Calculating Acquisition Behavior for Completely Digital Phase-Locked Loops

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The acquisition behavior of completely digital first and second-order phase-locked loops is considered. It is shown that modeling the loop by its difference equations allows the acquisition behavior to be computed using a procedure wherein the number of computations as well as the required storage grow only linearly with the size of the state space. It is also shown that the procedure can be easily modified to include the effects of doppler, finite length accumulators, and initial phase estimate jamming.

I. Introduction

At present very little is known analytically about the acquisition behavior of phase-locked loops. The primary reason for this is that the linearizing or simplifying assumptions used in calculating steady-state performance parameters are not applicable when the loop is in the acquisition mode. However, if one considers completely digital phase-locked loops (Refs. 1, 2), then any implementation of such a loop is necessarily a finite state machine. In such cases if one is given an initial state distribution vector, then one can, at least in principle, compute any subsequent time–state distribution vector by considering the loop as a finite state Markov chain and multiplying the initial state distribution vector by the appropriate power of the Markov transition matrix. Unfortunately the number of computations as well as the storage required for this procedure both increase as the square of the size of the state space, making such a procedure quite unattractive for all but the smaller state spaces. For example, Chadwick (Ref. 3) used this technique to compute the acquisition behavior of a digital tracking loop which, even after restricting himself to a most likely subset of the state space, resulted in a one-step Markov transition matrix having 6.25×10^8 entries.

The important point, which seems to have been overlooked in the past, is that for tracking loops of the kind described in Refs. 1 and 2 most of the entries in the Markov transition matrix are zeros. Consequently, if one characterizes the loop by its difference equations then only the essential computations need be performed. This results in a tremendous savings both in computations and storage. In this article we will use the difference equation approach for computing acquisition performance of first- and second-order digital loops. We will see that this approach can be easily modified to include the effects of finite accumulators (for the second-order loop) and doppler. Finally, we will show how this approach can be used to evaluate the merits of initial phase estimate jamming.

II. First-Order Loop

Consider the completely digital first-order phase-locked loop shown in Fig. 1. In this loop the input signal (a squarewave signal immersed in additive white gaussian noise) is filtered, sampled at the Nyquist rate and is applied to the transition sample selector. The transition sample selector, which is controlled by the phase shifter, extracts the sample of the input signal which is believed to have occurred at the transition of the received signal and corrects for the expected sign of this sample. The result is then accumulated until a sufficiently high signal-to-noise ratio is obtained to estimate (with high probability) whether the loop reference supplied by the phase shifter is leading or lagging the received signal. If the reference is leading the received signal, the accumulator will accumulate positively whereas it will accumulate negatively if the reference is lagging. After the accumulation has been completed, the sign of the accumulator output is used to shift or bump the phase of the loop reference signal one step in the direction to reduce the expected phase error, and the accumulator is reset.

The steady-state behavior of such a loop has been amply studied by Holmes (Ref. 1). For the acquisition behavior let us consider that the phase bump size is $2\pi/K$ radians. Then the loop can be considered as a Markov chain having K states. Let us consider (without loss of generality) that the phase of the received signal is zero. Furthermore, we will take the pessimistic approach by assuming that the phase error when the loop is in the kth state ϕ_k is

$$\phi_k = \frac{(2k-1)\pi}{K} \text{ radians} \tag{1}$$

for $k = 1, 2, \cdots, \frac{K}{2}$, and

$$\phi_k = \frac{(2K-2k-1)\pi}{K} \text{ radians} \tag{2}$$

for

$$k = \frac{K}{2}+1, \frac{K}{2}+2, \cdots, K$$

A state diagram for this loop is shown in Fig. 2.

We now define the state probability vector

$$\mathbf{s}_t = \begin{bmatrix} s_t(1) \\ s_t(2) \\ \cdot \\ \cdot \\ s_t(K) \end{bmatrix} \tag{3}$$

where $s_t(k)$ is the probability of being in state k at time t. Since the loop is discrete, we assume that t belongs to the index set (i.e., $t \in \{0, 1, 2, \cdots\}$). Then, from Fig. 2 and the theory of Markov chains we know that

$$\mathbf{s}_t = [\mathbf{P}]\,\mathbf{s}_0 \tag{4}$$

where $\mathbf{s}_0$ is the initial state distribution vector and $\mathbf{P}$ is the K by K one-step Markov transition matrix given by

$$\mathbf{P} = \begin{bmatrix} 0 & p & & & & & & & & & p \\ q & 0 & p & & & & & & & & \\ & q & 0 & p & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & \\ & & & q & 0 & p & & & & & \\ & & & & q & 0 & q & & & & \\ & & & & & q & 0 & q & & & \\ & & & & & & p & 0 & q & & \\ & & & & & & & \ddots & \ddots & \ddots & \\ & & & & & & & & p & 0 & q \\ p & & & & & & & & & p & 0 \end{bmatrix} \tag{5}$$

In this equation p is the probability that the accumulation will cause a phase bump in the direction to decrease the phase error, $q = 1 - p$, and the blanks are assumed to be filled with zeros.

In order to compute the state probability distribution after one accumulation using this procedure we must store (aside from the distribution vectors themselves) a matrix having K^2 entries and perform K^2 computations. However, if one characterizes the loop by its stochastic difference equations, a tremendous savings in both storage and computations is achieved. In particular we note that the loop can be characterized by

$$s_{t+1}(k) = \begin{cases} ps_t(k+1) + ps_t(k-1+K); & k = 1 \\ ps_t(k+1) + qs_t(k-1); & k \varepsilon [2, K/2 - 1] \\ qs_t(k+1) + qs_t(k-1); & k = K/2, K/2+1 \\ qs_t(k+1) + ps_t(k-1); & k \varepsilon [K/2+2, K-1] \\ ps_t(k+1-K) + ps_t(k-1); & k = K \end{cases} \tag{6}$$

By using this characterization the one-step probability distribution vector can be computed with only $2K$ computations and with a storage only slightly greater than that required to store the probability vectors themselves.

III. Second-Order Loop

The first-order loop of Fig. 1 can be converted to a second-order loop by inserting a filter of the form shown in Fig. 3 between the accumulator sign detector and the phase shifter. The effect of this filter is to cause the phase shifter to be bumped by

$$u \cdot n + x_2$$

units where $u = \pm 1$ is the output of the accumulator sign detector, n is an integer gain factor and x_2 is the value (after u is added) of the loop filter accumulator. For such loops one must specify the values of two state variables in order to characterize the state of the loop. We shall use the instantaneous phase error, as we did in the first-order case, for one of these variables and the output x_2 of the loop filter summer as the other.

It is convenient to view the state space of the second-order loop as a series of rings similar to the one shown in Fig. 2, where each ring corresponds to all possible values of instantaneous phase when the second state variable is fixed at some value. Note, however, that for the second-order loop no transitions within a given ring are allowed (except possibly at the extremes which will be considered later) and that only certain transitions between rings are allowed. One can further visualize this series of rings as forming a tube with each ring on the tube being rotated about the tube axis by an amount dependent on the location of the ring (value of x_2) as well as the value of n.

With this picture in mind, let us define $s_t(k, \ell)$ as the probability of being in the kth error state at time t with a loop filter value of ℓ. We can further consider a ring probability vector

$$\mathbf{s}_t(\ell) = \begin{Bmatrix} s_t(1,\ell) \\ s_t(2,\ell) \\ \vdots \\ s_t(k,\ell) \end{Bmatrix} \tag{7}$$

The loop stochastic difference equations are (assuming no restriction on ℓ):

(a) $|\ell| \leq n$

$$s_{t+1}(k,\ell) = \begin{cases} ps_t(k+\ell+n, \ell-1) + ps_t(k+\ell+K-n, \ell+1); & k\varepsilon [1, n-\ell] \\ ps_t(k+\ell+n, \ell-1) + qs_t(k+\ell-n, \ell+1); & k\varepsilon \left[n-\ell+1, \dfrac{K}{2}-n-\ell\right] \\ qs_t(k+\ell+n, \ell-1) + qs_t(k+\ell-n, \ell+1); & k\varepsilon \left[\dfrac{K}{2}-n-\ell+1, \dfrac{K}{2}+n-\ell\right] \\ qs_t(k+\ell+n, \ell-1) + ps_t(k+\ell-n, \ell+1); & k\varepsilon \left[\dfrac{K}{2}+n-\ell+1, K-n-\ell\right] \\ ps_t(k+\ell+n-K, \ell-1) + ps_t(k+\ell-n, \ell+1); & k\varepsilon [K-n-\ell+1, K] \end{cases} \tag{8a}$$

(b) $\ell < -n$

$$s_{t+1}(k,\ell) = \begin{cases} qs_t(k+n+\ell+K,\ell-1) + ps_t(k+K+\ell-n,\ell+1); & k\varepsilon\,[1,-n-\ell] \\ ps_t(k+n+\ell,\ell-1) + ps_t(k+\ell-n,\ell+1); & k\varepsilon\,[1-n-\ell,n-\ell] \\ ps_t(k+n+\ell,\ell-1) + qs_t(k+\ell-n,\ell+1); & k\varepsilon\left[n-\ell+1,\frac{K}{2}-n-\ell\right] \\ qs_t(k+n+\ell,\ell-1) + qs_t(k+\ell-n,\ell+1); & k\varepsilon\left[\frac{K}{2}-n-\ell+1,\frac{K}{2}+n-\ell\right] \\ qs_t(k+n+\ell,\ell-1) + ps_t(k+\ell-n,\ell+1); & k\varepsilon\left[\frac{K}{2}+n-\ell+1,K\right] \end{cases} \tag{8b}$$

and

(c) $\ell > n$

$$s_{t+1}(k,\ell) = \begin{cases} ps_t(k+\ell+n,\ell-1) + qs_t(k+\ell-n,\ell+1); & k\varepsilon\left[1,\frac{K}{2}-n-\ell\right] \\ qs_t(k+\ell+n,\ell-1) + qs_t(k+\ell-n,\ell+1); & k\varepsilon\left[\frac{K}{2}-n-\ell+1,\frac{K}{2}+n-\ell\right] \\ qs_t(k+\ell+n,\ell-1) + ps_t(k+\ell-n,\ell+1); & k\varepsilon\left[\frac{K}{2}+n-\ell+1,K-n-\ell\right] \\ ps_t(k+\ell+n,\ell-1) + ps_t(k+\ell-n,\ell+1); & k\varepsilon\,[K-n-\ell+1,K+n-\ell] \\ ps_t(k+\ell+n-K,\ell-1) + qs_t(k+\ell-n-K,\ell+1); & k\varepsilon\,[K+n-\ell+1,K] \end{cases} \tag{8c}$$

where it is assumed that the interval (set) $[a,b]$ is empty if $b < a$.

As in the first-order case, we see that the amount of storage necessary to accommodate these equations is essentially the same as the amount of storage necessary to store the state probabilities. Therefore, the storage grows linearly with the range of ℓ $(=x_2)$. Note also that the number of computations grows linearly with the size of the state space.

IV. Practical Considerations

(a) Accumulator Truncation

In any implementation of a discrete second-order phase-locked loop, the accumulator in the loop filter will have a finite range. For example, we will let $x_2\varepsilon\,[L_{min}, L_{max}]$. This is equivalent to placing saturating boundaries at the L_{max} and L_{min} rings of the tubular state space. Any attempt to penetrate beyond the $L_{max}\,(L_{min})$ ring results in a transition back onto a state in the $L_{max}\,(L_{min})$ ring.

The difference equations of the second-order loop can be easily modified to accommodate this saturation of the state space. First, we note that each of the expressions in Eq. (8) is of the form

$$s_{t+1}(k,\ell) = \alpha s_t(k',\ell-1) + \beta s_t(k'',\ell+1)$$

over the appropriate range of k. If ℓ is one of the saturating boundaries, then one of the two rings $(\ell+1)$ or $(\ell-1)$ does not exist. In fact, the only way to get to the ℓth ring is to either be on the interior ring next to ℓ or to be on the ℓth ring already. As a consequence, we see that if $\ell = L_{max}$ then we can add the set of equations

$$s_{t+1}(k,L_{max}) = \alpha s_t(k',L_{max}-1) + \alpha s_t(k',L_{max}) \tag{9}$$

over the appropriate range of k and where the quantities α and k' are determined by the range of ℓ in equation 8 corresponding to $\ell = L_{\max}$. Similarly, if $\ell = L_{\min}$ then we add the equations

$$s_{t+1}(k,L_{\min}) = \beta s_t(k'',L_{\min}) + \beta s_t(k'',L_{\min}+1) \tag{10}$$

again over the appropriate ranges of k. Thus, the Eqs. (8) apply to all interior rings and one need only add the set of modified equations for each boundary ring.

(b) Doppler

The effects of doppler can be quite easily handled provided one is willing to use a finite-state approximation approach. For example, if a loop has K phase error states and phase bumps every t_{up} seconds then one can approximate a doppler offset of D/Kt_{up} Hz (D an integer) by cyclically shifting each ring probability vector by D positions after each phase bump interval. If D is not an integer then one can either bound the desired result by using the appropriate neighboring integer or can further approximate by periodically selecting the amount of cyclic shift from a set of neighboring integers.

V. Initial Phase Estimate Jamming

Let us now assume that the received signal

$$s(t,\xi) = A\,\text{sgn}\,\{\sin[wt + \xi]\} \tag{11}$$

where

$$\xi \varepsilon \left[-\frac{\pi}{16}, \frac{\pi}{16}\right]$$

and is uniformly distributed in this interval. Assume also that $s(t,\xi)$ is to be correlated over the interval

$$t \varepsilon \left[-\frac{M^*T}{2}, \frac{M^*T}{2}\right]$$

with each of the reference signals

$$s_i(t) = \text{sgn}\,\{\sin[wt + \xi_i]\}; \qquad i = -7, -6, \cdots, 7, 8 \tag{12}$$

where

$$\xi_i = \frac{\pi i}{8} \text{ and } T = \frac{2\pi}{W}$$

If we denote

$$R_{\xi(i)} = \int_{\frac{-M^*T}{2}}^{\frac{M^*T}{2}} s(t,\xi)s_i(t)dt \tag{13}$$

and perform the integration, we obtain

$$R_\xi(i) = \begin{cases} A_i M^*T\left(1 - \dfrac{2}{\pi}\left|\xi_i - \xi\right|\right); & i = -7, -6, \cdots, 6, 7 \\ & \text{or } i = 8 \text{ and } \xi > 0 \\ -A_i M^*T\left(1 - \dfrac{2}{\pi}\left|\xi\right|\right); & i = 8 \text{ and } \xi < 0 \end{cases} \tag{14}$$

We note that if $A_i = A$ Eq. (14) does not include the effects of doppler. To include doppler, one need only replace A_i by $\alpha_d(i)A$ where

$$\alpha_d(i) = \frac{B(i,\lambda)}{B(i,\lambda = 0)} \tag{15}$$

is the effective signal loss factor due to doppler. Expressions necessary for computing $\alpha_d(i)$ are given in the Appendix.

We next allow $s(t,\xi)$ to be received in additive white gaussian noise $n(t)$ of one sided spectral density N_0 W/Hz. We also define the noisy correlator output by

$$g_i(\xi) = \int_{\frac{-M^*T}{2}}^{\frac{M^*T}{2}} [s(t,\xi) + n(t)]s_i(t)dt; \qquad i = -7, -6, \cdots, 7, 8 \tag{16}$$

We note that the $g_i(\xi)$ can be expressed as

$$g_0(\xi) = R_\xi(0) + \sum_{i=1}^{8} n_i - \sum_{i=1}^{8} n_{-i} \tag{17}$$

$$g_1(\xi) = R_\xi(1) + \sum_{i=1}^{8} n_i - \sum_{i=1}^{8} n_{-i} + 2n_{-1} - 2n_8 \tag{18}$$

etc., where the n_i's, i = $\pm 1, \pm 2, \cdots, \pm 8$ are independent zero-mean gaussian random variables with variance

$$\frac{N_0 M^*T}{32}$$

The purpose of all this is to select the ξ_i corresponding to the most likely phase of $s(t,\xi)$ so that the loop can be initially set (i.e., jammed) to that phase. Of primary interest is the mean square phase error immediately after the phase jamming. We shall be satisfied with computing a close bound to this mean square error. Toward this end, we note that

$$\Pr\{g_0 = \max_j g_j\} \le \Pr\{g_0 > g_1\} \Pr\{g_0 > g_{-1}\}$$

$$= \left[\tfrac{1}{2} + \tfrac{1}{2}\operatorname{erf}\left(\frac{R_\xi(0) - R_\xi(1)}{\sqrt{\frac{N_0 M^* T}{2}}}\right)\right]$$

$$\times \left[\tfrac{1}{2} + \tfrac{1}{2}\operatorname{erf}\left(\frac{R_\xi(0) - R_\xi(-1)}{\sqrt{\frac{N_0 M^* T}{2}}}\right)\right] \tag{19}$$

as well as

$$\Pr\{g_i = \max g_j\} \le \Pr\{g_i > g_0\}$$

$$= \left[\tfrac{1}{2} + \tfrac{1}{2}\operatorname{erf}\left(\frac{R_\xi(i) - R_\xi(0)}{\sqrt{\frac{N_0 M^* T|i|}{2}}}\right)\right]; \qquad i \ne 0 \tag{20}$$

Finally, we note that

$$E\{(\xi - \hat{\xi})^2 | \xi\} \le E\left\{(\xi - \hat{\xi})^2 | \xi = \frac{\pi}{16}\right\}$$

$$= \frac{\pi^2}{(16)^2}\left\{\sum_{i=-7}^{7} (1 - 2i)^2 \Pr\{g_i = \max_j g_j\}\right.$$

$$\left. + 225 \Pr\{g_8 = \max_j g_j\}\right\} \tag{21}$$

where $\hat{\xi}$ is the estimated phase. Substituting (19) and (20) into (21) yields the final bound.

VI. Examples

(a) First-Order Loop

The acquisition performance of a first-order loop having 256 states is shown in Figs. 4 and 5. It is assumed that the loop is updated (bumped) every M cycles of the received signal and that the initial phase estimation is based on an observation of the signal for $M^* = NM$ cycles. For $N = 0$ (i.e., no initial estimation), the initial state distribution vector is uniform. For $N > 0$, the mean square phase error due to phase estimation is first computed using the results of Section V. The initial state distribution is then taken as a quantized gaussian distribution with zero mean and this mean square error as a variance. The time axes have been normalized by the time required for M cycles (t_{up}).

Figure 4 represents the acquisition behavior when the loop update SNR

$$\rho = \frac{MTA^2}{N_0 4} = 2.5$$

and when no doppler is present. Figure 5 illustrates the behavior under the same conditions, except that a doppler offset was simulated by cycling the state probability vector one position every two updates.

(b) Second-Order Loop

Figures 6 to 8 demonstrate the acquisition behavior of a second-order loop similiar to the one described in Ref. 4. Here the phase variable has 256 discrete values and the proportional control gain factor $n = 4$. The range of the second state variable x_2 was restricted to the interval $[-3, 4]$. Figure 6 shows the acquisition with no doppler. Figures 7 and 8 illustrate the acquisition behavior when the numbers of doppler-generated shifts per loop update are 2 and 4, respectively.

(c) Discussion

By examination of Figs. 4 to 8, we see that if one is interested in fast acquisition then a large portion of the acquisition time should be allocated to phase estimate jamming. This is particularly true for the first-order loop, since the closed-loop time constant is generally quite large. For the second-order loop much less is to be gained by jamming unless the doppler offset is quite large. Note, however, that the doppler cannot be allowed to increase indefinitely since, as is shown in the appendix, if the estimator integration time equals the time needed to accumulate one complete doppler shifted cycle, the received signal will be orthogonal to all estimator reference signals.

VII. Summary

Described herein is a procedure for computing acquisition performance of first- and second-order digital loops which requires only a linear increase in storage and computations with the size of the state space. The procedure was found to be easily modified to account for finite accumulators and doppler. Expressions for the initial state mean square error after phase jam estimation were developed and several examples were presented to show the relative merits of phase jamming as an initial part of acquisition.

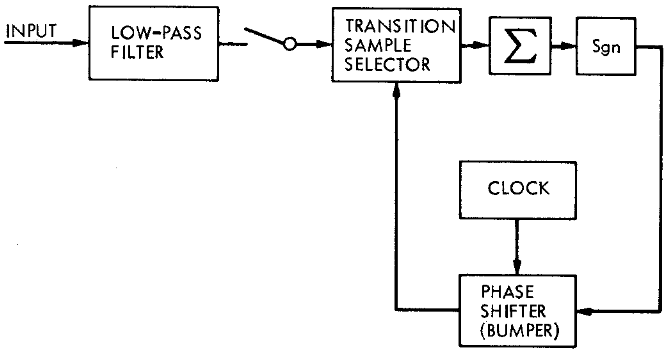

Fig. 1. First-order phase-locked loop block diagram

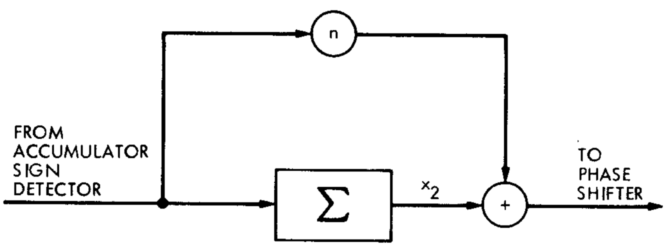

Fig. 3. Second-order digital PLL loop filter

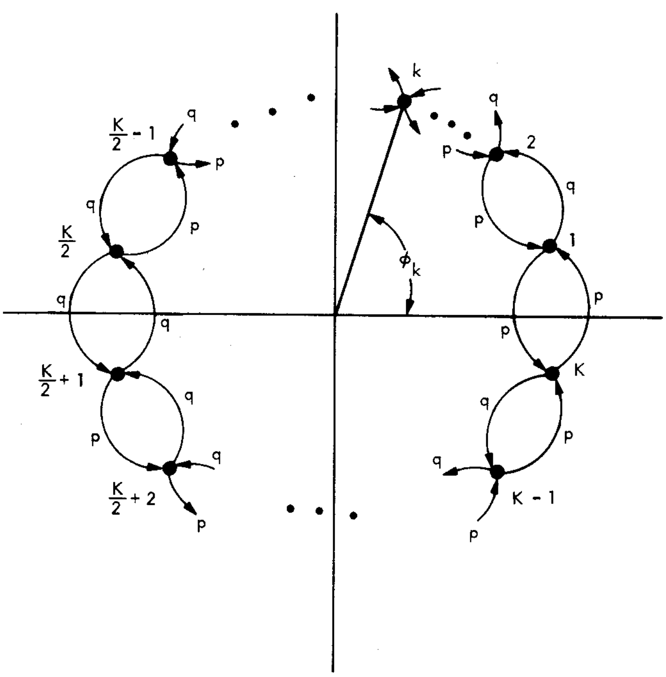

Fig. 2. Representation of state space for first-order PLL

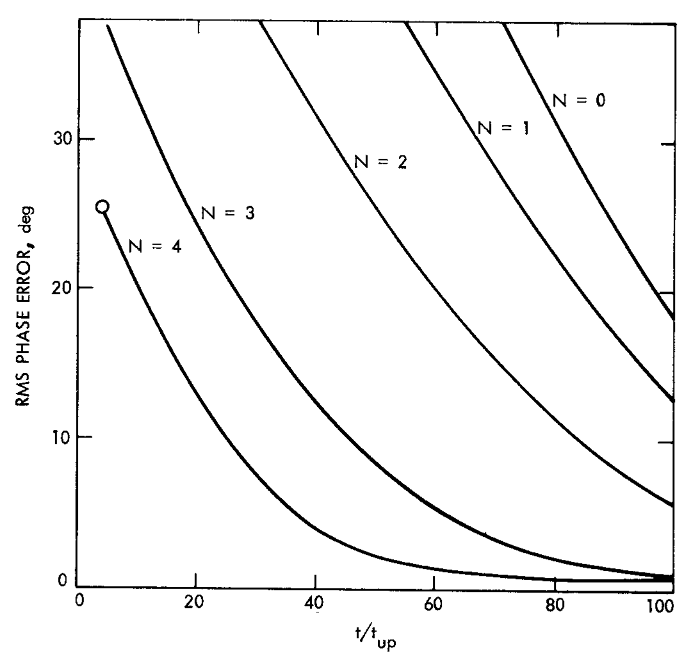

Fig. 4. Acquisition performance of first-order digital loop (no doppler)

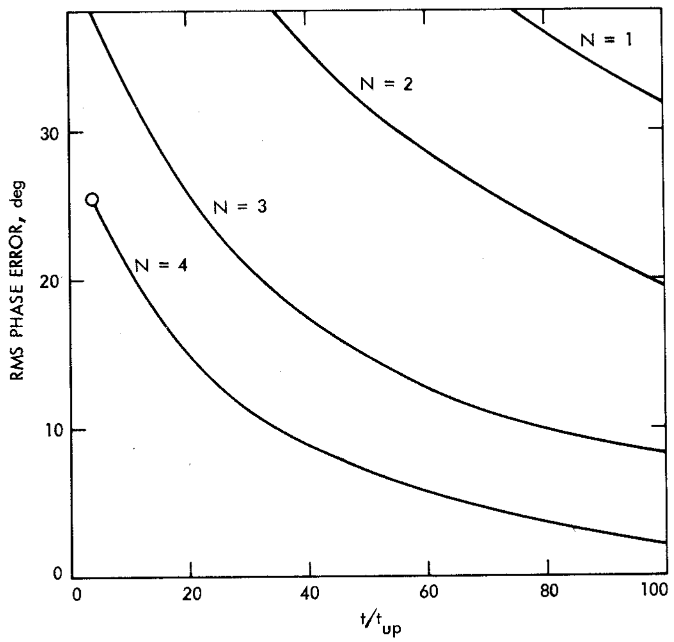

Fig. 5. Acquisition performance of first-order digital loop (one doppler state shift every two updates)

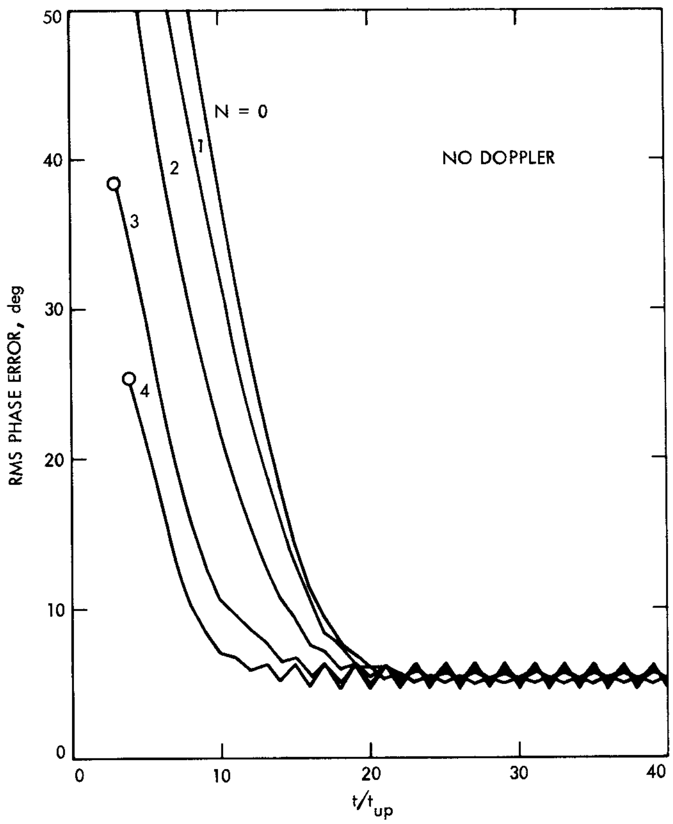

Fig. 6. Acquisition performance of second-order loop (no doppler)

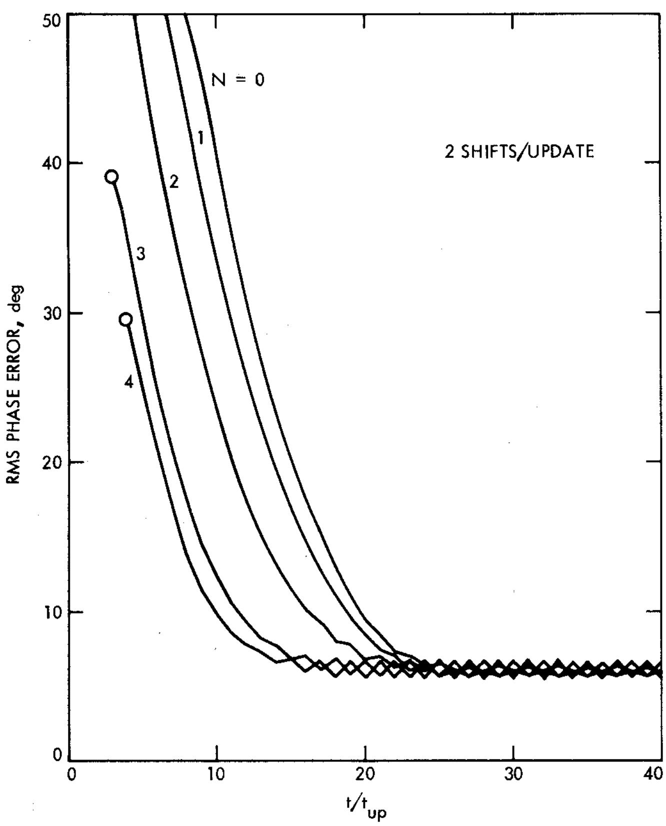

Fig. 7. Acquisition performance of second-order loop (2 doppler shifts per update)

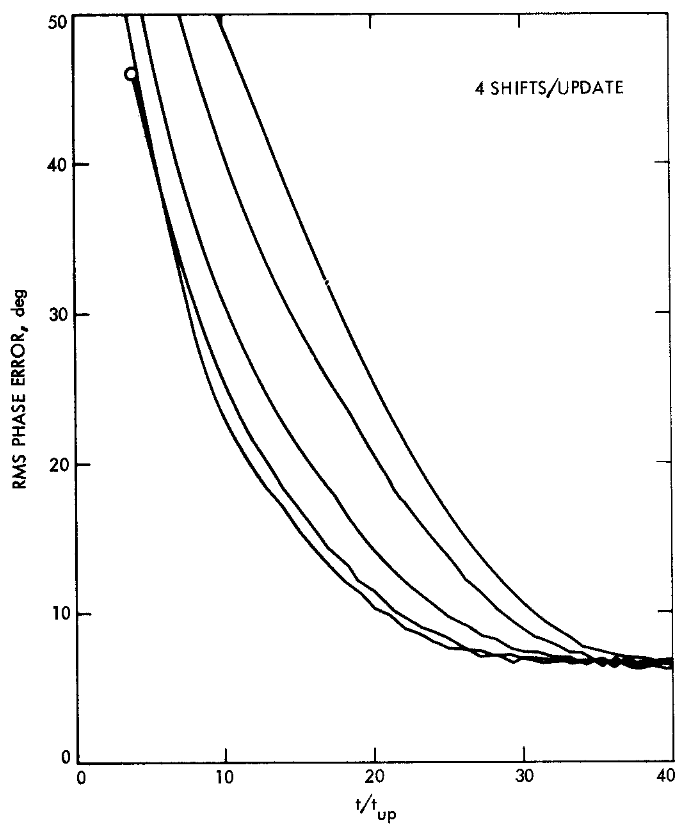

Fig. 8. Acquisition performance of second-order loop (4 doppler shifts per update)

Appendix

Signal Correlation in the Presence of Doppler

Consider a reference signal $s(t)$ of the form

$$s(t) = \text{sgn}\,\{\sin\,[W_0(t - \theta)]\}$$

and a received signal of the form

$$s'(t) = A\,\text{sgn}\,\{\sin\,[W_1(t - \theta')]\}$$

Let the two signals be correlated for a time interval of M^*T seconds where $T = 2\pi/W_0$ and M^* is an even integer. The correlator output at the end of this interval will be

$$\int_{-\frac{M^*T}{2}}^{\frac{M^*T}{2}} s(t)s'(t)dt = 2 \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \frac{C_k C'_\ell \sin\left[(kW_0 + \ell W_1)\dfrac{M^*T}{2}\right]}{kW_0 + \ell W_1} \qquad \text{(A-1)}$$

where

$$C_k = \begin{cases} \dfrac{2 \exp(-jkW_0\theta)}{jk\pi} & ; \quad k \text{ odd} \\ 0 & ; \quad k \text{ even} \end{cases}$$

and

$$C'_k = \begin{cases} \dfrac{2A \exp(-jkW_1\theta')}{jk\pi} & ; \quad k \text{ odd} \\ 0 & ; \quad k \text{ even} \end{cases}$$

Substituting the C's into (A-1) and defining

$$\delta = \frac{W_1 - W_0}{W_0}$$

gives

$$\int_{-\frac{M^*T}{2}}^{\frac{M^*T}{2}} s(t)s'(t)dt = -32A \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{\infty} \frac{\sin\,[\ell\delta M^*\pi]}{k\ell\pi^2\,[k^2W_0^2 - \ell^2W_1^2]} \left[kW_0 \cos(kW_0\theta)\cos(\ell W_1\theta') + \ell W_1 \sin(kW_0\theta)\sin(\ell W_1\theta')\right] \qquad \text{(A-2)}$$

Now let us assume that θ' is uniformly distributed and define for $T' = 2\pi/W_1$ the intervals

$$\Delta'_i = \left\{\theta' \colon \frac{(2i-1)T'}{32} \leq \theta < \frac{(2i+1)T'}{32}\right\} i = 0,1,2,\cdots,15$$

Furthermore, assume that if $\theta' \epsilon \Delta'_i$

then

$$\theta = \frac{(i-J)T}{16} \tag{A-3}$$

for $J \epsilon \{0,1, \cdots ,15\}$. Note that for δ small, then $T' \approx T$ and θ corresponds to the midpoint of one of the 16 intervals Δ'_i. Next, define $B_i(J,\delta,M^*)$ as the average over $\theta' \epsilon \Delta'_i$ when θ is given by (A-3). Performing this average yields

$$B_i(J,\delta,M^*) = -\frac{(16)^2}{\pi^3} A \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{\infty} \frac{\sin[\ell\delta M^*\pi]\sin\left(\frac{\ell\pi}{16}\right)}{k\ell^2[k^2W_0^2 - \ell^2W_1^2]}$$

$$\times\left[(kW_0 - \ell W_1)\cos\left(\frac{(k+\ell)\pi_i}{8} - \frac{kJ\pi}{8}\right) + (kW_0 + \ell W_1)\cos\left(\frac{(k-\ell)\pi_i}{8} - \frac{kJ\pi}{8}\right)\right] \tag{A-4}$$

It is our intent to average (A-4) over the 16 values of i. However, in most practical systems some band limiting exists, and hence the use of equations which display unlimited harmonic content is quite unnecessary. Therefore, we will restrict equation (A-4) to summations up to the seventh harmonic. Then, averaging over i we get

$$B(J,\delta,M^*) = (16)^2 AM^*T\left\{\sum_{\substack{k=1 \\ k \text{ odd}}}^{7} \frac{\sin((8-k)\delta M^*\pi)\cos\left(\frac{k\pi}{16}\right)\cos\left(\frac{kJ\pi}{8}\right)}{2\pi^4k(8-k)^2[8+\delta(8-k)]M^*} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{7} \frac{\sin(k\pi\delta M^*)\sin\left(\frac{k\pi}{16}\right)\cos\left(\frac{kJ\pi}{8}\right)}{2k^4\pi^4\delta M^*}\right\}$$

However, if δ is small, the second term dominates the equation so that

$$B(J,\delta,M^*) = AMT\sum_{\substack{k=1 \\ k \text{ odd}}}^{7} \frac{(16)^2\sin(k\pi\lambda)\sin\left(\frac{k\pi}{16}\right)\cos\left(\frac{kJ\pi}{8}\right)}{2(k\pi)^4\lambda}$$

$$= B(J,\lambda) \tag{A-5}$$

where

$$\lambda = \delta M^*$$

Equation (A-5) represents the average signal correlation of a reference signal and a signal which is uniformly distributed in an interval J sixteenths of a period away from the reference.

Finally, we note that if the doppler goes to zero then

$$B(J,\lambda=0) = AMT\sum_{\substack{k=1 \\ k \text{ odd}}}^{7} \frac{(16)^2\sin\left(\frac{k\pi}{16}\right)\cos\left(\frac{kJ\pi}{8}\right)}{2(k\pi)^3}$$

The ratio

$$\frac{B(J,\lambda)}{B(J,\lambda=0)}$$

can be considered as the effective signal reduction factor due to doppler.

Figure A.1 illustrates the behavior of $|B(J,\lambda)|$ (after normalization) as a function of λ, the number of doppler-shifted cycles contained in the signal correlation integration time. It is instructive to note that as λ increases the peaks of the triangular squarewave correlation curve are suppressed more than the interior points. This causes the correlation function or S-curve to become more rounded. This rounding continues until the integration time encompasses one complete doppler-shifted cycle at which time the received signal is orthogonal to all of the reference signals.

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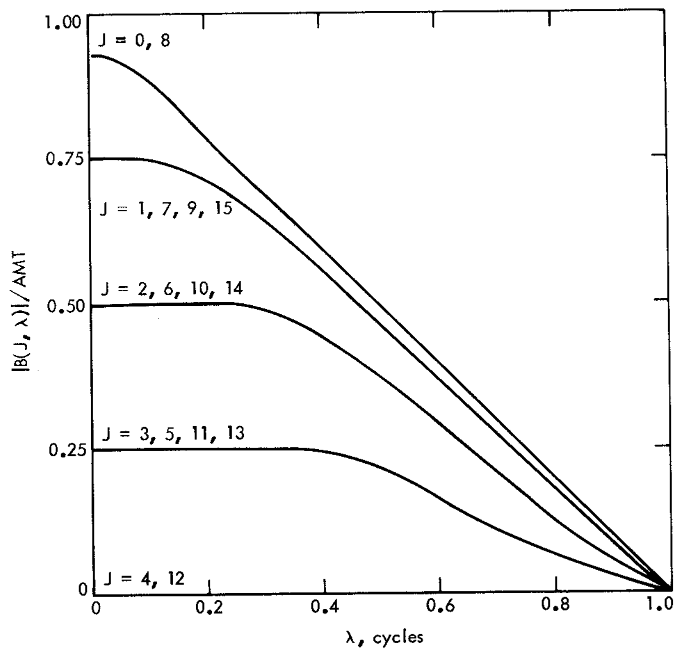

Fig. A1. Signal correlation loss due to doppler