Power Spectrum for Binary NRZ Data With Less Than 50% Transitions

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Spacecraft-to-ground telemetry received by the DSN can often be modelled as binary NRZ data with independent transitions of probability p $\leq$ ½. *In this paper, a simple expression is derived for the power spectrum of this type of data modulation; this formula is used to investigate how rapidly the data bandwidth decreases as* p *gets smaller.*

I. Introduction

The performance of a coherent communication link will be degraded by data modulation that falls within the carrier tracking loop bandwidth. Satellite telemetry links must operate under imposed bandwidth constraints. Crosstalk can occur when two adjacent data channels overlap in the frequency domain. With these considerations in mind, a space telecommunications designer is interested in determining the power spectra of his transmitted data.

For many deep space telemetry links supported by the DSN, the data modulation is in a binary NRZ format, and the data source suggests an independent transition model. When the transition rate is ½, it is well known (e.g., Ref. 1) that the data power spectrum is given by

$$S_d(f) = \frac{1}{R}\left[\frac{\sin(\pi f/R)}{\pi f/R}\right]^2 \tag{1}$$

where R is the data rate. In this paper, a simple expression is derived for the data power spectrum when the transition rate is less than ½; this formula is then used to measure the data bandwidth dependence on the transition rate.

II. Derivation of Data Power Spectrum

Consider the unit-amplitude, binary NRZ data stream $d(t)$, shown in Fig. 1. The data is assumed to be generated by a Markov source: data transitions occur independently, with stationary probability p. That is,

$$Pr[d_i = -d_{i-1}] = p, \forall_i. \tag{2}$$

As a preliminary step to computing the power spectrum of $d(t)$, we can calculate its autocorrelation function

$$R_d(\tau) \equiv \overline{d(t)\,d(t+\tau)} \tag{3}$$

It is easy to show that for $k/R \leq |\tau| \leq (k+1)/R$, $k = 0,1,2,\ldots$

$$R_d(\tau) = (1-2p)^k\,[(1+2kp) - 2p|\tau|R] \tag{4}$$

which yields the piecewise-linear function of Fig. 2. For example, using Fig. 3 for $1/R \leq |\tau| \leq 2/R$, it is evident that

$$\begin{aligned} R_d(\tau) &= \left(\frac{\frac{2}{R} - |\tau|}{\frac{1}{R}}\right) \overset{(1-p)-\mathrm{p}}{\overline{d_i d_{i+1}}} \\ &\quad + \left(\frac{|\tau| - \frac{1}{R}}{\frac{1}{R}}\right) \overset{(1-p)^2+p^2-2p(1-p)}{\overline{d_i d_{i+2}}} \\ &= (1-2p)\,[(1+2p) - 2p|\tau|R] \end{aligned} \tag{5}$$

The data power spectrum $S_d(f)$ is simply the Fourier transform of $R_d(\tau)$. Using the construction diagram of Fig. 4, we can write

$$\begin{gathered} R_d(\tau) = \sum_{k=1}^{\infty} R_k(\tau) \\ \Updownarrow \\ S_d(f) = \sum_{k=1}^{\infty} S_k(f) \end{gathered} \tag{6}$$

The parameters a_k and $b_k \equiv a_{k-1} - a_k$ in Fig. 4 are specified by

$$\begin{aligned} a_k &= (k+1)\,q^k - kq^{k+1} \\ b_k &= kq^{k-1}\,(1-q)^2 \end{aligned} \tag{7}$$

where

$$q \equiv 1 - 2p.$$

But $R_k(\tau)$ is the autocorrelation function of binary NRZ data with 50% transitions and data rate R/k; modifying Eq. (1), we have

$$\begin{aligned} S_k(f) &= \frac{Rb_k}{k}\left[\frac{\sin(k\pi f/R)}{\pi f}\right]^2 \\ &= \frac{Rq^k(1-q)^2\,[1-\cos(2k\pi f/R)]}{2(\pi f)^2} \end{aligned} \tag{8}$$

$$\therefore S_d(f) = \frac{R(1-q)^2}{2q(\pi f)^2}\left[\overset{\frac{q}{1-q}}{\sum_{k=1}^{\infty} q^k} - \overset{\frac{q(\cos 2\pi f/R - q)}{1 - 2q\cos 2\pi f/R + q^2}}{\sum_{k=1}^{\infty} q^k \cos(2k\pi f/R)}\right] \tag{9}$$

where the last summation is found in Ref. 2. Simplifying Eq. (9) yields the power spectrum formula[1]

$$\boxed{\begin{aligned} S_d(f) &= \frac{1}{R}\left[\frac{\sin(\pi f/R)}{\pi f/R}\right]^2 \\ &\quad \times \left[\frac{1-(1-2p)^2}{1 - 2\,(1-2p)\cos(2\pi f/R) + (1-2p)^2}\right] \end{aligned}} \tag{10}$$

III. Results

Normalized data power spectra are plotted in Fig. 5 for several transition rates $p \leqslant 1/2$, using Eq. (10). As expected, the bandwidth narrows as p decreases. The functional dependence of the one-sided bandwidth BW on p is illustrated in Fig. 6 for cases where 90% and 95% of the total data power lies in the frequency domain $(-BW, BW)$.

[1]Equation (10) agrees with a prior unpublished result obtained by M. K. Simon of JPL Section 339 following an entirely different approach based on Chapter 1 of Ref. 1.

References

1. Lindsey, W. C., and Simon, M. K., *Telecommunication Systems Engineering*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1973, Eq. 1-21.

2. Gradshteyn, I. S., and Ryzhik, I. M., *Table of Integrals, Series, and Products.* Academic Press, New York, New York, 1965, p. 40.

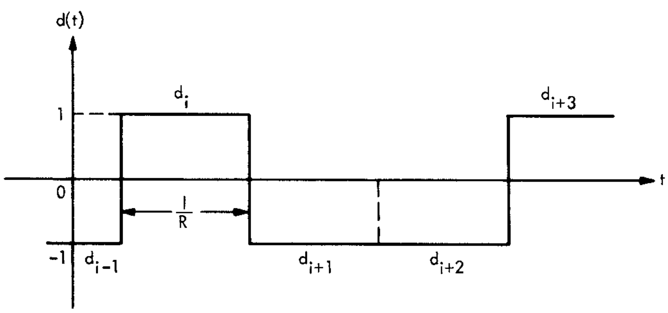

Fig. 1. Unit-amplitude, binary NRZ data

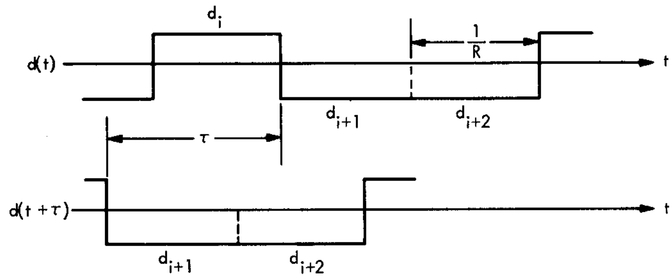

Fig. 3. Comparison of $d(t)$ and $d(t+\tau)$ for $1/R \le \tau \le 2/R$

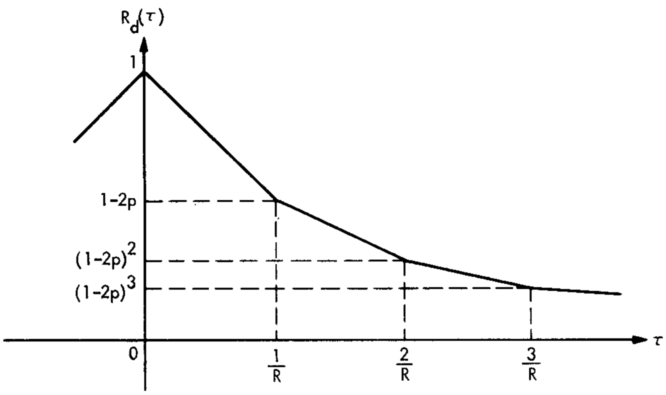

Fig. 2. Autocorrelation function of $d(t)$, with transition probability p

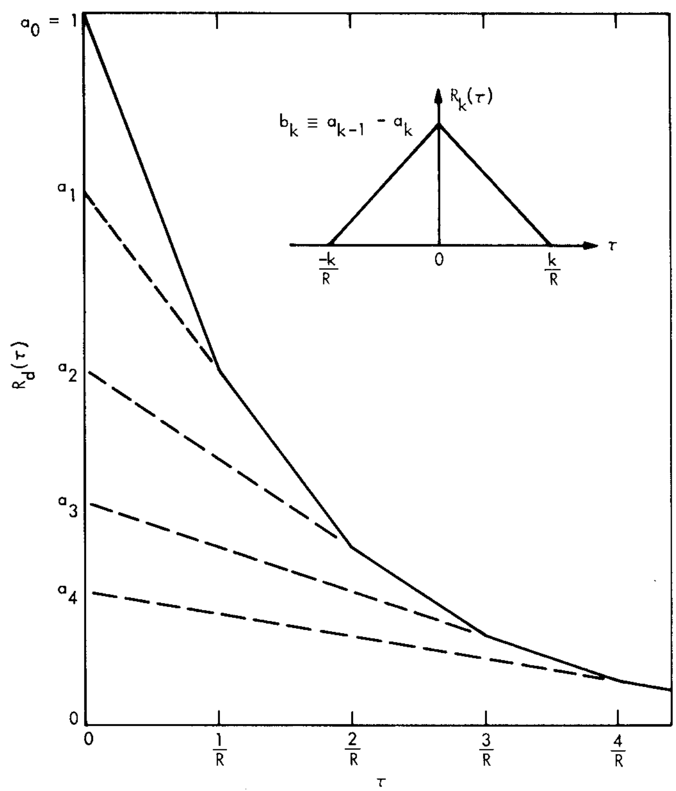

Fig. 4. Decomposition of $R_d(\tau)$ into $\sum_{k=1}^{\infty} R_k(\tau)$

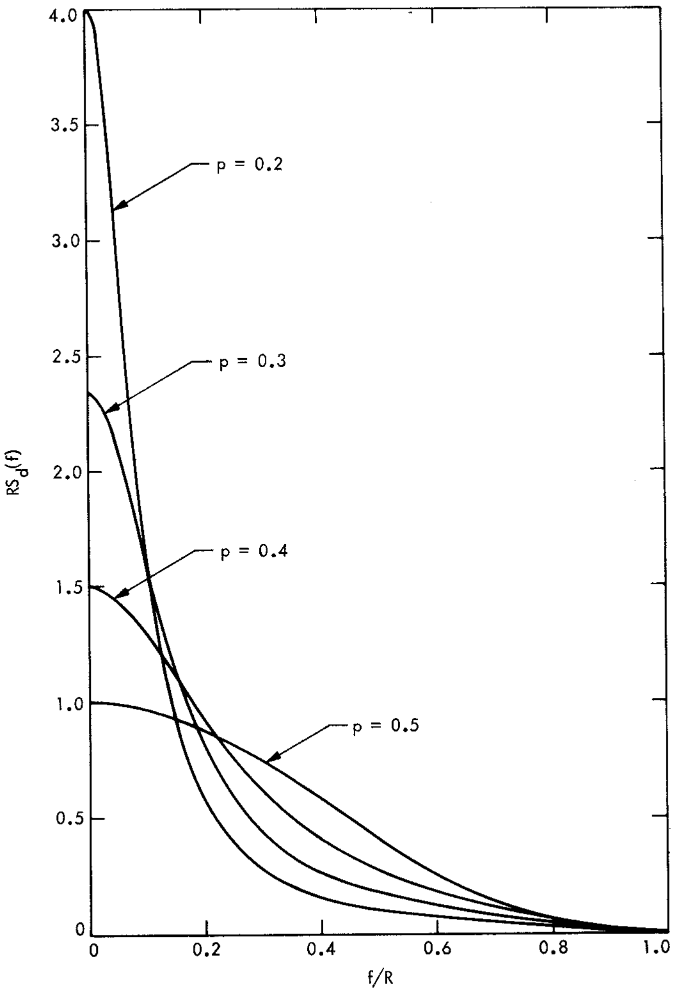

Fig. 5. Normalized power spectra of $d(t)$ vs transition probability $p \leq 1/2$

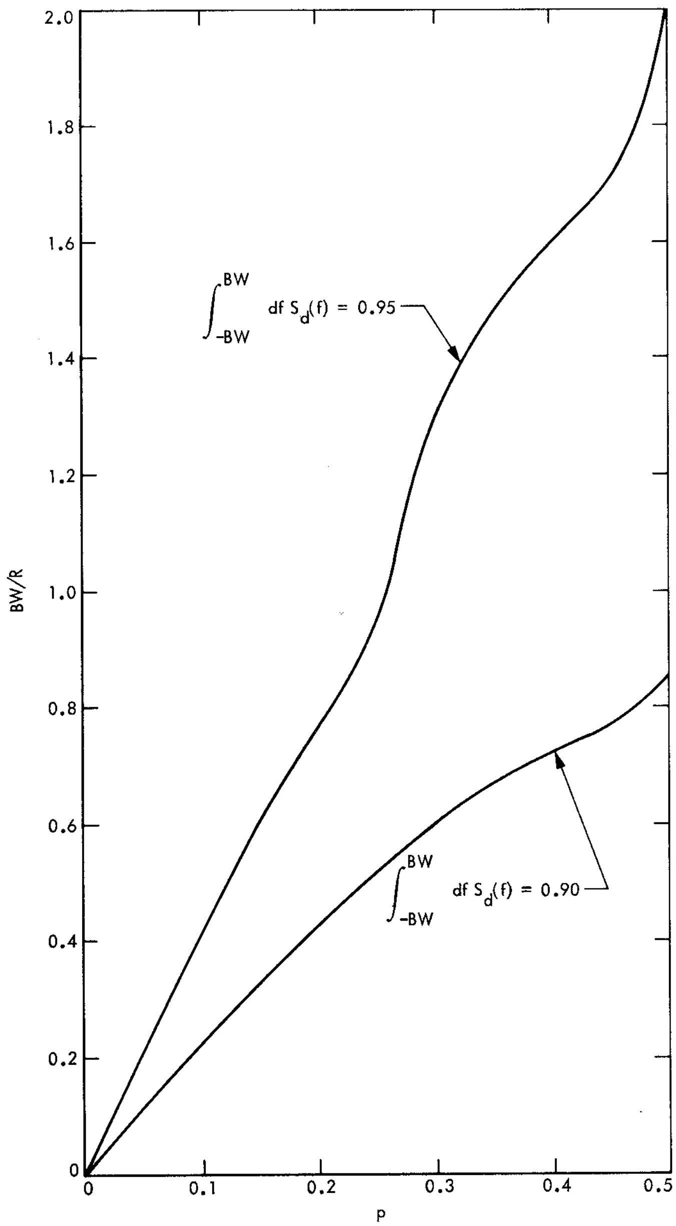

Fig. 6. Functional behavior of one-sided bandwidth BW of $d(t)$ vs transition probability p, for 90% and 95% of total data power contained in $(-BW, BW)$