Graphs With Multiplace Nodes for Printed Circuitry

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A problem in the design of printed circuits is recast into a problem in applied graph theory. A catalog of diagrams that can be used to obtain efficient realizations of many circuit diagrams is presented.

I. Introduction

Arranging the terminals of a printed circuit so that none of the connecting lines cross each other can be taken as a problem in applied graph theory.

To agree with the terminology of graph theory let us call terminal codes nodes, connecting lines edges, and when some nodes and edges are given let us call the resulting collection a graph.

A given graph, then, presents a question as to whether or not it is possible to draw a picture of the graph on one side of one sheet of paper, for example, with no edges crossing.

Usually the answer is no. So the next question is how to get around it. One possibility is to draw a picture with the number of crossings minimized and then drill holes in the board. The literature has a few studies of the problem of minimizing the number of crossings (Ref. 1) but these results may still be far from best possible.

Another possibility is to use several sheets of paper, picturing the nodes over again on each sheet but each edge on only one sheet. Here the problem is to minimize the number of sheets. The literature has better results on this problem, and in some cases the best possible (Ref. 2).

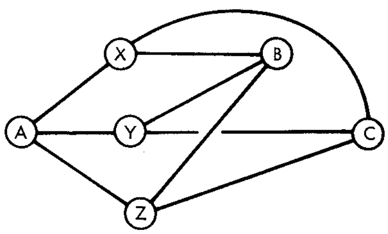
A third possibility is the subject of the present study. It is to picture the graph on only one sheet of paper by allowing each node to appear in several places. Thus by using multiplace nodes any graph can be pictured on one sheet with no edges crossing. The problem is to minimize the number of places. There seems to be little mention of this problem in the literature between 1890 when Heawood (Ref. 3) drew the basic example of 12 two-place nodes with all 66 edges, and quite recently (Ref. 4) when some interesting cases are mentioned as unsolved, but with no new results since Heawood.

In actually realizing some circuitry there is also the question of making connection to the nodes from the outside, and presumably this could present real difficulties if the circuitry is to be miniaturized. One separate sheet for the nodes would be enough if each node could be spread out in a strip of conducting material, and then connected at certain places to the sheet on which the graph is pictured.

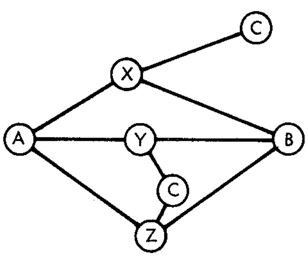
Two sheets will suffice for any graph: one sheet for access to the nodes from the outside, and one sheet to picture the graph with multiplace nodes.

These are the first entries for a catalog of pictures that hopefully will be of use to the circuit designer. Any one of these graph picturing problems is quite likely to be very troublesome by itself. The designer can look in the catalog and find a picture close enough to what he needs to at least keep it on only two sheets and maybe economize on multiplace nodes as well.

To start with a well-known example, suppose it is required to draw all the edges from nodes A,B,C to nodes X,Y,Z. After a few trials it will look impossible as in this sketch:



But if we allow C to be a 2-place node, it looks too easy, as in this sketch:



So to have a more difficult example, suppose we require all the edges from nodes A,B,C,D,E,F,G to nodes 1,2,3,4, 5,6,7,8 allowing each node to be 2-place. This graph is designated $K_{7,8}$ and pictured 2-place in Fig. 1.

The notation $K_{a,b}$ refers to a graph having $a + b$ nodes with all the edges from a of them to b of them. Figure 1 shows $K_{7,8}$ pictured with 2-place nodes and that will go into the catalog. What makes it interesting is the fact that, with nodes allowed no more than 2 places each, $K_{8,8}$ is impossible. This bounding impossibility, and most of the others like it, will become easy to prove using the famous formula $V - E + F = 2$, often called Euler's formula.

II. $V - E + F = 2$

By common usage E is the number of edges, F is the number of faces, and V is the number of vertices. In the present context one place at which a node is pictured will count as one vertex.

An undivided region will count as one face. A line segment drawn with a vertex at each end will count as one edge. Let it be understood from here on that an edge is not meant to cross, nor even touch, another edge in the picture.

The formula holds for any connected configuration of vertices, edges, and faces on a surface like the sphere, plane, or one side of a sheet of paper. This is one of those situations where minimum conditions result in the simplest proof. So here is proof by induction that $V - E + F = 2$ holds for any connected configuration of vertices, edges, and faces satisfying *conditions 0* and *1*:

0. An edge which has both ends on the same vertex will divide the surface into two regions.

1. $V > 0$.

The induction is on the number of edges. When $E = 0$, connectedness requires $V < 2$; so, we can only have $V = 1$, $F = 1$, and $V - E + F = 1 - 0 + 1 = 2$. When $E = 1$ there are two possible configurations, namely, A—B with $V = 2$, $E = 1$, $F = 1$, and A with $V = 1$, $E = 1$, and as we know by *conditions 0*, $F = 2$.

Now for the inductive step let us be given a connected configuration with E edges, V vertices, and F faces and presume the formula holds for all connected configurations with fewer than E edges, under *condition 0* and *1*. Select any edge and consider two cases.

Case I. The selected edge has different vertices at its two ends. In this case we shrink the edge to nothing and merge the two vertices into one vertex, thereby reducing the number of edges by one and reducing the number of vertices by one. This keeps the configuration connected and preserves *conditions 0* and *1* so by the inductive presumption $(V - 1) - (E - 1) + F = 2$, and $V - E + F = 2$.

Case II. The selected edge has the same vertex at both ends. In this case erasing the edge will reduce the number of edges by one and reduce the number of faces by one because of *condition 0.* But the configuration will remain connected and *conditions 0* and *1* will be unaffected. So by our inductive presumption $V - (E - 1) + (F - 1) = 2$ and therefore $V - E + F = 2$.

Thus $V - E + F = 2$ for any connected configuration on a surface which satisfies *condition 0.*

Observe that a configuration with several parts on a surface satisfying *condition 0* would still satisfy the inequality $V - E + F \geq 2$, because it would still have $V - E + F = 2$ for each connected part.

III. $K_{a,b}$ with 2-place Nodes

We can lose no generality by considering only pictures in which every face has three or more edges, and now that we want to find bounds on the most that can be pictured with $K_{a,b}$ we can derive even more limitation. In $K_{a,b}$ an edge only connects one of the a nodes with one of the b nodes so that a face will have an even number of edges. Thus for $K_{a,b}$ we can count the number of incidences of a face with one side of an edge and find from the edges that the number is $2E$, while from the faces the number is $\geq 4F$.

The inequality $2E \geq 4F$ together with $4V - 4E + 4F \geq 8$ boils down to a very useful relationship: $E \leq 2V - 4$, which applies to $K_{a,b}$ as long as $a + b > 2$. Again, counting edges is easy for $K_{a,b}$, and we have $E = ab$.

Now here are some bounding examples of what graphs $K_{a,b}$ can be pictured using m-place nodes in case we require $m \leq 2$. It makes $V \leq 2(a + b)$ and so it would be not impossible only for values of a and b satisfying:

$$ab \leq 4(a + b - 1) \quad .$$

The impossibility of picturing $K_{8,8}$ with 2-place nodes is now established because the formula is not satisfied with a and b both ≥ 8.

The values allowed by the formula are as follows:

$$a = 7 \quad \text{requires } b \leq 8$$

$$a = 6 \quad \text{requires } b \leq 10$$

$$a = 5 \quad \text{requires } b \leq 16$$

and then $a \leq 4$ allows b as large as space permits.

With Fig. 2 showing $K_{7,8}$, Fig. 3 showing $K_{6,10}$, Fig. 4 showing $K_{5,16}$, and Fig. 5 showing a scheme for $a \leq 4$ and arbitrary b, we have sharp bounding examples.

IV. K_n, One of the Worst Cases

The complete graph, denoted K_n, has n nodes with an edge for every pair of distinct nodes. Regarding an arbitrary graph as a subgraph of the complete graph on the same nodes makes it clear that no graph on n nodes can demand more vertices than K_n to be pictured with multiplace nodes.

Thus we get a rough outline by asking for the minimum number of vertices needed for K_n. Table 1 and the corresponding figures show the best obtained so far.

The question marks in Table 1 indicate tried without success, whereas the blank spaces just mean not tried yet.

Now to explain the column headings in Table 1. The numbers under t are taken from Ref. 2. Given n the number of nodes in the complete graph, t is the minimum number of sheets that would be needed. Beineke gives

$$\left[\frac{n + 7}{6}\right]$$

for large n except for the question marks when $n \equiv 4 \pmod 6$ or $n = 9$. We could put the separate sheets side by side on one sheet and always have true min $V \leq nt$.

The true min V is the actual minimum number of vertices for any picture of K_n using multiplace nodes.

The number in the picture V column gives the number of vertices in a multiplace picture of K_n which has actually been drawn.

The formula V has been calculated from $E = \binom{n}{2}$ as the smallest integer satisfying

$$V \geq \frac{E+6}{3}.$$

This lower bound for the minimum V has been derived from our formula $V - E + F = 2$ as follows.

A picture minimizing V will need at least $E = \binom{n}{2}$ edges, and any more edges could only get in the way. So, without loss of generality, we can erase any edge with the same vertex at both ends, or any extra edges between the same pair of vertices—with the result that any face will touch at least three edges. Counting edge-face incidences gives us the inequality $3F \leq 2E$. Putting this together with the formula $3V - 3E + 3F \geq 6$ gives us $3V - 3E + 2E \geq 6$ which says $3V \geq E + 6$. Thus in general we have formula $V \leq$ true min $V \leq nt$.

A word of explanation about the figures. $w \cdot 1 + x \cdot 2 + y \cdot 3 + z \cdot 4 = V$ means that the picture is using w 1-place nodes, x 2-place nodes, y 3-place nodes, and z 4-place nodes, where of course, $w + x + y + z = n$.

When the same figure is listed for several values of n it means that a picture for the smaller case can be obtained by simply erasing some vertices and edges. For example to get a picture of K_9 with $V = 3 \cdot 1 + 6 \cdot 2$ just erase the 3 vertices each, of nodes A,B, and C from Fig. 8, together with the edges which end on A, B, or C.

V. The Impossibility Proofs

Proof is required when the true min V is larger than the formula V.

For K_6 it is almost immediate that $V = 7$ is impossible. It could only be pictured with $V = 5 \cdot 1 + 1 \cdot 2$; but then erasing the 2-place node and edges to it would leave K_5 pictured with $V = 5$. Similarly the impossibility of picturing K_7 with $V = 9$ is proved by reducing to the previous case.

By contrast the proof for K_9 is difficult—in fact, it is mentioned as an unsolved problem in (Ref. 4). Since a more elegant proof may be presented in part II, only a brief sketch is given here, as follows.

First, the previous cases would force a picture of K_9 to have $V = 4 \cdot 1 + 5 \cdot 2$.

Next, naming the 1-place nodes W,X,Y,Z, and the 2-place nodes 1,2,3,4,5, the result in (Ref. 5) helps to rule out all but a few apportionments of the numbered vertices into the four regions formed by the six edges connecting the lettered vertices.

Then an exhaustive comparison of the valences reduces it to the two partial pictures in Fig. 10. And by some further direct exhaustion, it turns out that these cannot be completed.

References

1. Saaty, T. L., "On Polynomials and Crossing Numbers of Complete Graphs," *J. Combin. Theory*, Vol. 10, No. 2, March 1971.

2. Beineke, L. W., "The Decomposition of Complete Graphs into Planar Sub-Graphs," in *Graph Theory and Theoretical Physics*, pp. 139–153, F. Harary, ed., Academic Press, 1967.

3. Heawood, P. J., "Map-colour Theorem," *Quart. J. Math*, Vol. 24, No. 1890, pp. 332–338.

4. White, A. T., *Graphs, Groups, and Surfaces*, North Holland Pub. Co., distributed by American Elsevier Pub. Co., Inc., New York, 1973.

5. Battle, J., Harary, F., and Kodama, Y., "Every Planar Graph with 9 Points has a Non-Planar complement," *Bull. Amer. Math. Soc.*, Vol. 68, pp. 569–571, 1962.

Table 1. Compared bounds for K_n

t	n	E	Formula V	Picture V	True min V	Reference
1	4	6	4	4	4	Fig. 5
2	5	10	6	6	6	Fig. 7
2	6	15	7	8	8	Fig. 7
2	7	21	9	10	10	Fig. 7
2	8	28	12	12	12	Fig. 7
3	9	36	14	15	15	Fig. 8
3	10	45	17	18	?	Figs. 8 and 9
3	11	55	21	21	21	Fig. 8
3	12	66	24	24	24	Figs. 8 and 6
3	13	78	28	28	28	
3	14	91	33	33	33	
3	15	105	37	38	?	
?	16	120	42			
4	17	136	48			
4	18	153	53	53	53	Fig. 11
4	19	171	59	59	59	Fig. 11
4	20	190	66			
4	21	210	72			
?	22	231	79			
5	23	253	87			
5	24	276	94			
5	25	300	102			

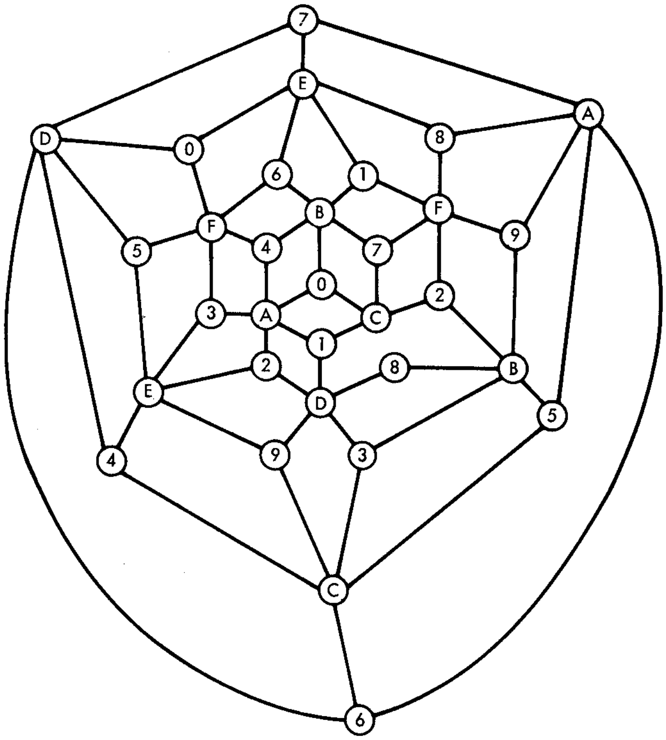


Fig. 1. $K_{7,8}$ with 2-place nodes

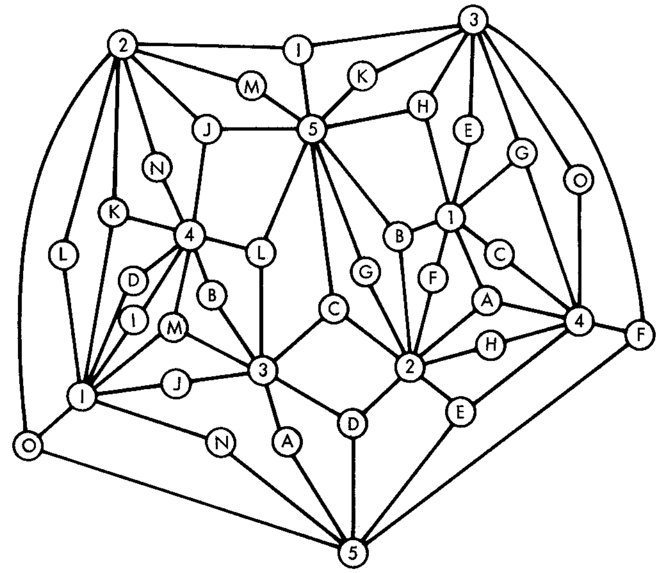


Fig. 3. $K_{5,16}$ with 2-place nodes

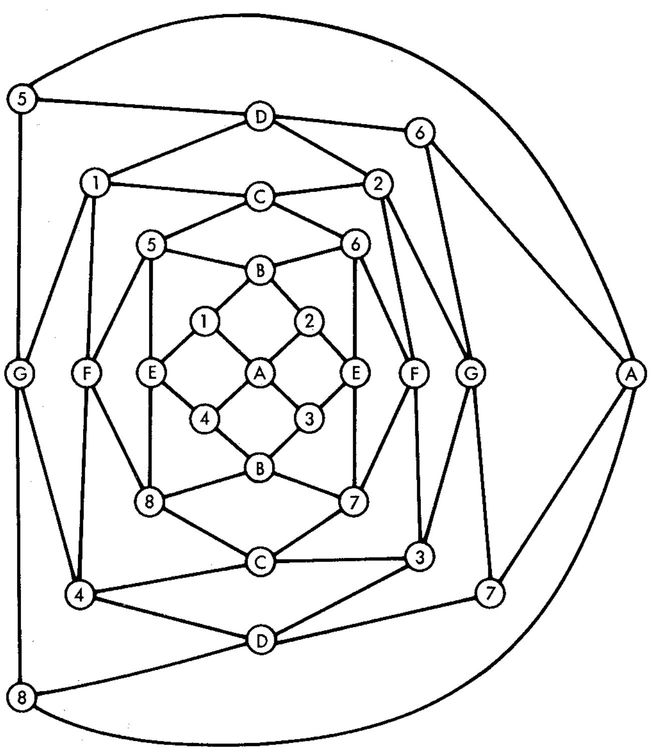


Fig. 2. $K_{6,10}$ with 2-place nodes

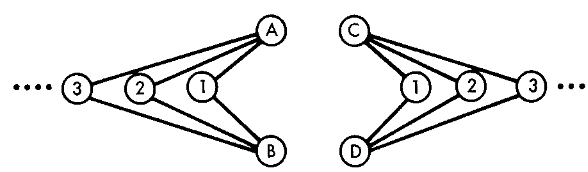


Fig. 4. $K_{4,b}$ with 2-place nodes

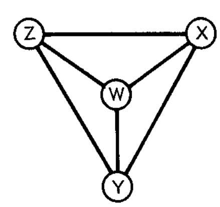


Fig. 5. $V = 4 \cdot 1$

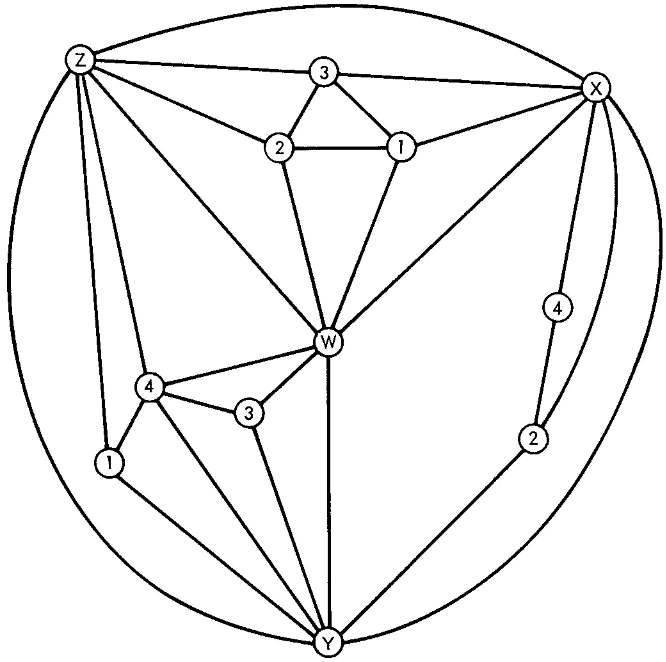


Fig. 7. $V = 4 \cdot 1 + 4 \cdot 2$

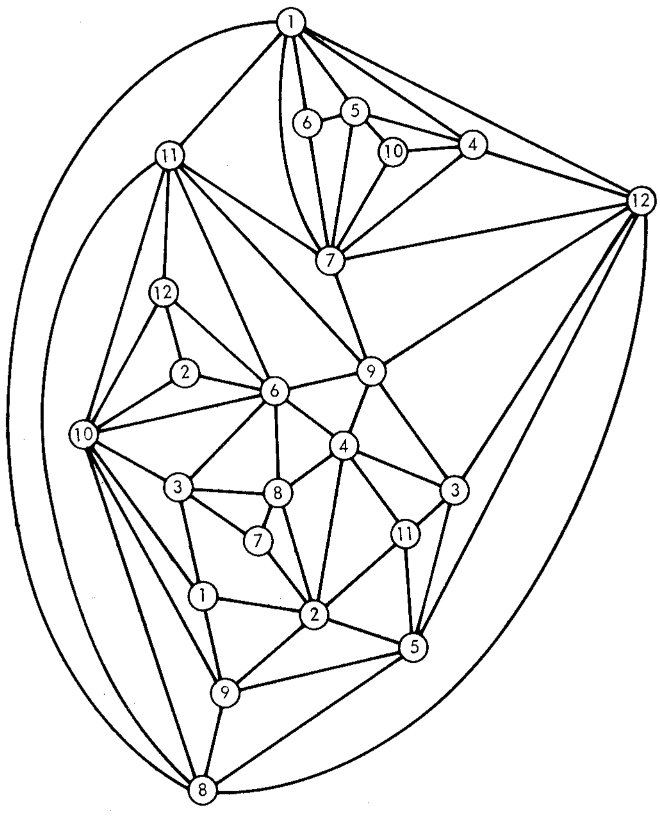


Fig. 6. $V = 0 + 12 \cdot 2$

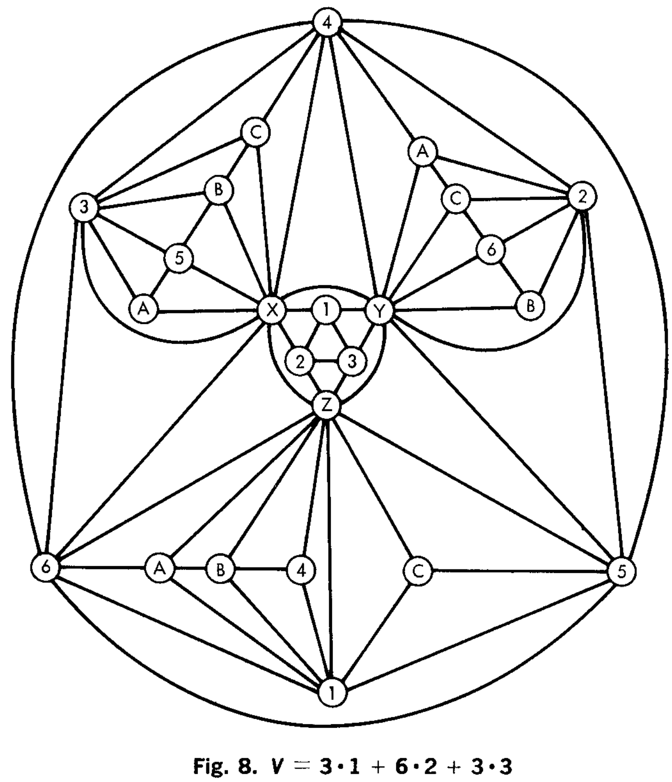


Fig. 8. $V = 3 \cdot 1 + 6 \cdot 2 + 3 \cdot 3$

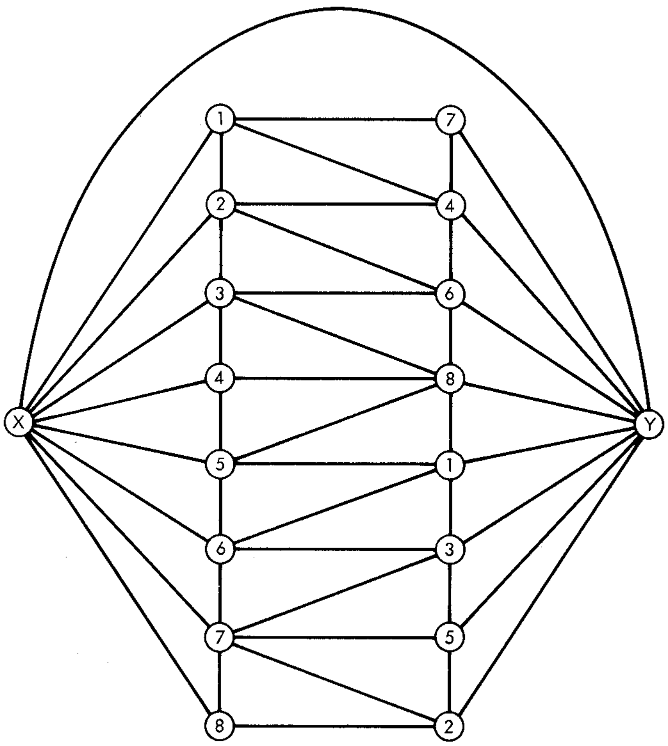


Fig. 9. $V = 2 \cdot 1 + 8 \cdot 2$

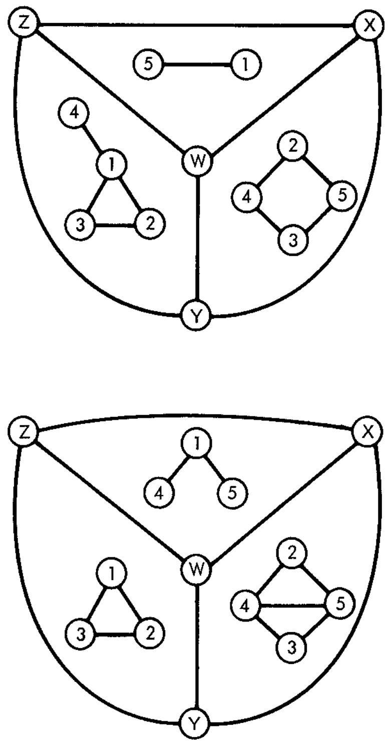


Fig. 10. Last stage of the impossibility proof for K_9

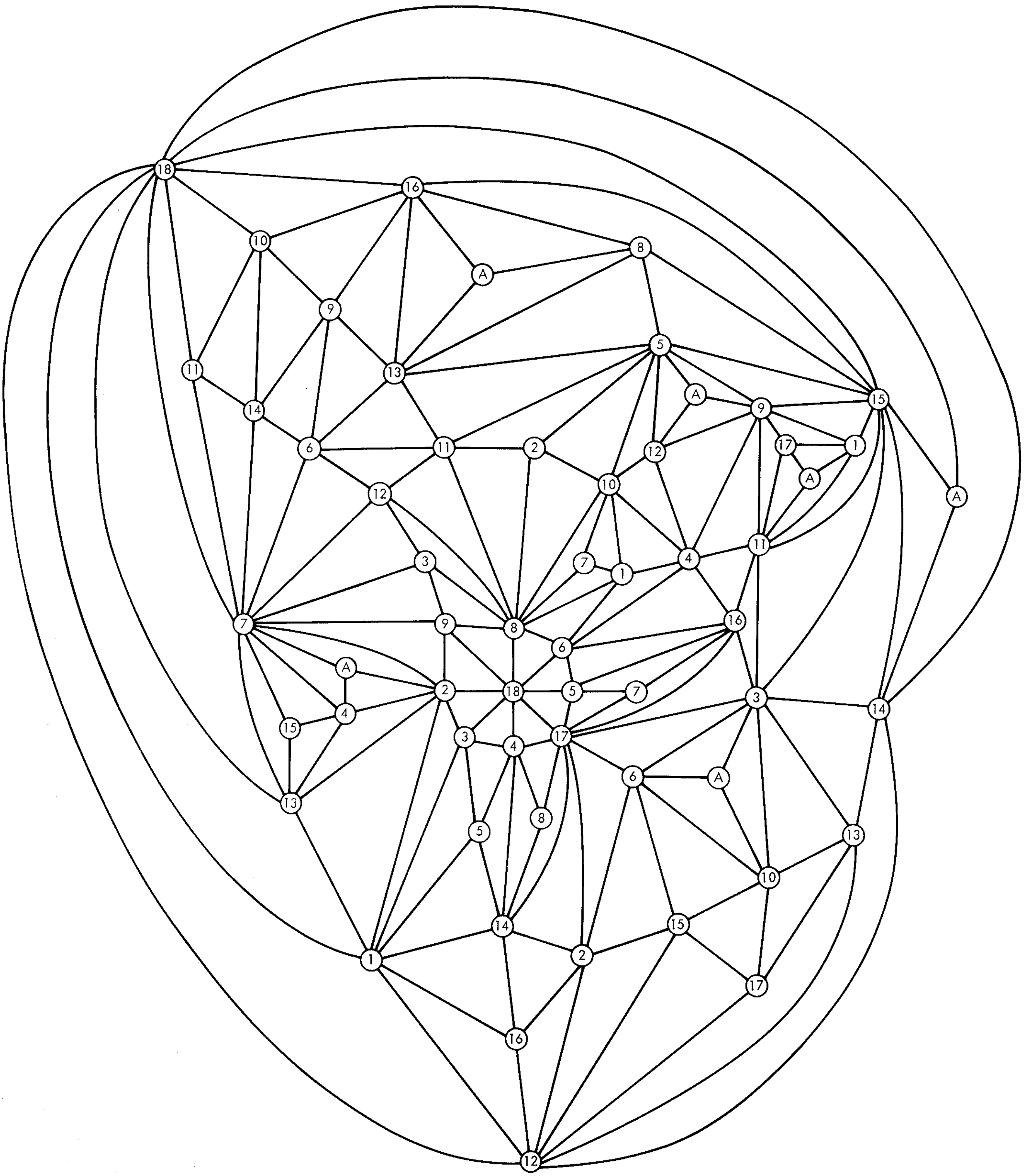


Fig. 11. $V = 0 + 1 \cdot 2 + 17 \cdot 3 + 1 \cdot 6$