Confidence Bounds on Failure Probability Estimates

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One common way of estimating the failure probability p of a system is to simulate the system until a fixed number, say r, failures are observed and then set $p = r/N$, where N is the number of trials required. In this paper this technique is analyzed, and, in particular, the relationship between r and the reliability of the estimator p is studied.

I. Introduction

It is often necessary to analyze the behavior of a complex system by simulation rather than by direct analysis. That is, we make a probabilistic model for the system inputs and drive the system with pseudo-random inputs generated according to the model. We then estimate various performance parameters from the result of the simulation.

Now suppose the results of the system's behavior on a particular set of inputs can be classified as either "success" or "failure." Success could be, for example, "program execution time is less than one second," "antenna converged to target in 10 seconds," or "decoder correctly recognized command," etc. In such a case we might use binomial sampling, i.e., perform a fixed number n simulations of the system, and estimate the probability p of failure by $p = k/n$, where k is the number of observed failures in n trials. But without some *a priori* knowledge of p, the number of trials n required to obtain good estimate of p will not be known. Thus a better way is to perform trials until a desired number of failures, say r, have been observed, and then set $p = r/N$, where N is the number of trials required. It is this technique for estimating p that we shall analyze in this paper.

The sequence of successes and failures may be thought of as Bernoulli sequence in which a '0' represents a success and a '1' represents a failure. The number of successes, say X, before the r failures occur is a random variable distributed according to the negative binomial, i.e.,

$$P(X = x) = \binom{r + x - 1}{x} p^r (1 - p)^x; \qquad x = 0,1,2,\cdots \tag{1}$$

A number of methods is provided for constructing $1 - \gamma$ interval estimates for p. A *two-sided $1 - \gamma$ confidence interval* with limits p and p for p is defined as an interval (p,p) such that

$$P(p < p < p) = 1 - \gamma \tag{2}$$

The length of the interval is $L = p - p$. If $p = 0$, p is called the upper $1 - \gamma$ confidence limit. The interval (p,p) is said to be the shortest $(1 - \gamma)$ interval if its length L_s is minimum among all the intervals satisfying Eq. (2). Finally a two-sided $1 - \gamma$ confidence interval (p,p) is called *symmetric* if

$$P(p < p) = P(p > p) = \gamma/2$$

If the number of successes before the r failures is $X = n$ and Z is the total number of trials required to obtain the r failures, then our first method gives as $1 - \gamma$ confidence limits p and p defined by

$$I_p(r,n+1) = \gamma/2$$

and (3)

$$I_p(r,n) = 1 - \gamma/2$$

where $I_x(A,B)$ is the Incomplete Beta Function with parameters A and B; $0 \leq x \leq 1$. Our computationally easier second method gives

$$p = \frac{a}{2Z}, \quad p = \frac{b}{2Z} \tag{4}$$

where

$$P(\chi^2_{2r} < a) = \gamma/2 = 1 - P(\chi^2_{2r} < b),$$

χ^2_{2r} being a chi-square variate with $2r$ degrees of freedom. Limits in Eq. (4) are good for all p small enough that $-\ln(1-p) \approx p$. In Section III we show that the shortest $1 - \gamma$ interval is $L_s = (b-a)/2Z$ where a,b are such that $h_{2r}(a) = h_{2r}(b)$, $a \neq b$, $h_{2r}(t)$ is the χ^2_{2r}-density function. A table of a and b is provided for $r = 1,2,\cdots,$ 120 and $\gamma = 0.01(0.01)0.05$. This table is believed to be new.

It is important to know how good p is as an estimate of p. Suppose it is desired that $p = r/Z$ not be more than 10% greater than p, i.e., $p < 1.1xp$, where Z as in Eq. (4) is the number of trials required to obtain r failures. In Section IV it is shown in general that if $p/p < \alpha, \alpha > 1$ is desired such that $P(p < p) = 1 - \gamma$, then the number of failures required is bounded by

$$r \geq q\frac{\alpha^2}{(\alpha-1)^2}\phi^2_{1-\gamma}; \qquad q = 1 - \mathrm{p} \tag{5a}$$

where $\phi_{1-\gamma}$ is the lower $1 - \gamma$ point of the unit normal distribution. The minimum value of r that is good for all p is thus

$$r = \frac{\alpha^2}{(\alpha-1)^2}\phi^2_{1-\gamma} \tag{5b}$$

The exact values of r for given α,p and γ is

$$\sum_{r}^{\left\lfloor \frac{r}{p\alpha} \right\rfloor} \binom{Z-1}{r-1} p^r q^{Z-r} = \gamma \tag{6}$$

but it is not of much use because p is unknown, hence the importance of the estimate in Eq. (5b).

We shall illustrate many of our techniques with the example $p = 0.005$. This value of p is the classical maximum acceptable for the bit error probability in a deep-space television picture. For example to achieve a p which is only 10% greater than p and such that $P(p < p) = 1 - \gamma$, $\gamma = 5\%$, by Eq. (5b) we need $r \geq 327$ whereas by Eq. (6) $r \geq 313$ would do. This shows that r estimates in Eq. (5b) are very good. For comparison a graph of r values given by Eq. (6) for $\gamma = 0.1$, $\alpha = 1.1$ (i.e., 10% error) and $0.001 \leq p \leq 0.05$ is plotted.

II. Methods of Construction of Confidence Limits

Method A

A usual method for constructing confidence intervals based on Eq. (1) directly is to use the maximum likelihood estimate of p, $p = r/r + x$ (Refs. 1 and 2). If the experiment results in $X = n$ successes before r failures are obtained, then the $1 - \gamma$ symmetric confidence limits p and p for p are, respectively, the solutions to the equations

$$F_x(n,r,p) = \sum_{x=0}^{n} \binom{r+x-1}{x} p^r(1-p)^x = \gamma/2$$

and

$$\sum_{x=n}^{\infty} \binom{r+x-1}{x} p^r(1-p)^x = 1 - F_x(n-1,r,p) = \gamma/2 \tag{7}$$

It is known (see Ref. 3 for example) that the negative binomial distribution is related to the Incomplete Beta Function by

$$F_x(n,r,p) = I_p(r,n+1); \qquad 0 < r < \infty$$

where

$$I_p(m,\nu) = \frac{1}{B(m,\nu)} \int_0^p u^{m-1}(1-u)^{\nu-1}\,du$$

$$B(m,\nu) = \int_0^1 u^{m-1}(1-u)^{\nu-1}\,du \tag{8}$$

Thus p and p are such that

$$I_p(r,n+1) = \gamma/2$$

and

$$I_p(r,n) = 1 - \gamma/2 \tag{9}$$

We can read off the values p and p from the table of Incomplete Beta Functions in (Ref. 4) if r and n are fairly small, say. For larger values of r and n it is convenient to use the tables of the F-distribution and the relation

$$P(F_{v_2}^{v_1} > F_0) = I_p\left(\frac{v_2}{2}, \frac{v_1}{2}\right) \tag{10}$$

where

$$p = \frac{v_2}{v_2 + v_1 F_0}$$

and $F_{v_2}^{v_1}$ is an F-variate with v_1,v_2 degrees of freedom. Applying Eq. (10) to Eq. (9) we see that

$$p = \frac{r}{r + (n+1)\,F_{2r}^{2(n+1)}\,(\gamma/2)}$$

and

$$p = \frac{F_{2n}^{2r}\,(\gamma/2)}{n + r\,F_{2n}^{2r}\,(\gamma/2)}$$

where

$$P(F_{v_2}^{v_1} > F_{v_2}^{v_1}\,(\gamma/2)) = \gamma/2)$$

For example, for $p = 0.005$, $r = 10$, the expected value of n is 1990. $F_{2r}^{2(n+1)}\,(0.025) = 2.09$, $F_{2n}^{2r}\,(0.025) = 1.71$ and the expected 95% symmetric confidence limits for p are

$$p = 0.0024$$

$$p = 0.0085.$$

Method B

Apart from its computational simplicity, the following construction is better suited for the range of p of interest here.

Let X_j, $j = 1, \cdots, r$ be the number of trials after the $(j-1)^{st}$ failure to the occurrence of the j^{th} failure. Then X_j has the geometric distributions with parameter p, i.e.,

$$P(X_j \le n_j) = 1 - (1-p)^{n_j}; \qquad n_j = 1,2,\cdots$$

The geometric distribution may be approximated by exponential distribution

$$P(X_j \le x) = 1 - e^{-\lambda x}$$

such that $\lambda = \ell n\ (1-p)$. With this approximation, the sequence $X_1,X_2,\cdots,X_j,\cdots,X_r$ becomes a sequence of exponentially distributed random variables with a common parameter λ. It is known that

$$Z = \sum_1^r X_j$$

is then distributed according to the Gamma distribution such that

$$W = 2\lambda Z$$

has the χ_{2r}^2 distribution with $2r$ degrees of freedom with density function given by

$$h_{2r}(x) = \frac{1}{2^r\Gamma(r)}\,x^{r-1}\,e^{-x/2}; \qquad x \ge 0. \tag{11}$$

Thus a $1-\gamma$ confidence interval for λ is given by

$$\lambda = a/2Z, \qquad \lambda = b/2Z$$

such that

$$P(a < 2\lambda Z < b) = 1 - \gamma. \tag{12}$$

a and b may be obtained from the χ_{2r}^2 table in Ref. 4. For small p we may take $p = \lambda = -\ell n(1-p)$. The effect of this approximation on the confidence intervals for p is very negligible indeed.

Suppose we sample till $r = 10$ failures occur and $Z = 2000$, then we may conclude that with probability 0.95

$$p < 0.00785$$

or that with probability 0.99

$$p < 0.00939$$

Because tables of χ^2-distribution exist for high degrees of freedom this method is useful when p is small so that r is large.

By Eq. (12) it is clear that if, corresponding to a given r, it is required to have $Z_0 = X + r$ trials to achieve level $1 - \gamma$, we may stop sampling before the r failures are observed and conclude that Eq. (12) holds if the number of trials Z is already greater than Z_0.

III. Shortest Confidence Intervals

If for a fixed γ, confidence intervals of lengths L_1 and L_2 are constructed for p such that $L_1 < L_2$, we would normally prefer L_1 to L_2. In this section a method of constructing shortest confidence intervals will be given (see Ref. 5).

From Eq. (12), the length of the interval for λ is

$$L = \frac{1}{2Z}(b - a). \tag{13}$$

We want to minimize L subject to

$$\int_a^b h_{2r}(t)\,dt = 1 - \gamma \tag{14}$$

where $h_{2r}(t)$ is given by Eq. (11). Partially differentiating Eqs. (13) and (14) we have

$$\frac{\partial L}{\partial a} = \frac{1}{2Z}\left(\frac{db}{da} - 1\right)$$

and

$$h_{2r}(b)\frac{db}{da} - h_{2r}(a) = 0$$

which gives

$$\frac{\partial L}{\partial a} = \frac{1}{2Z}\left(\frac{h_{2r}(a)}{h_{2r}(b)} - 1\right)$$

$\partial L/\partial a$ is minimum when

$$h_{2r}(a) = h_{2r}(b) \tag{15}$$

That is, the a and b which gives minimum confidence length L_s are the pair satisfying Eq. (15) for $a \neq b$.

Table 1 gives the values of a and b satisfying Eq. (15) for $\gamma = 0.01(.01).05$ and $r = 1, \cdots, 120$. We think this table is new.

Let us find the *expected* length of the interval in the case $r = 10$, $p = 0.005$. In this case expected $Z = 2000$ and a,b, from the table, giving 95% confidence on p, are

$$a = 8.5841, \qquad b = 32.607393$$

Thus

$$p = a/2Z = 0.002146, \qquad p = b/2Z = 0.0081518$$

and the length $L_s = 0.0060058$.

The symmetric case given by Eq. (12) makes

$$a = 9.59083, \qquad b = 34.1696$$

from χ^2_{20} table. Thus

$$p = 0.0023977, \qquad p = 0.0085424$$

and the length $L = 0.0061447$. In this case the shortest interval results in a modest 2% gain over the symmetric 95% confidence interval.

The table we have constructed is useful not only in the case of Bernoulli parameter p but generally in constructing shortest confidence intervals for λ whenever the statistic to be employed has the χ^2_n-distribution. In all these cases the a and b giving the minimum interval and satisfying Eq. (12) can be read off from the table. When, however, the Bernoulli p is so large that $p \approx \lambda = -\ln(1 - p)$ we need to be able to get the shortest intervals directly for p. In such a case we may use the fact that by Eq. (12):

$$P(1 - e^{-a/2Z} < p < 1 - e^{-b/2Z}) = 1 - \gamma \tag{16}$$

The length of the confidence interval $L = e^{-a/2Z} - e^{-b/2Z}$ and just as in the case for λ, we want to minimize L subject to Eq. (14). It turns out that L is minimized if

$$\frac{e^{-b/2Z}}{h_{2r}(b)} = \frac{e^{-a/2Z}}{h_{2r}(a)} \tag{17}$$

If the number of trials to give r failures, Z, is large enough, which is the case either when r is larger or p is small, then Eq. (17) reduces to Eq. (15) for λ.

IV. Minimum r Required for Prescribed Estimate Accuracy

In a number of applications it is enough to show that the true value of p is almost surely less than the critical value for the system (i.e., with probability $1-\gamma$). Since our estimate of p is based only on the value of r, the number of system failures observed and Z the number of system operations required to observe r, then it becomes important to be able to estimate the minimum number of failures that would give an estimate $p = r/z$ which is only slightly greater (5%, say) than p with very high probability. Put equivalently, the problem is to find the minimum value of r that would give an estimate p within $100\beta\%$ of p with probability $1-\gamma$, $\beta > 0$, and small. In our terminology, this reduces to finding the least r such that

$$P\left(p\alpha > \frac{r}{Z}\right) = 1-\gamma, \qquad \alpha = 1+\beta \tag{18}$$

In other words we want the probability that the true value of the parameter be less than $p\alpha^{-1}$ to be small $(=\gamma)$.

Since Eq. (18) can be written as

$$P\left(Z > \frac{r}{p\alpha}\right) = 1-\gamma$$

then, we can use the distribution of Z in Eq. (1) to calculate r for a given β and γ and each $0 < p < 1$. Thus Eq. (18) becomes

$$\sum_{r}^{\lfloor \frac{r}{p\alpha} \rfloor} \binom{Z-1}{r-1} p^r q^{Z-r} = \gamma \tag{19}$$

where $\lfloor r/p\alpha \rfloor$ is the largest integer smaller than $r/p\alpha$. We iterate the left-hand side of Eq. (19) until a value of r satisfying the equation is obtained. We note that any larger value of r would give a sum less than γ.

This method is not convenient, however. Even if the computation of r in Eq. (19) were easy, the volume of table of r values to be required to cover all p such that $0 < p < 1$, β and γ of interest, would be so large as to discourage such effort. Moreover, the value of p to use is not known beforehand. Hence we need to have a method of evaluating r that is independent of the particular value of p. It turns out that a good bound on the minimum value of r can be obtained by considering the complementary requirement to Eq. (18) that

$$P\left(p < \frac{r}{Z}\alpha\right) = 1-\gamma, \qquad \alpha = 1+\beta \tag{20}$$

That is to say that the probability that the true value of the parameter is ever greater than $p\alpha$ be very small $(=\gamma)$.

We may now use the fact that the maximum likelihood estimate of p, $p = r/Z$ is, for large r, approximately normally distributed with mean p and variance

$$\sigma_p^2 = -\frac{1}{r\,E\left(\dfrac{\partial^2}{\partial p^2}\ln f(x,p)\right)} \tag{21}$$

where

$$\begin{aligned} f(x,p) &= P(X_j = x) \\ &= p(1-p)^{x-1}; \qquad x = 1,2,3,\cdots \end{aligned}$$

X_j, $j = 1,2,\cdots,r$ being the number of trials after the $(j-1)^{st}$ failure to the occurrence of the j^{th} failure. Writing $L = \ln f(x,p)$ we have

$$E\left(\frac{\partial^2 L}{\partial p^2}\right) = -\frac{1}{p^2 q} \tag{22}$$

and

$$\sigma_p^2 = \frac{p^2 q}{r}$$

Thus $(r/Z)\alpha$ is, for large r, approximately normally distributed with mean αp and variance $\alpha^2 p^2 q/r$ and hence

$$\left(\alpha\frac{r}{Z} - \alpha p\right)\sqrt{\frac{r}{\alpha^2 p^2 q}}$$

has the unit normal distribution.

Now Eq. (20) is equivalent to

$$P\left[p(1-\alpha)\sqrt{\frac{r}{\alpha^2 p^2 q}} < \left(\alpha\frac{r}{Z} - \alpha p\right)\sqrt{\frac{r}{\alpha^2 p^2 q}}\right] = 1-\gamma \tag{23}$$

Hence

$$p(1-\alpha)\sqrt{\frac{r}{\alpha^2 p^2 q}} = -\phi_{1-\alpha} \tag{24}$$

where $\phi_{1-\alpha}$ is the lower $(1-\alpha)$ percentile of the unit normal distribution. From Eq. (24) we see that

$$r = q\frac{\alpha^2}{(\alpha-1)^2}\phi_{1-\alpha}^2, \qquad \alpha = 1+\beta \tag{25}$$

That is the value of r good for all p is

$$r = \frac{\alpha^2}{(\alpha-1)^2}\phi_{1-\alpha} \tag{26}$$

If we denote by r_D the r obtained by the direct method in Eq. (19) and by r_{est} (estimate) and r_{asymp} (asymptotic), respectively, those of Eqs. (25) and (26), to achieve an estimate error of less than 10% ($\beta = 0.1$) with 90% confidence ($\gamma = 0.1$) on $p = 0.005$ we need to wait for at least $r_D = 190$ successes. The corresponding $r_{est} = 199$, $r_{asymp} = 200$. Values of r for $p = 0.005$ and some values of β and γ are:

		r_D	r_{est}	r_{asymp}
$\gamma = 0.1$	$\beta = 0.1$	190	199	200
	$\beta = 1.05$	710	724	728
$\gamma = 0.05$	$\beta = 0.1$	313	326	327

A graph of r_D for $\gamma = 0.1$, $\beta = 0.1$ and $0.001 \leq p \leq 0.05$ is plotted below.

V. Conclusion

A sequential sampling technique has been applied to the problem of estimating the failure probability of those systems whose operation history can be modeled by the Bernoulli sequence with small failure probability p. Methods of obtaining $1-\gamma$ confidence bounds on p are presented, and several tables for constructing the limits of these bounds are indicated. The desirability of obtaining short $1-\gamma$ confidence intervals prompts us to investigate the conditions for the shortest such intervals in this case. A new table for use in constructing these intervals for usual values of γ and a wide range of r is provided. The question of the minimum number of system failures required to achieve a prescribed accuracy of the p estimate is answered.

References

1. Hwang, D., and Buehler, R. J., "Confidence Intervals for Some Function of Several Bernoulli Parameters with Reliability Applications," *JASA*, Vol. 68, No. 341, March 1973.
2. Mood, A. M., *Introduction to the Theory of Statistics*, McGraw-Hill Book Co., New York, 1950.
3. Patil, G. P., "On the Evaluation of the Negative Binomial Distribution with Examples," *Technometrics*, Vol. 2, pp. 501–505, Nov. 1960.
4. Pearson, E. S., and Hartley, H. O., *Biometrika Tables for Statisticians*, Vol. I, Cambridge University Press, Massachusetts, 1956.
5. Guenther, W. C., "Shortest Confidence Intervals," *The American Statistician*, Vol. 23, No. 1, pp. 22–25, Feb. 1969.

Table 1. Values of *a* and *b* for the shortest confidence intervals

Table of a and b such that $h_{2r}(a) = h_{2r}(b)$ where

$$h_{2r}(x) = \frac{x^{r-1}}{2^r\Gamma(r)} e^{-x/2}, x \geq 0$$

and

$$\int_a^b h_{2r}(t)dt = 1\text{-}\gamma.$$

a is the first entry and b is the second for each γ and $n = 2r$. For example, if $\gamma = 0.05$ and $n = 2r = 20$, then

$$a = 8.5841, \qquad b = 32.607393$$

When $r = 1$, $h_2(x)$ reduces to the exponential distribution with parameter $\lambda = 1/2$.

	γ				
$n = 2r$	0.05	0.04	0.03	0.02	0.01
4	0.0848 9.5282394	0.068 10.062034	0.0514 10.734574	0.0345 11.684643	0.0175 13.281327
6	0.607 12.802453	0.539 13.388847	0.464 14.128164	0.376 15.164491	0.264 16.900628
8	1.425 15.896601	1.3083 16.525274	1.1738 17.325344	1.01 18.436192	0.7856 20.295983
10	2.4139 18.860515	2.2528 19.531733	2.0643 20.383955	1.8296 21.565627	1.4978 23.533077
12	3.5161 21.729099	3.3147 22.439061	3.0767 23.339668	2.7771 24.583977	2.3444 26.653195
14	4.7004 24.524892	4.462 25.270476	4.178 26.216824	3.818 27.520418	3.291 29.684001
16	5.9477 27.263128	5.6743 29.04377	5.348 29.030965	4.9315 30.390159	4.316 32.641633
18	7.2452 29.954814	6.9395 30.766807	6.573 31.793831	6.103 33.20639	5.4041 35.540208
20	8.5841 32.607393	8.2477 33.449596	7.8434 34.513723	7.323 35.975951	6.5444 38.389569
22	9.9575 35.227546	9.592 36.098361	9.1515 37.198528	8.5834 38.707429	7.7289 41.196762
24	11.3613 37.81774	10.9675 38.717077	10.493 39.850243	9.8786 41.405594	8.9515 43.967203
26	12.7908 40.383458	12.3704 41.308935	11.8627 42.475229	11.204 44.074835	10.2073 46.705532
28	14.243 42.927308	13.797 43.878022	13.2574 45.076108	12.5564 46.717492	11.492 49.416424
30	15.7155 45.451379	15.2446 46.426908	14.6744 47.655306	13.9324 49.337469	12.8033 52.10051

Table 1 (contd)

	γ				
$n = 2r$	0.05	0.04	0.03	0.02	0.01
32	17.2061 47.957866	16.7113 48.957043	16.1115 50.214788	15.3298 51.936489	14.1379 54.762048
34	18.7132 50.448126	18.1951 51.470521	17.5665 52.756972	16.7464 54.516847	15.4935 57.403297
36	20.2352 52.923807	19.6945 53.968712	19.0381 55.282712	18.1806 57.079891	16.8685 60.02534
38	21.7708 55.386172	21.2082 56.452832	20.5246 57.79397	19.6308 59.627288	18.261 62.630384
40	23.3189 57.836276	22.735 58.924084	22.025 60.29148	21.0959 62.159881	19.6696 65.219739
42	24.8786 60.274932	24.2739 61.383428	23.538 62.776738	22.5745 64.679327	21.0934 67.793854
44	26.449 62.703059	25.824 63.831749	25.063 65.250007	24.0657 67.186391	22.5314 70.353626
46	28.029 65.122002	27.384 66.270743	26.599 67.712431	25.568 69.683141	23.981 72.903478
48	29.619 67.530537	28.955 68.69785	28.145 70.165218	27.082 72.167619	25.447 75.432389
50	31.217 69.731697	30.535 71.116189	29.701 72.607796	28.606 74.642475	26.917 77.965244
52	32.824 72.323363	32.122 73.528802	31.265 75.043303	30.14 77.107124	28.405 80.472543
54	34.438 74.708599	33.718 75.931784	32.839 77.467702	31.683 79.562951	29.9 82.976087
56	36.06 77.085409	35.322 78.326562	34.42 79.885977	33.235 82.009512	31.401 85.477027
58	37.688 79.456858	36.932 80.716237	36.01 82.294267	34.795 84.448257	32.914 87.963049
60	39.323 81.821041	38.55 83.097149	37.605 84.699231	36.362 86.880625	34.435 90.44157
62	40.964 84.179387	40.174 85.472433	39.208 87.095386	37.936 89.30627	35.965 92.910289
64	42.611 86.5317	41.804 87.841859	40.818 89.484241	39.518 91.723148	37.505 95.36705
66	44.264 88.877814	43.44 90.205233	42.433 91.868955	41.108 94.131061	39.047 97.826457
68	45.922 91.219172	45.082 92.56239	44.055 94.245964	42.701 96.53847	40.598 100.27512
70	47.585 93.555623	46.729 94.914796	45.682 96.618434	44.301 98.938184	42.158 102.7129
72	49.253 95.887035	48.381 97.262298	47.314 98.986182	45.907 101.33176	43.721 105.15047
74	50.925 98.214848	50.038 99.604764	48.954 101.34417	47.519 103.71903	45.291 107.58031

Table 1 (contd)

	γ				
$n = 2r$	0.05	0.04	0.03	0.02	0.01
76	52.602 100.53739	51.7 101.94208	50.595 103.70367	49.139 106.09652	46.871 109.99698
78	54.284 102.85459	53.366 104.27571	52.242 106.05641	50.759 108.47747	48.451 112.41801
80	55.969 105.16942	55.036 106.60554	53.894 108.40391	52.386 111.85002	50.041 114.8256
82	57.659 107.47873	56.711 108.92993	55.55 110.74766	54.017 113.21905	51.631 117.23707
84	59.352 109.78549	58.391 111.24882	57.211 113.08599	55.564 115.58116	53.231 119.6349
86	61.049 112.08811	60.072 113.56826	58.876 115.42039	57.294 117.94115	54.831 122.03622
88	62.749 114.38802	61.758 115.88202	60.546 117.74921	58.939 120.29566	56.441 124.42379
90	64.453 116.68366	53.448 118.1916	62.217 120.07862	60.589 122.64463	58.051 126.81451
92	66.161 118.975	65.142 120.49693	63.893 122.40231	62.241 124.99279	59.671 129.1914
94	67.871 121.26493	66.939 122.80096	65.573 124.72176	63.901 127.33049	61.291 131.57118
96	69.585 123.55046	68.538 125.10063	67.255 127.04	65.561 129.67038	62.911 133.95363
98	71.301 125.83446	70.241 127.39739	68.942 129.35236	67.222 132.01074	64.541 136.3221
100	73.021 128.11398	71.948 129.68971	70.632 131.66186	68.892 134.33895	66.171 138.69305
102	74.743 130.39189	73.658 131.97902	72.326 133.96693	70.562 136.66907	67.811 141.05
104	76.468 132.66669	75.369 134.26824	74.022 136.27055	72.237 138.99323	69.451 143.40928
106	78.196 134.93835	77.084 136.5529	75.721 138.57116	73.915 141.3145	71.091 145.77072
108	79.926 137.20827	78.801 138.8359	77.421 140.87169	75.595 143.63436	72.74 148.11972
110	81.659 139.475	80.521 141.11573	79.125 143.16762	77.275 145.95581	74.389 150.47074
112	83.395 141.7385	82.244 143.39237	80.832 145.46042	78.964 148.26503	76.039 152.82208
114	85.132 144.00159	83.969 145.66725	82.542 147.75005	80.653 150.57575	77.698 155.16094
116	86.872 146.26141	85.697 147.93888	84.254 150.03796	82.343 152.88638	79.357 157.5106
118	88.615 148.51792	87.427 150.20867	85.968 152.3241	84.039 155.18931	81.017 159.8424

Table 1 (contd)

			γ		
$n = 2r$	0.05	0.04	0.03	0.02	0.01
120	90.359 150.77393	89.159 152.47661	87.688 154.60261	85.736 157.49208	82.677 162.18483
122	92.107 153.0252	90.895 154.7398	89.408 156.88222	87.436 159.79162	84.347 164.51316
124	93.856 155.2759	92.632 157.00251	91.128 159.16289	89.136 162.09239	86.017 166.84305
126	95.606 157.52606	94.371 159.26329	92.854 161.43588	90.842 164.3854	87.687 169.17441
128	97.359 159.77281	96.111 161.52353	94.58 163.70986	92.549 166.67809	89.367 171.49172
130	99.114 162.01754	97.855 163.77895	96.308 165.98191	94.258 168.96893	91.047 173.81046
132	100.87 164.26162	99.6 166.03379	98.038 168.252	95.968 171.25937	92.727 176.13056
134	102.629 166.50226	101.347 168.28661	99.772 170.51726	97.683 173.54349	94.407 178.45194
136	104.398 168.74223	103.096 170.5374	101.506 172.78339	99.399 175.82717	96.096 180.76083
138	108.151 170.98011	104.846 172.78754	103.243 175.04608	101.116 178.11035	97.776 183.08464
140	107.915 173.21589	106.599 175.0342	104.982 177.30674	102.836 180.39011	99.466 185.3944
142	109.681 175.44957	108.353 177.28018	106.722 179.56678	104.556 182.67077	101.166 187.6902
144	111.448 177.6825	110.109 179.52406	108.462 181.82757	106.282 184.94363	102.856 190.00224
146	113.216 179.91465	111.867 181.76583	110.208 184.08063	108.008 187.21736	104.556 192.30029
148	114.986 182.14466	113.626 184.00686	111.954 186.33443	109.735 189.49048	106.256 194.59942
150	116.759 184.37116	115.386 186.24713	113.702 188.58612	111.465 191.76011	107.956 196.89957
152	118.532 186.59821	117.15 188.4825	115.451 190.83709	113.195 194.03052	109.656 199.2007
154	120.307 188.82308	118.914 190.71846	117.201 193.08733	114.925 196.3017	111.356 201.50277
156	122.082 191.04847	120.679 192.95363	118.952 195.3368	116.66 198.56646	113.066 203.79091
158	123.861 193.26896	122.447 195.18524	120.707 197.58134	118.4 200.82484	114.776 206.07999
160	125.641 195.48861	124.216 197.41604	122.462 199.82648	120.135 203.09105	116.486 208.36997
162	127.421 197.70874	125.986 199.64601	124.222 202.0653	121.875 205.3584	118.201 210.65344

Table 1 (contd)

	γ				
$n = 2r$	0.05	0.04	0.03	0.02	0.01
164	129.203 199.92665	127.756 201.87648	125.977 204.31161	123.615 207.6113	119.911 212.94511
166	130.987 202.14234	129.531 204.10068	127.738 206.5502	125.36 209.86536	121.626 215.23025
168	132.771 204.35847	131.305 206.32672	129.499 208.78933	127.105 212.12006	123.346 217.50888
170	134.556 206.5737	133.08 208.55188	133.264 211.02351	128.85 214.37539	125.066 219.7883
172	136.345 208.78402	134.858 210.77346	133.029 213.25822	130.6 216.6243	126.786 222.06848
174	138.133 210.99609	136.637 212.99416	134.794 215.49343	132.35 218.87382	128.506 224.34939
176	139.923 213.2059	138.416 215.21529	136.56 217.72777	134.1 221.12393	130.231 226.62378
178	141.713 215.41611	140.196 217.43551	138.33 219.95715	135.85 223.37459	131.956 228.89887
180	143.506 217.62273	141.979 219.65214	140.1 222.187	137.605 225.61883	133.681 231.17464
182	145.299 219.82972	143.762 221.86918	141.87 224.41731	139.358 227.8664	135.411 233.44388
184	147.092 222.03708	145.545 224.08661	143.643 226.644	141.114 230.11032	137.141 235.71378
186	148.888 224.24084	147.331 226.30044	145.416 228.87113	142.874 232.34923	138.871 237.98432
188	150.684 226.44495	149.12 228.51067	147.192 231.09464	144.63 234.59417	140.601 240.25547
190	152.483 228.64547	150.906 230.72525	148.968 233.31858	146.39 236.83407	142.341 242.51295
192	154.282 230.84633	152.695 232.93622	150.744 235.54293	148.15 239.07445	144.071 244.7853
194	156.081 233.04751	154.487 235.14358	152.524 237.7623	149.913 241.7623	145.811 247.04399
196	157.883 235.24508	156.276 237.35526	154.304 239.98206	151.676 243.54935	147.551 249.30326
198	159.685 237.44298	158.069 239.562	156.084 242.20222	153.442 245.78187	149.291 251.5631
200	161.487 239.64118	159.862 241.76907	157.864 244.42275	155.208 248.01584	151.031 253.82349
202	163.29 241.83837	161.659 243.97121	159.648 246.6383	156.978 250.24477	152.771 256.08441
204	165.094 244.03455	163.452 246.17893	161.432 248.85421	158.744 252.47959	154.511 258.34584
206	166.901 246.22713	165.249 248.38172	163.219 251.06649	160.514 254.70936	156.261 260.5937

Table 1 (contd)

	γ				
$n = 2r$	0.05	0.04	0.03	0.02	0.01
208	168.708	167.049	165.003	162.284	158.011
	248.42	250.58088	253.28311	256.93955	262.84209
210	170.515	168.846	166.79	164.054	159.751
	250.61313	252.78428	255.49607	259.17012	265.10501
212	172.325	170.646	168.58	165.831	161.501
	252.80266	254.98405	257.70539	261.39159	267.35438
214	174.132	172.446	170.367	167.601	163.251
	254.99633	257.18411	259.91902	263.62293	269.60423
216	175.945	174.246	172.157	169.381	165.001
	257.18252	259.38446	262.12899	265.84112	271.85454
218	177.755	176.049	173.947	171.155	166.751
	259.37283	261.58117	264.33928	268.06778	274.10531
220	179.568	177.852	175.741	172.932	168.511
	261.55953	263.77817	266.54459	270.29076	276.3426
222	181.381	179.655	177.535	174.712	170.261
	263.74648	265.97543	268.75021	272.51005	278.59426
224	183.194	181.462	179.329	176.492	172.021
	265.93367	268.16776	270.95613	274.72969	280.83245
226	185.008	183.269	181.123	178.272	173.771
	268.11981	270.36036	273.16234	276.94968	283.08493
228	186.825	185.073	182.92	180.052	175.531
	270.30234	272.55711	275.3649	279.17000	285.32397
230	188.642	186.883	184.717	181.832	177.291
	272.4851	274.74633	277.56774	281.39064	287.56343
232	190.459	188.69	186.514	183.612	179.051
	274.66809	276.93968	279.77086	283.61159	289.8033
234	192.276	190.5	888.311	185.402	180.811
	276.85129	279.1294	281.97425	285.8195	292.04355
236	194.093	192.31	190.111	187.182	182.571
	279.0347	281.31936	284.17398	288.04107	294.28419
238	195.912	194.12	191.911	188.972	184.341
	281.21451	283.50956	286.37397	290.24962	296.51147
240	197.733	195.933	193.711	190.759	186.101
	283.39452	285.69612	288.57422	292.46246	298.75286

We are grateful to I. Eisenberger for assistance in computing this table.

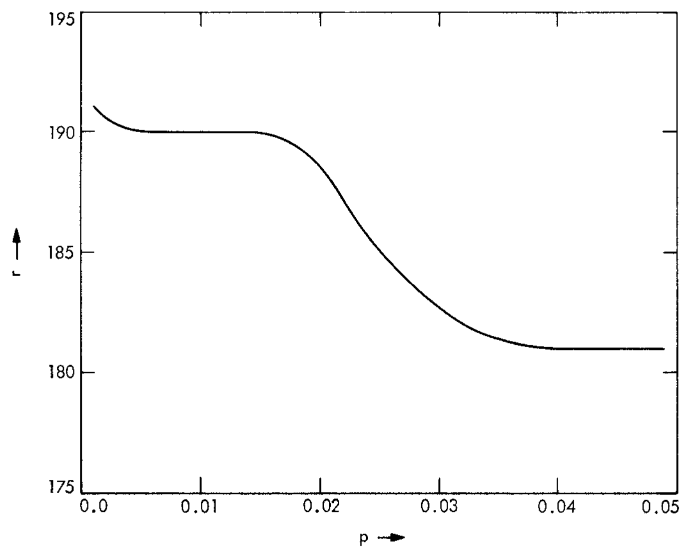

Fig. 1. Minimum number of system failures r giving 90% (i.e., $\gamma = 0.1$) confidence that estimate error is within 10% ($\beta = 0.1$), $r_{asymp} = 200$