simulation data are available, a more accurate model can be produced by discarding Eq. (5) and selecting T_m such that the model conforms to the experimental deletion rate behavior.

Table 1. $\{A_{n'r}\}$ for Helios data

r	n = −1	0	1
0	2.397	8.824	−0.9887
1	−0.5331	−6.788	1.569
2	0.02303	0.8848	−0.8543

Table 2. Required P_T/N_0 to achieve $P(\varepsilon) = 10^{-3}$ at optimum modulation angle θ_{opt}, assuming no losses other than noisy reference and fading

R_B, bits/s	Uncoded case, Bit error rate = 10^{-3} bits/s				Sequential coding, Deletion rate = 10^{-3} bits/s			
	No fading		Log-normal fading		No fading		Log-normal fading	
	P_T/N_0, dB	θ_{opt}, deg	P_T/N_0, dB	θ_{opt}, deg	P_T/N_0, dB	θ_{opt}, deg	P_T/N_0, dB	θ_{opt}, deg
16	22.6	52	23.3	52	21.3	42	22.4	45
64	27.3	59	28.0	59	26.1	47	27.2	50
128	29.8	63	30.4	63	28.5	50	29.6	53
256	32.3	67	33.0	67	30.8	53	32.1	57

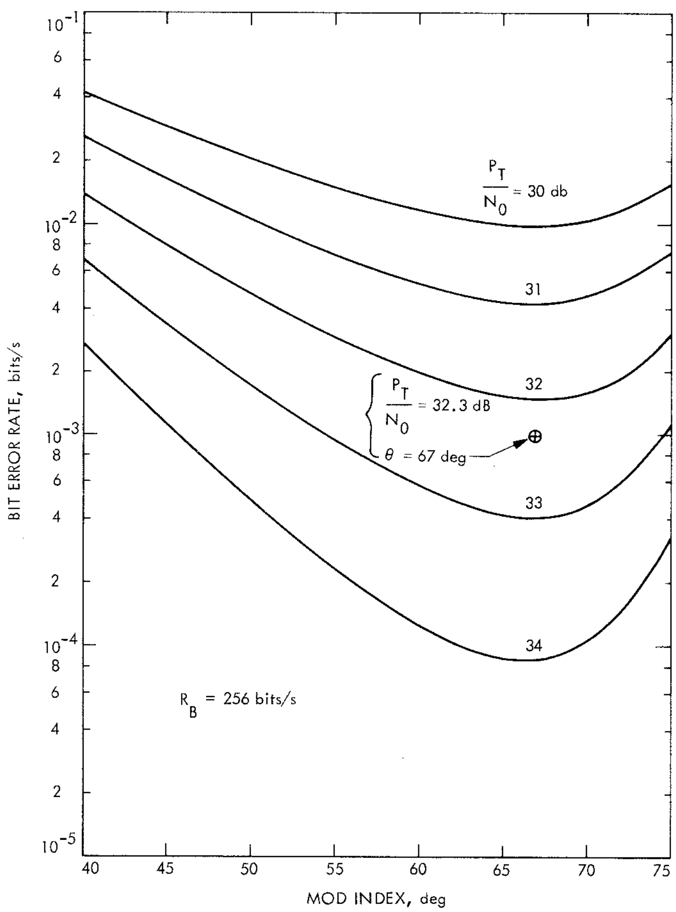

Fig. 1. Uncoded binary PSK, no fading

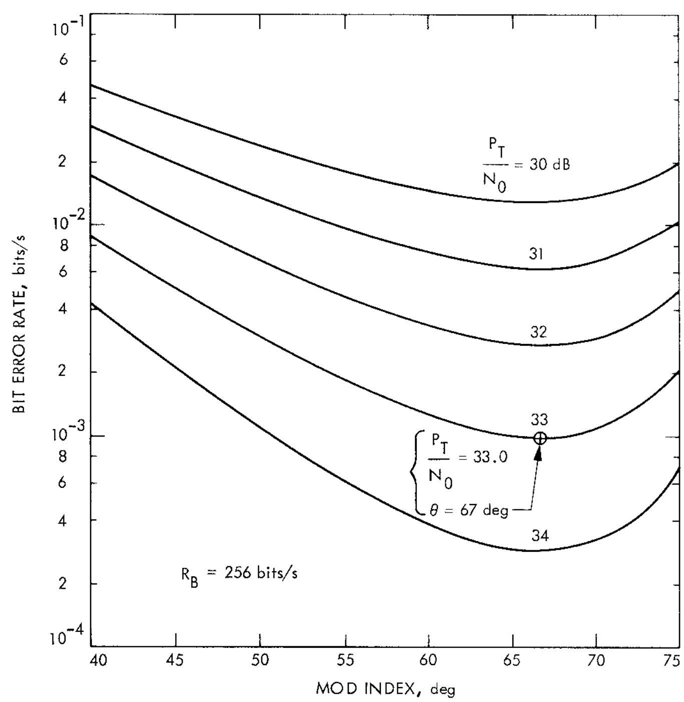

Fig. 2. Uncoded binary PSK, log-normal fading

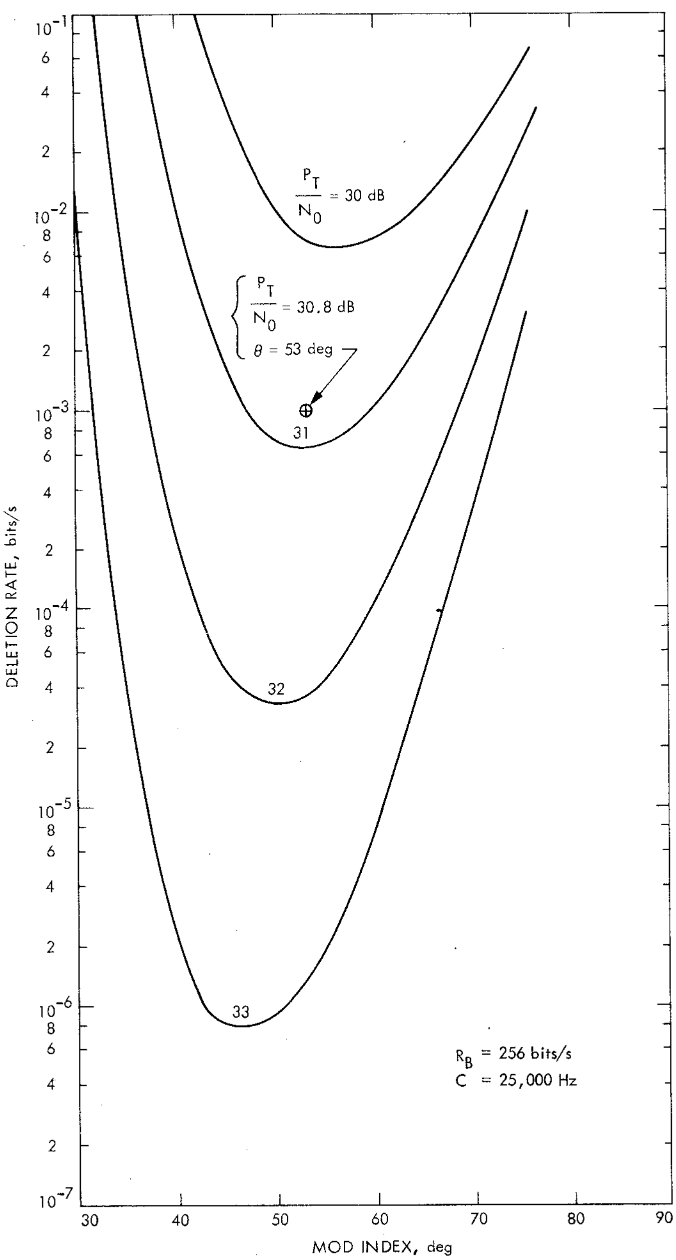

Fig. 3. Sequential decoding, no fading

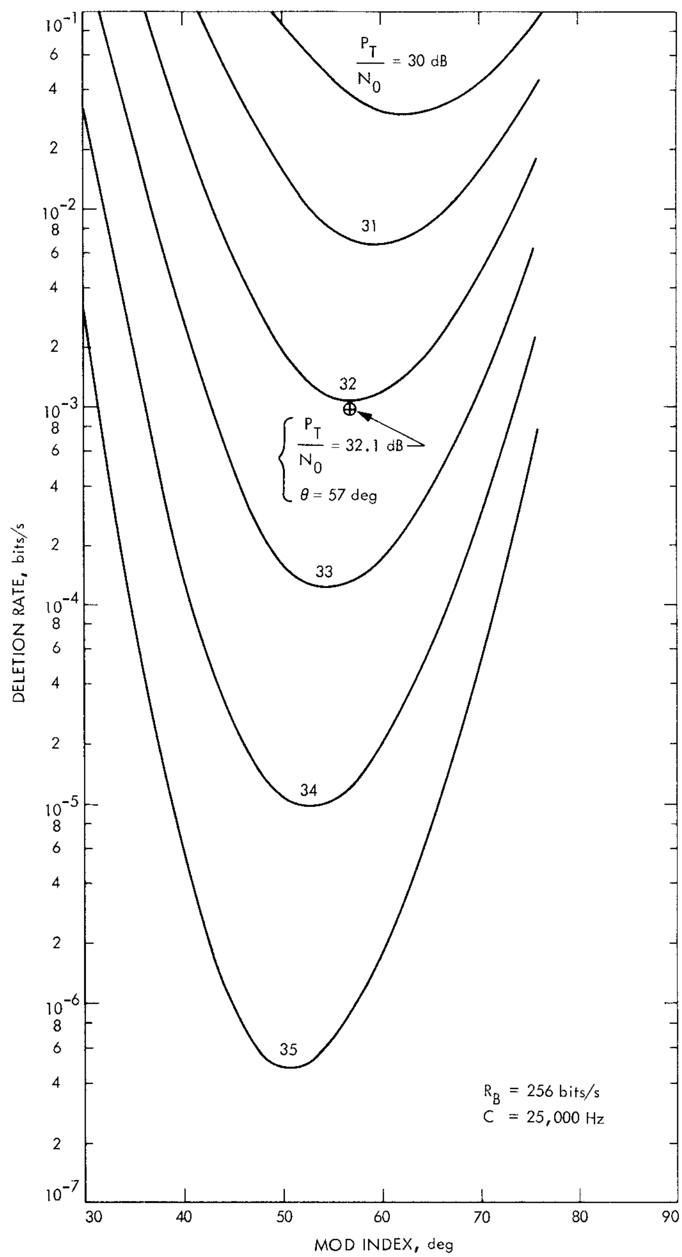

Fig. 4. Sequential decoding, log-normal fading

Appendix A

Linearized 2nd Order Phase-Locked Loop Preceded by a Bandpass Limiter in the Presence of Log-Normal Fading

This section documents approximate expressions used to compute the steady state phase jitter and noise bandwidth of a phase-locked loop receiver with log-normal channel fading.

Initially, suppose there is no fading. Linear phase-locked loop theory shows that the one-sided loop noise bandwidth B_L varies with the received signal-to-noise level (Ref. 7, Eqs. (5-18) and (8-14)):

$$B_L \cong K_1(r+1) \qquad \text{(A-1)}$$

where

$$r = K_2 \alpha_l \qquad \text{(A-2)}$$

and K_1 and K_2 are loop constants which need not be defined here. The limiter signal amplitude suppression factor α_l may be approximated by (Ref. 7, Eq. (8-13)):

$$\alpha_l \cong \sqrt{\frac{0.7854\eta_l + 0.4768\eta_l^2}{1 + 1.024\eta_l + 0.4768\eta_l^2}} \qquad \text{(A-3)}$$

in terms of the input signal-to-noise ratio

$$\eta_l = \frac{P_c}{N_0 B_l}$$

in the (fixed) limiter bandwidth B_l; P_c is the input power to the tracking loop, and N_0 is the one-sided input noise spectral density.

The tracking loop operating point is often defined in telemetry design control tables by the fictitious signal-to-noise ratio

$$\eta = \frac{P_c}{N_0\, 2B_{L0}} \qquad \text{(A-5)}$$

The subscript 0 on B_L above, and on other loop parameters, traditionally denotes the threshold design point, defined to occur when $\eta = 1$:

$$P_{c0} = N_0\, 2B_{L0} \qquad \text{(A-6)}$$

Using the threshold condition as a reference point, we can write

$$\eta = \frac{P_c}{P_{c0}} = \frac{\eta_l}{\eta_{l0}} \qquad \text{(A-7)}$$

$$\frac{B_L}{B_{L0}} = \frac{r_0\left(\frac{\alpha_l}{\alpha_{l0}}\right) + 1}{r_0 + 1} \qquad \text{(A-8)}$$

Typically, B_l is sufficiently large that the tracking loop is operating in the limiter suppression region, $\eta_l << 1$. Also, $r_0 = 2$ in a DSS receiver. Then we have

$$B_L \cong \left(\frac{2\sqrt{\eta} + 1}{3}\right) B_{L0} \qquad \text{(A-9)}$$

The effective signal-to-noise ratio in the operating loop bandwidth is

$$\rho_L = \frac{P_c}{\Gamma N_0 B_L} \cong \frac{6\eta}{\Gamma(2\sqrt{\eta} + 1)} \qquad \text{(A-10)}$$

where Γ is the limiter performance factor, which can be approximated by (Ref. 8, Eq. 21)

$$\Gamma \cong \frac{1 + \eta_l}{0.862 + \eta_l}; \qquad B_l > 10\, B_L \qquad \text{(A-11)}$$

In the limiter suppression region, we can write

$$\rho_L \cong \frac{5.172\eta}{2\sqrt{\eta} + 1} \qquad \text{(A-12)}$$

If ρ_L is large, the loop phase error $\phi(t)$ is essentially a zero-mean Gaussian random process with variance $\sigma_\phi^2 = 1/\rho_L$ (Ref. 7, Eq. (8-17)). For intermediate operating levels ($\rho_L > 3$), quasi-linear loop theory yields (Ref. 3, p. 84):

$$\sigma_\phi^2 = \frac{1}{\rho_L} \exp\left(\frac{\sigma_\phi^2}{2}\right) \qquad \text{(A-13)}$$

Now consider the effect of perturbing the system with log-normal channel fading. If the phase fading process is sufficiently narrowband relative to B_L, as is expected for PV'78, it will be tracked by the phase-locked loop receiver. The degradation in the tracking loop performance then results from the lognormal amplitude fading process, $e^{\chi(t)}$ (Ref. 2, Eq. 2). Equations A-9 and A-12 are applicable to the fading case if η is replaced by the random process $\eta e^{2\chi(t)}$, wherein η is now regarded as the signal-to-noise ratio in the absence of fading:

$$B_L \cong B_{L0}\left[\frac{2\sqrt{\eta}\, e^{\chi(t)} + 1}{3}\right] \tag{A-14}$$

$$\rho_L \cong \frac{5.172\eta\, e^{2\chi(t)}}{2\sqrt{\eta}\, e^{\chi(t)} + 1} \tag{A-15}$$

For a communication system employing binary PSK modulation, with modulation angle θ,

$$P_c = P_T \cos^2 \theta \tag{A-16}$$

where P_T is the total received power in the modulated carrier. Then Eq. (A-5) is

$$\eta = \frac{P_T \cos^2 \theta}{N_0\, 2B_{L0}} \tag{A-17}$$

Equations (A-13) to (A-15) and (A-17) define the tracking loop performance in terms of the system parameters P_T/N_0, B_{L0}, and θ and the fading process $e^{\chi(t)}$.

Appendix B
Medium Rate Model

In the main text, it is shown that the effects of log-normal channel fading and a noisy carrier reference depend on the random variable (Eq. 11)

$$\alpha \equiv \frac{1}{T_m}\int_0^{T_m} dt\, e^{\chi(t)} \cos\phi(t) \tag{B-1}$$

The fading term $\chi(t)$ is a stationary Gaussian random process, with mean $m_\chi = -\sigma_\chi^2$ (Ref. 7), and power spectral bandwidth B_χ. For the PV'78 study, Woo established that $\sigma_\chi^2 = 0.014$ (Ref. 8, Eq. 14), and $B_\chi \sim 1$ Hz (Ref. 2, Fig. 3). As discussed in Appendix A, $\phi(t)$ is a non-stationary, zero-mean Gaussian random process, whose variance σ_ϕ^2 and bandwidth B_L depend on $\chi(t)$.

Typically, σ_ϕ^2 is small enough to warrant the approximation $\cos\phi(t) \cong 1 - \phi^2(t)/2$:

$$\alpha \cong \frac{1}{T_m}\int_0^{T_m} dt\, e^{\chi(t)} - \frac{1}{2T_m}\int_0^{T_m} dt\, \phi^2(t)\, e^{\chi(t)} \tag{B-2}$$

The first integral is the time average of $e^{\chi(t)}$ over the effective memory interval $(0, T_m)$, which can be approximated by the log-normal random variable e^γ (Ref. 3):

$$e^\gamma \cong \frac{1}{T_m}\int_0^{T_m} dt\, e^{\chi(t)} \tag{B-3}$$

where γ has the probability density function

$$P(\gamma) = \frac{1}{\sqrt{2\pi\sigma_\gamma^2}} \exp\left(-\frac{(\gamma - m_\gamma)^2}{2\sigma_\gamma^2}\right) \tag{B-4}$$

with

$$\sigma_\gamma^2 = \ln\left\{1 + \frac{2\sigma_\chi^2}{\beta_\chi^2}\left[\exp(-\beta_\chi) - 1 + \beta_\chi\right]\right\} \tag{B-5}$$

$$m_\gamma = -\frac{1}{2}(\sigma_\chi^2 + \sigma_\gamma^2) \tag{B-6}$$

$$\beta_\chi \equiv 2\pi B_\chi T_m \tag{B-7}$$

For PV'78, we are concerned with data rates above 16 bps, so that $T_m \gtrsim 1/8$ sec. Therefore, $e^{\chi(t)}$ varies slowly over $(0, T_m)$ such that $\sigma_\gamma^2 \approx \sigma_\chi^2$. On the other hand, if $2B_{L0} = 12$ Hz and the tracking loop is operating sufficiently above threshold, B_L can be of the order of 50 Hz. Thus $\phi^2(t)$ varies much more rapidly than $e^{\chi(t)}$. To approximate the second integral in Eq. (B-2), assume $e^{\chi(t)}$ is relatively constant over $(0, T_m)$, having the value of its time average e^γ:

$$\frac{1}{2T_m}\int_0^{T_m} dt\, \phi^2(t)\, e^{\chi(t)} \cong \frac{e^\gamma}{2T_m}\int_0^{T_m} dt\, \phi^2(t) \tag{B-8}$$

Over $(0, T_m)$, conditioned on e^γ, σ_ϕ^2 and B_L are given by

$$\sigma_\phi^2 = \frac{1}{\rho_L}\exp\left(\frac{\sigma_\phi^2}{2}\right) \tag{B-9}$$

$$\rho_L \cong \frac{5.172\eta\, e^{2\gamma}}{2\sqrt{\eta}\, e^\gamma + 1} \tag{B-10}$$

$$B_L \cong B_{L0}\left(\frac{2\sqrt{\eta}\, e^\gamma + 1}{3}\right) \tag{B-11}$$

Define

$$\mu \equiv \frac{1}{\sigma_\phi^2 T_m}\int_0^{T_m} dt\, \phi^2(t) \tag{B-12}$$

The statistical behavior of the random variable μ has been determined (Ref. 2):

$$p(\mu|\gamma) = \begin{cases} \sqrt{\dfrac{a}{\pi\mu}} \exp\left(-a\mu - \dfrac{b}{\mu} + 2\sqrt{ab}\right); & \mu \geq 0 \\ 0; \quad \mu < 0 \end{cases} \tag{B-13}$$

$$a = \frac{\beta_\phi}{4}\left(1 + \sqrt{1 + \frac{4}{\beta_\phi}}\right) \tag{B-14}$$

$$b = a + \frac{1}{4a} - 1 \tag{B-15}$$

$$\beta_\phi = \frac{\beta_L}{1 - \dfrac{1}{4\beta_L}[1 - \exp(-4\beta_L)]} \tag{B-16}$$

$$\beta_L \equiv 2B_L T_m \tag{B-17}$$

Now, we have

$$\alpha \cong e^\gamma\left(1 - \frac{\sigma_\phi^2 \mu}{2}\right) \tag{B-18}$$

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