Life Cycle Costing of Long-Term Capability With a Discount Rate

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This article studies life-cycle costing for a capability needed for the indefinite future. The two costs considered are reprocurement cost and maintenance and operations (M&O) cost. The reprocurement price is assumed known, and the M&O costs are assumed to be a known function of the time since last reprocurement, in fact an increasing function. The problem is to choose the optimum reprocurement time so as to minimize the quotient of the total cost over a reprocurement period divided by the period. Or one could assume a discount rate and try to minimize the total discounted costs into the indefinite future. It is shown that the optimum policy in the presence of a small discount rate hardly depends on the discount rate at all, and leads to essentially the same policy as in the case in which discounting is not considered. An algorithm for finding the optimum reprocurement time is presented as implemented in an MBASIC program.

I. Introduction

Suppose that one is planning to provide a capability into the indefinite future. The initial procurement cost is given. In addition, the annual maintenance and operations costs are known and are a nondecreasing function of the time since the initial procurement. One has the option of replacing the equipment at the original price at any time, thus reverting to lower maintenance costs. However, costs in the future are discounted at a constant nonnegative rate. The total discounted cost of future procurements and maintenance and operations costs is called the Life-Cycle Cost Increment. The idea is to choose the reprocurement time so as to minimize this Life-Cycle Cost Increment. It is shown that as the discount rate approaches zero, the optimum reprocurement time approaches a limit, as does the minimum Life-Cycle Cost Increment divided by the discount rate. This ratio is shown to be a decreasing function as the discount rate increases. The procurement time limit is the optimum reprocurement time for zero discount rate, where the goal is to minimize the Life-Cycle Cost Rate, the average future expenditure per year. Finally, the limit of the minimum Life-Cycle Cost Increment divided by the discount rate is the minimum of the Life-Cycle Cost Rate. An MBASIC program has been designed and used to define optimum policies when one has the option of procuring "better" equipment, which perhaps costs more but results in lower maintenance and operations costs. The purpose of this article, then, is to create a framework for considering life-cycle costing with a discount rate, and, specifically, to show that the choice of discount rate does not materially affect optimum policy, but rather affects only the figure computed for discounted future costs. This latter cost is essentially given in arbitrary units, however, and we shall show how the proper units of life-cycle cost can be found.

The model of this paper is not overly typical of the DSN, which operates in an atmosphere of changing technology and new capability. Instead, we assume that we must provide a capability "forever." That capability is provided by a subsystem which costs P dollars to procure, and will always cost P dollars to procure. (The effect of inflation will be discussed shortly.)

Now comes the least certain assumption of this article. We assume that we know the M&O (maintenance and operations) cost rate for the subsystem (and assume this cost is independent of any other decision made). More particularly, the M&O cost rate in dollars/year depends on the age of the subsystem, i.e., the time since it was last reprocured. This function $M(t)$ is assumed to be nondecreasing with time, certainly a reasonable assumption. When $M(t)$ starts getting too large, we would reprocure at price P and start off again at the initial low M&O cost $M(0)$.

Now consider the discount rate α, a nonnegative number. Our model is built so that $\alpha = 0$ corresponds to not having a discount rate, and the definition of life cycle cost must then change. The discount rate α should represent the "social discount rate," the value of money for social investment. Thus, α might be, say, 2%, but should not be nearly as high as the inflation rate; that is, it should not be near 8%. We have assumed the reprocurement price P is constant forever, and also that $M(t)$ starts off again at the old $M(0)$, no matter how long we wait to reprocure. In other words, we have removed inflation from the picture, which explains the small values of α. Those who do not like to discount at all will be pleased to learn that the value of α in the problem considered here hardly matters (once α is small), and leads to the same policies as in the case of no discounting.

For mathematical convenience, and because the money market works that way, we are using "continuous compound interest." That is, the value of P dollars discounted time t years in the future is $Pe^{-\alpha t}$. For integer t, "classical" discounting would use $P/(1 + \alpha)^t$ instead, which is "annual discrete discounting." The relevant comparison is thus $e^{\alpha t}$ vs $(1 + \alpha)^t$. If discrete discounting is the policy, but continuous is used for mathematical convenience, a small discrepancy results. For example, for $\alpha = 2\%$ discrete, the correct continuous discount rate, say α', to use would be 1.98%. For $\alpha = 20\%$, the highest rate considered in this article, $\alpha' = 18.2\%$. We shall ignore this distinction from now on, but have mentioned it because of the Truth in Lending Act.

We assume that we have just paid P\$ for a new subsystem. Let us also assume that we have decided to buy an identical replacement subsystem every T years; thus T determines the policy. Note that we can restrict ourselves to considering only such periodic policies, since if it is wise to replace after the first T years it is always wise to do so because every reprocurement starts the samr process over.

When $\alpha > 0$, the Life-Cycle Cost Increment for policy determined by T is the total future costs, reprocurement plus M&O. But when $\alpha = 0$, we define instead the Life-Cycle Cost Rate (the other would be infinite) as the total cost (M&O plus reprocurement) over period T, including the reprocurement at price P at time T but excluding the original procurement. This number is then made into a cost rate by dividing by the period T. It will turn out that the minimum Life-Cycle Cost Increment, when multiplied by the discount rate α, approaches the Life-Cycle Cost Rate as α approaches 0. Thus, the proper units when discounting should be dollars per year, using the above normalization. Even though the Life-Cycle Cost Increment itself is in dollars, those dollars are really in a sense arbitrary units. This will be made more believable by our convergence results, to be given below.

II. Determining Life Cycle Cost

We first need the equation which displays the fact that if we reprocure at time T, we start over with $M(0)$ as the maintenance rate. Let $C_\alpha(T)$ be the Life-Cycle Cost Increment when $\alpha > 0$. Then

$$C_\alpha(T) = \int_0^T e^{-\alpha t} M(t)dt + e^{-\alpha T} [P + C_\alpha(T)], \alpha > 0 \tag{1}$$

(the "Renewal Equation"). The $e^{-\alpha t}$ term in the integral discounts the M&O cost. After one period of length T, we reprocure at price P and start over (renew) at Life-Cycle Cost Increment $C_\alpha(T)$. However, the cost $P + C_\alpha(T)$, being deferred T years, is discounted by $e^{-\alpha T}$. We solve for $C_\alpha(T)$ to find

$$C_\alpha(T) = \left[\int_0^T e^{-\alpha t} M(t)\,dt + e^{-\alpha t} P\right] \Big/ (1 - e^{-\alpha t}),\ \alpha > 0. \tag{2}$$

When $\alpha = 0$, let $C_{(0)}(T)$ be the Life-Cycle Cost Rate. Then straightforward calculation produces

$$C_{(0)}(T) = \frac{1}{T}\left(P + \int_0^T M(t)dt\right). \tag{3}$$

Equation (1) shows that $C_\alpha(T)$ and $C_{(0)}(T)$ are differentiable functions of T in $T > 0$:

$$C'_\alpha(T) = \frac{M(T) - \alpha\,[P + C_\alpha(T)]}{e^{\alpha T} - 1},\ \alpha > 0. \tag{4}$$

Likewise, from Eq. (3),

$$C'_{(0)}(T) = \frac{M(T) - C_{(0)}(T)}{T}. \tag{5}$$

Thus, if $C_\alpha(T)$ and $C_{(0)}(T)$ have a minimum in $T > 0$ (note that both are $+\infty$ when $T = 0$), then the minimum occurs at a stationary point, one where the derivative is 0. Thus, at a minimum,

$$C_\alpha(T) = \frac{M(T)}{\alpha} - P,\ \alpha > 0; \tag{6}$$

$$C_{(0)}(T) = M(T). \tag{7}$$

Define $C_{(\alpha)}(T)$ as $\alpha C_\alpha(T)$ for $\alpha > 0$, and call it the "Normalized Life Cycle Cost Increment." Then (6) and (7) become

$$C_{(\alpha)}(T) = M(T) - \alpha P,\ \alpha \geqq 0. \tag{8}$$

It will be shown in the next section that either $C_{(\alpha)}$ has no stationary point in $T > 0$ because it decreases forever, or else has a unique stationary point which is the minimum sought for.

Let us stop to study an example, which illustrates the results of the next section. Let $M(T) = kt$ for $t \geq 0$ (k in dollars/yr^2). Then we can evaluate $C_\alpha(T)$ and $C_{(0)}(T)$ exactly:

$$C_\alpha(T) = \frac{(k/\alpha^2)\,(1 - e^{\alpha T} - \alpha T e^{-\alpha T}) + e^{-\alpha T} P}{1 - e^{-\alpha T}}, \tag{9}$$

$$C_{(0)}(T) = \frac{1}{T}\left(P + \frac{kT^2}{2}\right). \tag{10}$$

Then $C_{(0)}(T)$ has zero derivative, and thus its minimum, at

$$T_{(0),min} = \sqrt{\frac{2P}{k}}, \tag{11}$$

with associated Minimum Life Cycle Cost Increment

$$C_{(0),min} = \sqrt{2Pk}. \tag{12}$$

For $\alpha = 0$, we find, differentiating (9), that the minimum Life Cycle Cost Increment $C_{\alpha,min}$ is achieved at $T_{\alpha,min}$, where

$$\alpha T_{\alpha,min} + e^{-\alpha T_{\alpha,min}} = 1 + \frac{\alpha^2 P}{k}. \tag{13}$$

Equation (13) does indeed have a unique solution in $T > 0$, since the function $\alpha T + e^{-\alpha T}$ is increasing in $T > 0$. For $\alpha \to 0$, we can show

$$T_{\alpha,min} = \sqrt{\frac{2P}{k}} + \frac{P\alpha}{3k} + O(\alpha^2), \tag{14}$$

so that $T_{\alpha,min} \to T_{(0),min}$ as $\alpha \to 0$.

And

$$C_{\alpha,min} \sim \frac{\sqrt{2Pk}}{\alpha}, \tag{15}$$

so $\alpha C_{\alpha,min} = C_{(\alpha),min}$ (definition) does indeed converge to $C_{(0),min}$. More tedious analysis would even show that $C_{(\alpha),min}$ is $C_{(0),min}$ plus α times a negative constant plus terms in α^2 or higher, as $\alpha \to 0$, since in fact $C_{(\alpha),min}$ is shown in the next section to be a decreasing function of α as α increases, for all α.

III. Key Results

This section lists key results without proof; proofs are found in Ref. 1. The proofs are not esoteric, merely using the fact that $e^{\alpha t} - 1$ looks like αt as $\alpha \to 0$. It is assumed that the maintenance cost rate function $M(t)$ (dollars/yr) is continuous nondecreasing. Also, P, the purchase price, is greater than 0. The results below are true for all $\alpha \geq 0$, including $\alpha = 0$, the case of no discounting.

Result 1: $C_{(\alpha)}(T)$ is either decreasing for all T ("minimum at ∞") or has a unique minimum $T_{\alpha,opt}$, with minimum $C_{(\alpha)}(T_{opt}) = C_{(\alpha),min}$, which is positive. For convenience of writing, we allow ∞ to be a minimum to avoid having to separately consider the special case in which $C_{(\alpha)}(T)$ decreases forever.

Result 2: $C_{(\alpha)}(T)$ is decreasing to the left of its minimum and increasing to the right of it.

Result 3: $C_{(\alpha)}(T)$ approaches ∞ as T decreases to 0. As $T \to \infty$, $C_{(\alpha)}(T)$ may or may not be bounded.

The next result is useful in obtaining a convenient algorithm for finding $T_{\alpha,opt}$:

Result 4: $C_{(\alpha)}(T)$ is convex upward to the left of $T_{\alpha,opt}$.

Result 5: $T_{\alpha,opt}$ is increasing in α as α increases, when $T_{\alpha,opt}$ is finite. When it is infinite for one α, it is infinite for larger α.

Result 6: $C_{(\alpha),min}$ is decreasing in α as α increases, and approaches $M(0)$ as $\alpha \to \infty$, where $M(0)$ is the maintenance cost rate at time 0.

The following are the two key results of this article. Result 7 shows that the optimum reprocurement time for small discount rate α is close to the optimum reprocurement time for zero discount rate. Result 8 shows that the minimum Normalized Life-Cycle Cost Increment $C_{(\alpha),min}$ for $\alpha > 0$ approaches the minimum Life-Cycle Cost Rate $C_{(0),min}$ as the discount rate α decreases to 0. Result 8 is the hardest result of this article to prove.

Result 7: $T_{\alpha,opt} \to T_{0,opt}$ as $\alpha \to 0$.

Result 8: $C_{(\alpha),min} \to C_{(0),min}$ as $\alpha \to 0$.

In fact, $T_{\alpha,opt}$ and $C_{(\alpha),min}$ are continuous functions of α in the region $\alpha \geq 0$, and Results 7 and 8 are the important special cases. What they mean is that the optimum policies for small α are close to the optimum policy for those who do not discount. The "discounters" get different answers from the "nondiscounters" only because they have failed to normalize $C_\alpha(T_{\alpha,opt})$ into $C_{(\alpha)}(T_{\alpha,opt})$ by failing to multiply by α. This normalization brings the minimum Life-Cycle Cost Increment to the more real units of dollars/year. The unnormalized Life-Cycle Cost Increment in dollars is really in arbitrary "α-money units," even though the same word "dollars" is used. The results will be numerically illustrated in a particular instance in the next section.

IV. Computing the Optimum

An interactive structured program for demonstration purposes has been produced in MBASIC according to the methodology of DSN Standard Practice 810-13, "Software Implementation Guidelines and Practices." The procedure to find $T_{\alpha,opt}$ first finds two times on either side of the optimum. This is easy to do because of Result 2, since we need only check the slopes—positive slope of $C_{(\alpha)}(T)$ means $T > T_{\alpha,opt}$, negative slope means $T < T_{\alpha,opt}$. Once we have "trapped" $T_{\alpha,opt}$, we converge to it (more precisely, we find a cycle time which is guaranteed to produce a cost within 0.5% of $C_{(\alpha),min}$) by taking advantage of the convexity Result 4. Full details on the procedure can be found in Ref. 1.

Figure 1 shows an exact output of the program. The variable TMAX is the largest cycle we are willing to consider, in this case 12 years—if $T_{\alpha,opt} > 12$ years, we use 12 years as the cycle (as a change from the value 50 years as declared before changes). The variable FINE is the quantization, for integration purposes, set as 1/4 year both for numerical reasons and because 3 months is considered the minimum time to which such life cycle policies can respond. This variable is not changed in the run shown. The variable BUMP is used in the sensitivity check within the format within the *'ed box—it is 1 year, unchanged by the user in this run. The variable DELTA, set at 2% (.02) in the program, is changed in this run to 0.5% (.005); it is the fractional accuracy we guarantee in $C_{(\alpha),min}$. The purchase price is $\$P$, entered as 12. Zero discount rate ALPHA was input. The maintenance cost rate $M(t)$ in \$/yr is entered as $1 + (X/24) \cdot [1 + (X/32)] = 1 + X/24 + X^2/(24 \cdot 32)$. (Here X is time t divided by FINE (FINE was 1/4) in order to agree with the definition of the array variable M as indexed from 0 to TMAX/FINE.) This quadratic function was used as being possibly typical of the as yet untested real world.

The answer then is TOPT = 8 years, with minimum Life-Cycle Cost Rate \$3.61/yr. This value is accurate to 0.5%, but the accuracy of $T_{0,opt}$ is neither specified nor relevant —we are trying to control costs, not times. We also find that if we increase the reprocurement time from 8 years to 9 years, the cost rate goes up only 4¢/yr, to 3.65 \$/yr. Decreasing the cycle to 7 years raises the cost rate only 3¢/yr. Thus, the optimum is rather broad, which is a useful property. We also print the percent of the cycle costs that are M&O costs, in this case 58.5%. For the example under discussion, Fig. 2 graphs $C_{(0)}(T)$ vs T to show this broad minimum. The graph was obtained from a modification of the program under discussion. Figure 3 graphs $C_{(\alpha),min}$ for the above parameters, but with ALPHA varying from 0 up to an outrageous 20% per year. Note that $C_{(\alpha),min}$ decreases from 3.61 \$/yr at zero discount rate down to 2.34 \$/yr at 20% discount rate. At the more reasonable 2% discount rate, $C_{(.02),min}$ is 3.50 \$/yr, only 14¢/yr different. More important, the optimum cycle time $T_{\alpha,opt}$ produced by the program is 8 years, from ALPHA = 0 up to ALPHA = 0.1, and then fluctuates near 8 years (see Table 1). If the 8 years of 0 discount rate were used for 2% rate, the $C_{(.02),min}$ would be within 1% of the true minimum. Even at 20% discount rate, the use of 8 years instead of the 10.25 years of Table 1 produces an increase in cost of only 2%, as a run of the program with BUMP = 2.25 shows. Thus, there appears to be little reason to use positive discount rates for the particular infinite-horizon life cycle problem considered in this article.

V. Allowing Purchase Price to Vary

There are of course situations where one has the option to lower M&O costs by paying more for the initial procurement (and thus for the reprocurements). Unfortunately, we don't always know exactly what the tradeoff is, but, in this section, we shall create an illustration in which we do know the tradeoff. We then find the best price P to pay so that when we subsequently minimize the Life-Cycle Cost Rate, we obtain the overall minimum Life-Cycle Cost Rate. (But for $\alpha > 0$, the proper cost to minimize is $P + C_\alpha(T)$, or, what gives the same choice of P and T, $\alpha P + C_{(\alpha)}(T)$.)

Suppose the tradeoff of purchase price P vs maintenance cost rate $M(t)$ is given by

$$M(t) = t\left(1 + \frac{1}{P^2}\right). \tag{16}$$

For very low P, $M(t)$ is outrageous. As P gets large, $M(t)$ settles down to t, so that very expensive "models" are not a good buy either. "Freshman Calculus" techniques show that $T_{0,opt}(P)$ is given, for a fixed P, by

$$T_{0,opt}(P) = P\left(\frac{2P}{P^2+1}\right)^{1/2} \tag{17}$$

with associated $C_{(0),min}(P)$ given by

$$C_{(0),min}(P) = \left[\frac{2(P^2+1)}{P}\right]^{1/2} \tag{18}$$

We seek the minimum of (18) as P varies. It is readily shown to occur at $P = 1$ with value $C_{(0),best}$ given by

$$C_{(0),best} = 2. \tag{19}$$

From Eq. (17) with $P = 1$, we find that the best reprocurement time, $T_{0,best}$, is

$$T_{0,best} = 1. \tag{20}$$

In other words the best overall policy, the one resulting in minimum Life-Cycle Cost Rate when P is allowed to vary, is to purchase units costing \$1 every (one) year.

We also used the computer program to vary P and find the overall minimum of $C_{(0),min}(P)$. We found that $C_{(0),best}$ was 2 to within 5 decimal places, occurring at $P = 1$ to within 3 decimal places. It took 11 uses of the original program to obtain the answer, varying P each time. Details are omitted.

VI. Future Work

Future areas for investigation suggest themselves from the preceding sections. The most important is the problem of how to learn what the maintenance cost rate function $M(t)$ is. This problem can be considered a sequential estimation problem, as in Ref. 2.

Unfortunately, the use of a similar approach is currently hampered by the fact that it is difficult to assign M&O costs to particular subsystems or assemblies. Equally important, it is very difficult to tell what the tradeoff is between procurement price and M&O costs. For example, we don't yet know how much to pay for increased semiautomatic operability of DSN subsystems, because we do not know in enough detail how the M&O cost rate function $M(t)$ is built up.

In another vein, a rational policy on life cycle costing ought perhaps to take into account availability of the tracking station or Network. Thus, we could decide to lower $M(t)$ by increasing the probability of downtime. Or, for a given acceptable downtime, there would be a combined procurement specification and reprocurement time which results in lower overall costs.

Another area of investigation that could prove fruitful for the DSN involves modeling the M&O cost interactions of the various subsystems at a tracking station, acting in concert. It seems clear that costs of maintenance and operations are not actually additive by subsystem. Thus, one may wish to consider a combined policy on procurement intervals that takes total station or network costs into account. The techniques could even be statistical in nature for a system the size of a DSN tracking station.

The various problem areas above are indicative of the kinds of things one might want to know when adopting a life cycle cost policy, whether for the "infinite horizon" idealization of this paper or for the case more typical of much of the DSN, the fixed life cycle case (usually defined as initial procurement cost plus 10-year undiscounted M&O cost).

References

1. Posner, Edward C., "Life Cycle Costing with a Discount Rate," submitted to *Management Science*.
2. Lorden, G., "Sequential Tests for Exponential Distributions," in *The Deep Space Network*, Technical Report 32-1526, Vol. 5, pp. 82–90, Jet Propulsion Laboratory, Pasadena, Calif., Oct. 15, 1971.

Table 1. $T_{\alpha, opt}$ vs α

α, %	$T_{\alpha,opt}$, years
0	8.00
0.5	8.00
1	8.00
1.5	8.25
2	8.00
3	8.25
5	8.00
7.5	8.25
10	9.00
15	9.25
20	10.25

```
LOAD 'MAINTLFCY'
>RUNH
MBASIC 12/10/75    17:25:37


ENTER CHANGES TO TMAX,FINE,BUMP,AND DELTA,IF ANY. THEN TYPE  CON
>TMAX=12,DELTA=.005
>CON

ENTER PURCHASE PRICE F:12

ENTER DISCOUNT RATE ALPHA:0

ENTER MAINTENANCE COSTS INTO ARRAY FROM 0 TO TMAX/FINE;USE M( )=
THEN TYPE  CON
>M(X)=1+(X/24)*(1+(X/32)) FOR X=0 TO 48
>CON


********************************************************************************
*                                                                              *
* TOPT= 8.00 YRS                                                               *
* MIN COST: C(TOPT)= .361E+01 $                                                *
*                                                                              *
*                                                                              *
*  C(TOPT+1.00)= .365E+01 $                                                    *
*  C(TOPT-1.00)= .364E+01 $                                                    *
*                                                                              *
* 58.5 % OF LIFE CYCLE COST IS MAINTENANCE COST                                *
*                                                                              *
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MAINTLFCY OVER
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Fig. 1. Sample run

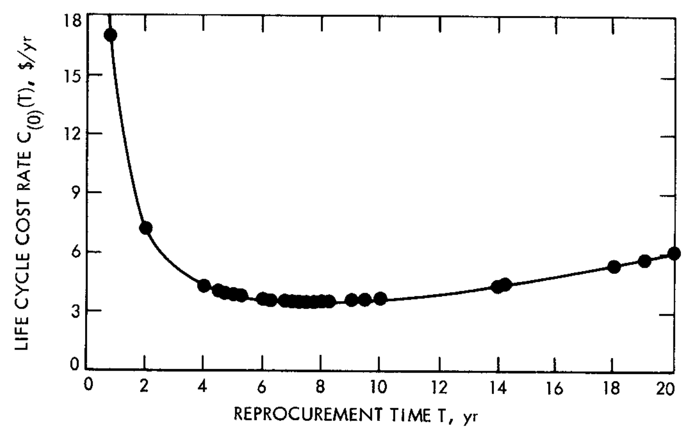

Fig. 2. Life-Cycle Cost Rate vs cycle length

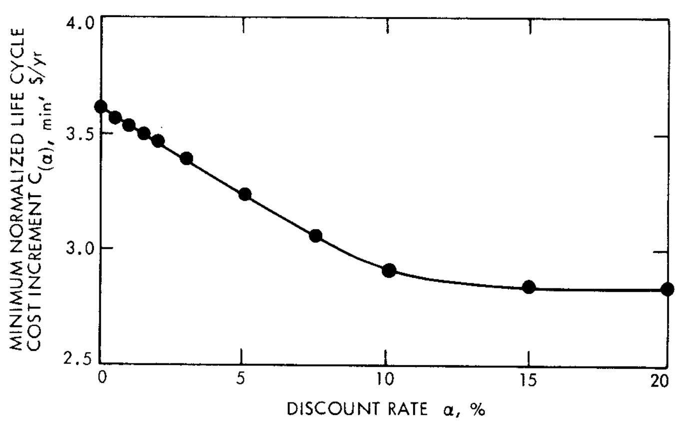

Fig. 3. $C_{(\alpha),min}$ vs α