DSN System Performance Test Doppler Noise Models; Noncoherent Configuration

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Recent DSN System Performance Test requirements (for pending missions) contain implied long-term measurements for which existing noncoherent test models are inadequate. The new models needed are those of the first two moments, mean and variance, of doppler noise under long-term conditions, and in noncoherent test mode. In this paper, the newer model for variance, the Allan technique, now adopted for testing, is analyzed in the subject mode. A model is generated (including considerable contribution from the station secondary frequency standard), and rationalized with existing data.

A mean-frequency model is subsequently proposed, based on data available, but this model is not considered rigorous. It is an introductory idea, to be evaluated and incorporated if proved valid. It uses a fractional-calculus integral to obtain a quasi-stable result, as observed, with integrable spectrum poles.

The variance model is definitely sound; the Allan technique mates theory and measure. The mean-frequency model is an estimate; this problem is yet to be rigorously resolved. The unaltered defining expressions are noncovergent, and the observed mean is quite erratic.

I. Introduction

DSN site doppler noise measurements, in noncoherent mode, do not fit existing models. New models for variance and mean-frequency are outlined in this paper.

Initially, the doppler counter and data processing are explained, and their expression as a frequency transfer function for Allan variance developed. Next, spectral densities are discussed, and the transfer function above combined with these to yield an Allan variance model. The model parameters are iterated with (DSS 11) data to define the variable model parameters, and a specific result for the site is plotted.

The mean-frequency function is then investigated (nonconvergent-integral), and a fractional filter proposed. Based on this filter, and an unknown parameter, an expression for the mean frequency is presented.

II. Doppler Counter Operation and Phase Data Reduction: Mean and Variance

The doppler counter input is a replica of the receiver carrier, phase-differenced with the exciter carrier (appropriately frequency translated), with mixing such that this input centers on a nominal 1.0 or 5.0 MHz bias frequency. All system phase variation, deterministic or random, appears as a composite time variation of phase, transformable to a phase-frequency spectrum, on the bias carrier. Phase variation (degrees, cycles, or radians) is referenced to the nominal phase of the bias frequency.

The counter ignores the amplitude of the input waveform. It simply registers, or counts, positive-going zero-crossings (cycles) of the input, and, upon receipt of a release data time pulse, resolves the (final) count to 10 nanoseconds, or 0.01 cycle at 1.0 MHz (3.6 deg), then releases data.

The time between release data pulses is the preset measurement period, T. The counter thus *integrates* the phase-frequency *continually*, releasing the integral to date at each N^{th} time-series increment NT. It does not automatically reset; the count is cumulative. The output count is thus a very large number, containing all cycles counted since last (arbitrary) manual reset. The format is a series of decimal digits:

$$\underbrace{XXXXXX}_{\text{time}} \qquad \underbrace{XXXXXXXXXX}_{\text{accumulated cycle count}} \cdot \underbrace{XXX}_{\text{resolver count}} = \Phi_N \text{ (readout)}$$

resolver count [1×10^{-9} s of bias cycle, resolution]

expressed as waveform functions (bias integral linear):

$$A(t) = \text{input waveform} = a \sin \left[\Phi_0 + Bt + \int_0^t \dot{\varphi}(t) dt \right]$$

$$\Phi_N = \Phi_0 + BNT + \int_0^{NT} \dot{\varphi}(t) \, \mathrm{dt} \text{ (resolver incorporated)}$$

The desired data consists of the difference between successive values of the trailing integral. This is the deviation from nominal; the integral of the phase frequency across a measurement period. This data time series is:

$$\begin{aligned} \varphi_N(NT) &= \Phi_N - \Phi_{N-1} - BT \text{ (by data reduction)} \\ &= \int_{(N-1)T}^{NT} \dot{\varphi}(t) \, \mathrm{dt} \end{aligned} \qquad (1)$$

The set $\{\varphi_N\}$ is processed to yield mean and variance of the data over a block of M measures [block period $\tau = MT$]:

$$\left.\begin{aligned} \hat{\varphi} &= \frac{1}{M} \sum_{N=1}^{M} (\varphi_N) \\ \hat{\sigma}^2(\varphi) &= \frac{1}{M-1} \left[\sum_{N=1}^{M} (\varphi_N)^2 - M(\hat{\varphi})^2 \right] \\ &\underline{\text{Also, for nonstationary mean:}} \\ \hat{F\phi} &= (\hat{\varphi})/\tau), \text{ Hz}, = (\Phi_M - \Phi_1 - MBT)/\tau \\ &\text{where } F\phi \text{ is the system phase-frequency offset.} \end{aligned}\right\} \qquad (2)$$

III. Allan Variance Data Processing

The Allan variance technique, newly adopted for DSN testing, is a variation of Eq. (2) where the mean, τ, is set to only $2T'$, and collective variance of sequential data pairs is averaged. This technique leads to expression of non-stationary data expectation by stationary spectral models; the variance measure is, in general, stabilized as a rational function of T. Variance becomes:

$$\begin{aligned} \hat{\sigma}^2_{AV}(\varphi) &= \frac{1}{M} \sum_{N=1}^{M} \left[\frac{(\varphi_N{}^2 + \varphi_{N-1}^2)}{2} - \left(\frac{\varphi_0 + \varphi_{N-1}}{2} \right)^2 \right] \\ &= \frac{1}{M} \sum_{N=1}^{M} (\varphi_N - \varphi_{N-1})^2 \end{aligned} \qquad (3)$$

Eq. (3), therefore, is the quantity to be predicted by the spectral model; further variance discussion will be limited to this technique alone.

IV. Allan Variance General Model and Filter Functions

Expected value of measures obtained from Eq. (3) can best be modeled as some combination of noise spectral densities of various orders (order in this paper is the order, or exponent, of the (predominant) denominator term of the density), as filtered by the counter-data-reduction process. To start, assume the variance contribution of an arbitrary noise source, as filtered:

$$E(\sigma_N^2) = K_N \int_{a_N}^{b_N} |H_N(j\omega)|^2 \cdot |F(j\omega)|^2 \, d\omega \quad (4)$$

where

N = Nth noise source index

K_N = noise constant

$H_N(j\omega)$ = spectral density function

$F(j\omega)$ = counter/data process frequency transfer function (Allan variance)

a_N, b_N = arbitrary integral frequency limits, or/cutoffs.

The filter function is invariant with noise spectrum, depending only on sample period T. We derive it from its Lapláce time transform for a single Allan measurement:

$$F(S) = \underbrace{\left[1 - e^{-ST}\right]}_{\text{sample}} - \frac{1}{2}\underbrace{\left[1 - e^{-2ST}\right]}_{\text{mean}} \quad (5)$$

This simply states that each measurement of the waveform is differenced with its value T earlier, with mean over $2T$ subtracted; a parallel to elements of Eq. (3). Eq. (5), by manipulation, leads to the desired transfer function:

$$|F(j\omega)|^2 = \frac{1}{4}[1 + \cos(2\omega T) - 2\cos(\omega T)]^2$$

$$+ \frac{1}{4}[\sin(2\omega T) - 2\sin(\omega T)]^2$$

$$= \left\{[\cos^2(\omega T)] + [\sin^2(\omega T)]\right\}[1 - \cos(\omega T)]^2$$

$$= [1 - \cos(\omega T)]^2 \text{ (Allan variance only)} \quad (6)$$

Expression (6) is remarkable in that its leading term is of fourth power, assuring convergence of integrals with spectral densities up to and including that order. Equation (6) thus enters Eq. (4) as the defined transfer characteristic, leaving the spectral densities and constants to be determined.

V. Noise Spectral Densities and Sources

When doppler noise is measured under strong-signal coherent conditions, variance is small, stationary, and consists only of system residual sources, lumped as φ_0^2. The standard deviation of this residual is only a few degrees (3 deg typical), and is assumed present in all test configurations. It is background noise, undoubtedly the rms combination of a number of minor sources. It is carried here as a single parameter, to be determined during model-data fit iteration. It is flat noise, order zero.

When the test configuration is altered to the noncoherent strong signal mode, very large nonstationary elements appear in the data. The predominant element is a strong quasi-stable mean. This mean, divided by the sampling period, represents a nearly stationary phase-frequency (Fϕ, Hz), or slope that often holds within ±5 percent for several hours, then suddenly changes state by ratios as much as 50:1. The mean behavior is analyzed later in this discussion. The Allan variance technique largely avoids these mean effects, as they influence variance; the mean is short-term.

The presence of the long-term mean does, however, indicate a dominant third-order spectrum, often called 1/F noise (referred to integrator inputs). However, during short measurement periods, the variance is approximately linear, indicating second-order source or sources. By reference to specifications, it was determined that the prime latter-source was the *secondary standard;* it literally swamped the variance for T of ten seconds or less, while giving third-order indications at longer periods.

To complete the set, assume some second-order contribution for system sources other than the standard. Before forming the model using these spectra, an analysis of spectral continuity of frequency standards in general is pertinent. The orders for this and the system are shown in Table 1.

VI. Frequency Standard Noise Models

The phase-noise contribution of frequency standards in the system is well understood to be large during short measurement periods, settling finally into a third-order spectral mode during longer periods.

Published frequency variance data indicate some combination of second- and third-order phase-power spectra. The question is whether both spectra are continuous and overlapping, with saturated integrals, or whether the composite spectral density crosses over from third order to second order form at a particular defined frequency; the second order integral subsequently continuous to saturation. When the latter is assumed, it is often replaced by a break-point time approximation. The frequency variance is considered log-linear with time before the break-point, and subsequently constant. The break-point measurement time varies from 30 seconds to 300 seconds with common standards of various quality.

The three models are outlined in Table 2, and results for the present DSN primary standard are shown in Fig. 1.

Model 2 (see Table 2) was obviously chosen, for it rides below the maximums, but is otherwise very close to the published break-point data. It appears that frequency standard noise is best modeled as sequential, rather than continuous, power spectral density orders.

During non-coherent test mode, results of Fig. 1 cannot be applied directly, for the site secondary standard is in use. The standard exhibits frequency variance that is much degraded over that of the primary standard. Also, definitive specifications are not available.

However, Model 2 was still assumed to apply, but with different cross-over frequency (F_c) and amplitude constant. These were given bounded ranges by degrading the primary standard data by one-half to one order of magnitude. Within these ranges, the parameters were iterated with the other system model estimates and data, as described later in the discussion.

VII. Final Allan Variance Model and Parameters

Since Model 2 of Table 2 was chosen for the standard, the same form (spectral cross-over) was finally chosen for the rest of the system. The choice was hardly trivial; the other models did not fit the data. The two models differed only in the selection of the amplitude constants and crossover frequencies. The final total model is (the spectral model of Eq. (3)):

$$
\begin{aligned}
E\{\sigma^0_{AV}\}^2 = {} & (\varphi_0)^2 \quad \text{(residual)} \\
& \left.\begin{aligned}
& + K_0 T^2 \int_0^{\omega_0 T} \frac{[1-\cos(x)]^2}{X^3}\,dx \\
& + \frac{K_0 T}{\omega_0} \int_{\omega_0 T}^{\infty} \frac{[1-\cos(x)]^2}{X^2}\,dx
\end{aligned}\right\} \text{system sources} \\
& \left.\begin{aligned}
& + K_s T^2 \int_0^{\omega_c T} \frac{[1-\cos x]^2}{X^3} \\
& + \frac{K_s T}{\omega_c} \int_0^{\omega_c T} \frac{[1-\cos x]^2}{X^2}\,dx
\end{aligned}\right\} \text{secondary standard}
\end{aligned}
\qquad (7)
$$

K_0, K_s = noise amplitude constants, $(\text{deg/sec})^2 \cdot (\text{rad})^2$

ω_0, ω_c = crossover frequencies, (rad/sec)

T = measurement time, sec

The four main constants (K_0, ω_0, K_s, ω_c) and φ_0 interact, and no solution would be possible unless their ranges could be bounded. Extrapolation of known frequency standard data to apparent secondary standard degradation gave such bounds to K_s and ω_c. It was obvious from data behavior that ω_0 was much greater than ω_c; system $1/F$ noise predominated over any possible standard contribution at long measurement intervals.

Four prime data points were used for iteration (to minimum rms error, normalized, model vs data), but several lower-confidence points were used as checks (all within ± 10%). The prime data points were (from DSS 11) as listed in Table 3.

Iteration was a manually aided machine variation of parameters technique, using Eq. (7) where, for each combination increment the sigma error of each point, and normalized rms error of all points, was calculated.[1] The points are those for a single site (DSS 11); other parameters would apply elsewhere.

The iteration data were as listed on Table 4.

Final fit of Eq. (7) to data, using Table 4 values, is shown in Fig. 2. The figure displays the long-term stationary parameter

[1] For use with Eq. (7), series solutions of the integrals were programmed.

σ/T, rather than σ directly. To obtain σ in degrees, the ordinate must be multiplied by T.

A program is planned to incorporate Eq. (7) and the iteration process into a station performance test algorithm. Finally, a discussion of $F\phi$ behavior and its implications follow.

VIII. Noncoherent Doppler Mean and Mean Frequency

The doppler phase noise mean value is simply the difference between the last and first total phase measure over a (long) block period, τ. The mean frequency, $F\phi$, is this measure divided by τ, normally expressed in Hz. The counter/data process filter transfer function for this quantity contains a sine-squared term:

$$F_M(s) = 1 - e^{-s\tau}$$

$$|F_M(j\omega)|^2 = [1 - \cos(\omega\tau)]^2 + [\sin(\omega\tau)]^2 \qquad (8)$$

$$= 4\sin^2\left(\frac{\omega\tau}{2}\right)$$

Concentrating on $F\phi$ (since the block mean is not used for Allan variance), the expressions consist of substituting Eq. (8) in Eq. (7) and dividing by τ.

This total model becomes (less φ, which is mean-zero), and expressed in Hz^2:

$$\left.\begin{aligned} E\left(\frac{MN}{\tau}\right)^2 &= E(F_0)^2 \\ &= \frac{K_0}{(360)^2}\int_0^{(\omega_0\tau/2)} \frac{\sin^2(x)}{X^3}\,dx \\ &\quad + \frac{K_0}{\omega_0\tau(360)^2}\int_{(\omega_0\tau)/2}^{\infty} \frac{\sin^2(x)}{X^2}\,dx \\ &\quad + \frac{K_s}{(360)^2}\int_0^{(\omega_c\tau/2)} \frac{\sin^2(x)}{X^3}\,dx \\ &\quad + \frac{K_s}{\omega_c\tau(360)^2}\int_{(\omega_c\tau/2)}^{\infty} \frac{\sin^2(x)}{X^2}\,dx \end{aligned}\right\} \qquad (9)$$

Now, when

$$1/\tau \ll \omega_0, \omega_c \quad [\tau > 100 \text{ sec}]$$

The x^2 (order 2) integrals become insignificant. Also, since $\omega_c \ll \omega_0$, in any definitive solution of Eq. (9), the initial integral will predominate. We can thus say, with reasonable certainty that – *$F\phi$ is a third-order system effect, essentially independent of all dynamic frequency standard spectral noise.*

This does *not* mean that the standards are free of contribution; $F\phi$ is a dc frequency offset; a phase accumulation. It simply means that dynamic (short-term-time-variant) frequency-standard contribution is ruled out: $F\phi$ and its variations are very-long-term processes, largely unspecified. In a mission sense, $F\phi$ is the station doppler (velocity) error, a contribution, during tracking, to the overall spacecraft velocity error.

Admitting to both third-order system effects, by model, and unmodeled standard frequency offsets (a third-order contribution), we restate Eq. (9), in Hz^2, as:

$$E(F\phi)_N^2\Big|_\tau = C_N \int_0^{(\omega_F\tau)} \frac{\sin^2(x)}{X^3}\,dx \qquad (10)$$

where

N = index of N^{th} value of $F\phi$

C_N = prevailing offset magnitude constant during measure N, (Hz)^2 · (rad)^2

ω_F = to coin a term, "the $F\phi$ noise bandwidth," or average frequency beyond which mean and $F\phi$ contributions are insignificant. If you placed a low-pass filter at ω_F, $F\phi$ would not be significantly affected.

Expression (10) has a magnificent drawback: it does not converge at the origin. It simply says "$F\phi$ is infinite" [-log (0)], a straight vertical phase slope, forever. This does not synchronize with reality.

The problem of this model non-convergence vs real measures has been investigated in many ways. The major conclusion has been that Eq. (10) does *not* represent any real spectrum when ω is very very small, and measures show that statistical generalizations no longer apply.

One approach, in particular, is to insert a sharp high-pass filter within Eq. (10). It attenuates the ambiguous region and converges the integral. Such a filter, generalized, has been most lately expressed in detail by Greenhall (Ref. 1) in connection with VCO spectral density estimates, where linear trends have been subtracted.

Assuming some-such general high-pass filter (steep enough to have no significant effect on the variance measure), we restate Eq. (10) as

$$\left. \begin{aligned} & E\,(F\phi)^2_N \Big|_\tau = C_N \int_0^{\omega_F \tau} \frac{\sin^2(x)}{X^3} \cdot G(x)\,dx \\ & G(x) = \text{a high-pass filter, plus added form to express observed data.} \\ & \text{for no effect on variance } G(x) = 1 \begin{Bmatrix} x > \Delta x \\ \Delta x \ll 1 \end{Bmatrix} \end{aligned} \right\} \quad (11)$$

The applicable form of $G(x)$ was approached here by a survey of DSN site $F\phi$ behavior, as gathered from noncoherent doppler test data.

$F\phi$ behaves in a peculiar manner (one look at Eq. (11) would suggest this). It maintains a fixed value for hours, then suddenly jumps to a new value, entirely different. The time integral, or phase waveform, is a series of constant slopes with sharp break points to new linear values. Such a break is plotted in Fig. 3, detected during a four-hour test at DSS 11.

Data are too scarce to estimate accurately any periodic sequence to this behavior. However, $F\phi$ appears to be absolutely statistically bounded; its various states seem to fall within probabalistic regions, with (at least close to) a mean of zero. The DSS 11 data showed a standard deviation of about 0.07 Hz. (6 samples; mean = 0.02 Hz; 4 month scattered data samples).

Since a given $F\phi$ value persists for such a long period, it appears quasi-stable, and any filter leading to its model must contain, upon integration, *a quasi-stable constant* component, applicable over (long) discrete time intervals.

The only known integrands that yield direct constants are those of fractional form. We thus assume that

Assumption 1: "$G(x)$, near the origin, results in an integral of fractional form, yielding a constant quasi-stable integration order."

This suggests that some system function fractionally integrates the low-frequency end of the spectrum. This source, if it exists, is indeterminate.

Accepting $F\phi$ as a mean-zero convergent random variable, its time waveform has, inherently, a finite Fourier transform, but resolveable over only over very long time periods (months). If mean-zero, the time-transform would, by assumption, show *no dc component.* Thus:

Assumption 2: "$G(x)$ must attenuate the pole at the origin (contain a high-pass filter) to represent the data. Data indicate an integrable pole at zero."

The third assumption, a virtual corallary to the first, is that the spectrum must contain a pole at all times, other than (or in place of) that attenuated at zero. The fractional integral of assumption 1 requires a definitive pole. Let the pole symbol be ω_p. The pole ω_p *need not be constant.* It is sufficient that its time period be that of the differential $F\phi$ break points. Thus, if a Fourier transform were taken over this interval, the ω_p component would far surpass all others in amplitude; that is, it would be the "pole of the measure." This is certainly sufficient for a daily tracking period in site operational use. The assumption is therefore:

Assumption 3: "$G(x)$ must contain a quasi-stable finite pole, defining $F\phi$ (daily) transient excursions as (daily) pole-frequency period."

Noting the above, the form chosen for $G(x)$, meeting all three assumptions, was unusually simple:[2]

$$G\,(x) = \left(\frac{x}{|(\omega_p \tau) \cdot x|} \right)^{\Delta} \qquad (12)$$

$$\Delta = \text{order of fractional integration}$$

Inserting this in Eq. (11), and separating integrals at the pole:

$$E\,(F\phi)^2_N \Big|_\tau = C_N \int_0^{\omega_{\rho_N} \tau} \left[\frac{\sin^2(x)}{x^2} \right] \cdot \frac{x^{\Delta - 1}}{(\omega_{\rho_N}\tau - x)^{\Delta}}\,dx$$

$$+ C_N \int_{\omega_{\rho_N}\tau}^{\omega_F \tau} \left[\frac{\sin^2(x)}{x^2} \right] \cdot \frac{x^{\Delta - 1}}{(x - \omega_{\rho_N}\tau)^{\Delta}}\,dx \qquad (13)$$

[2] The form is not rigorously established; measured data and source characteristics at these extremes are too scarce. The form is simply one (of possibly many) that meets the assumptions, as based on available data.

To obtain an upper bound (disposing of considerable numerical complication), let:

$$\frac{\sin^2(x)}{x^2} = 1$$

This gives (τ cancels out) the bound (with parameter interchange):

$$E\,\{(F\phi)^2\}_N < C_N\,\Gamma(\Delta) \int_0^{\omega_{P_N}} \frac{X^{\Delta}}{\Gamma(\Delta)\,[\omega_{P_N} - X]^{1-\Delta}}\,dX + C_N \int_{\omega_{P_N}}^{\omega_F} \frac{X^{\Delta-1}}{\left(X - \omega_{P_N}\right)^{\Delta}}\,dX \tag{14}$$

The first integral is of the fractional calculus form,[3] leading to a constant, as transiently observed. The second leads to a log ratio and trailing series, the latter insignificant if Δ is small. The slightest Δ converges the integral (solution of Eq. (10)):

$$\left.\begin{aligned} \frac{E\,\{(F\phi)^2\}_N}{C_N} &< \frac{\Gamma(1+\Delta)\,\Gamma(1-\Delta)}{\Delta} + \log\left[\frac{\omega_F}{\omega_{P_N}}\right] \\ &\quad + \underbrace{\Delta \sum_{K=1}^{\infty} \frac{\Gamma(K+\Delta)\left[1 - \left(\dfrac{\omega_F}{\omega_{P_N}}\right)^K\right]}{\Gamma(1+\Delta)\,(K)! \cdot K}}_{\text{normally insignificant, } \Delta \ll 1} \end{aligned}\right\} \tag{15}$$

Expression (15)[4] does not admit to small values of $F\phi$, occasionally observed (Fig. 3, early data), if Δ small, as required. It is but a maximum rms amplitude constant: some other snap-action parameter is determining it is quasi-period level. The parameter seems, at this point, to have three common states: small, large +, and large −. The resulting $F\phi$ model, with this unknown parameter, is therefore:

$$E\,\{F\phi\}_N \approx \left\{\sqrt{C_N}\,\sqrt{\frac{\Gamma(1+\Delta_N) + \Gamma(1-\Delta_N)}{\Delta_N} + \log\left[\frac{\omega_F}{\omega_{P_N}}\right]}\right\} \times \{A_N\}$$

N = index of period ($F\phi$ transition count)

C_N = $F\phi$ variance (site noise constant), during N.

Δ_N = spectrum fractional integration parameter ($\Delta_N \ll 1$, possibly site-constant) for DSN 11 data, $\Delta \approx 0.057$.

ω_F = noise bandwidth of mean power

ω_{P_N} = apparent pole frequency during N.

A_N = unknown step parameter with three apparent states: $(0, +1, -1) \pm 5\%$ (16)

To conclude, the $F\phi$ function is hardly rigorously determined. However Eq. (16) does provide a lead-in to its understanding, and represents a convergent solution to Eq. (10). For a more definitive model, extensive data would be required.

IX. Summary and Conclusions

(1) The Allan variance technique leads to predictable expected doppler noise values. The values depend upon DSN site-dependent parameters, and these numbers must be determined, for each location, in noncoherent mode. An appropriate algorithm is in process.

(2) The present DSN site secondary-frequency-standards severely contaminate – even mask – the true system noise, particularly during short measurement periods. The desired system measure is severely degraded by this source.

(3) The doppler mean-offset-frequency is quasi-stable, but erratic on a long-term scale. Available data are insufficient for a thorough analysis, and all models are non-convergent unless an arbitrary high-pass filter is

[3] The expression is a specific expression of the "Riemann-Liouville" fractional calculus integral (about 1850).

[4] Note that the Δ function is independent of ω_{P_N} first-term amplitude is not affected by pole location.

added. This parameter is a maverick, expressing the finite reality of an infinite theoretical prediction. The fractional-integration model herein is but one of many possible descriptions.

(4) The mean-offset frequency above has a large mission significance. It is the velocity error of the spacecraft, as contributed by the DSN sites. It would be expedient to locate and stabilize its sources.

Reference

1. Greenhall, C.A., "Models for Flicken Noise in DSN Oscillators," in *The Deep Space Network*, Technical Report 32-1526, Vol. XIII, pp. 183-193, Feb. 15, 1973.

Table 1. Doppler spectral elements

Order	$\lvert H(j\omega)\rvert^2$	Probable source
0	1	System residual φ_0
2	$1/\omega^2$	Secondary standard and system
3	$1/\omega^3$	

Table 2. Frequency standard models

Ordinate/symbol	Model	Expression/definition
$\sigma^2\left(\frac{\Delta\varphi}{T}\right)$ $\approx \alpha^2\left(\frac{\Delta F}{F}\right)$ Frequency standard variance	1	$K_0 \int_0^\infty \frac{F_c(X)}{X^3}\,dX + \frac{K_1}{T}\int_0^\infty \frac{F_c(X)}{X^2}\,dX$ $= K_0' + \frac{K_1'}{T}$
	2	$K_0 \int_0^{\omega_c T} \frac{F_c(X)}{X^3}\,dX + \frac{K_0}{\omega_c T}\int_{\omega_c T}^{\infty} \frac{F_c(X)}{X^2}\,dX$
	3	$\frac{K_1'}{T}$ if $T < T_c$; K_0' if $T > T_c$ — Break point (T_c) approximation
$F_c'(X)$	1, 2, 3	Sampling power function, Allan variance: $[1 - \cos(X)]^2$
K_0, K_1	1, 2	Third- and second-order noise constants, dimensioned to yield $[\text{deg/sec}]^2$
K_0', K_1'	1, 3	K_0 and K_1 with integral limit values included
ω_c	2	Cross-over frequency = $1/T_c$

Table 3. Prime variance data

Point	T, sec	$\sigma^0_{(\text{Allan})}$
1	1	15.56
2	10	49.34
3	60	219
4	120	504

Table 4. Parameter iteration using DSS II data

Parameter	Initial range	Initial value	Final value
φ_0	$0 < \varphi_0 < 10$	5	2.0
K_0	$10 < K_0 < 25$	20	13.52
ω_0	$0.05 < \omega_0 < 0.5$	0.05	0.1486
K_s	$5 < K_s < 20$	20	11.87
ω_c	$0.01 < \omega_c < 0.05$	0.01	0.0294
	rms error	0.2850	0.0116

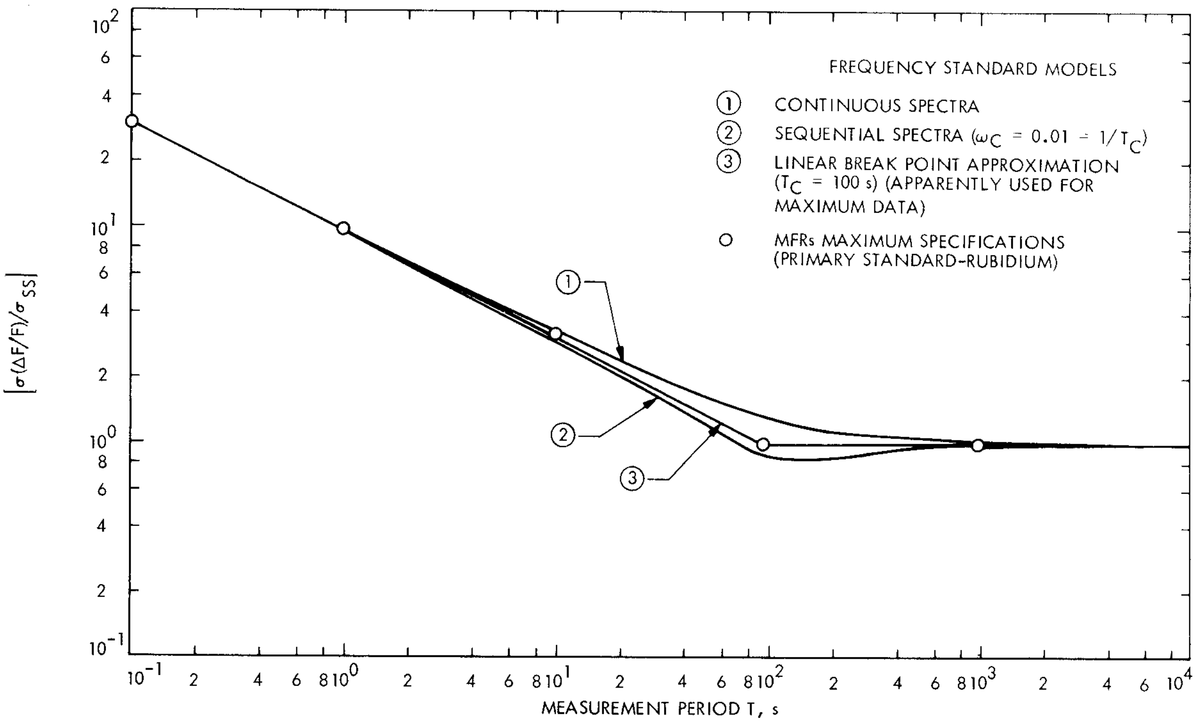

Fig. 1. Comparison of frequency standard noise models

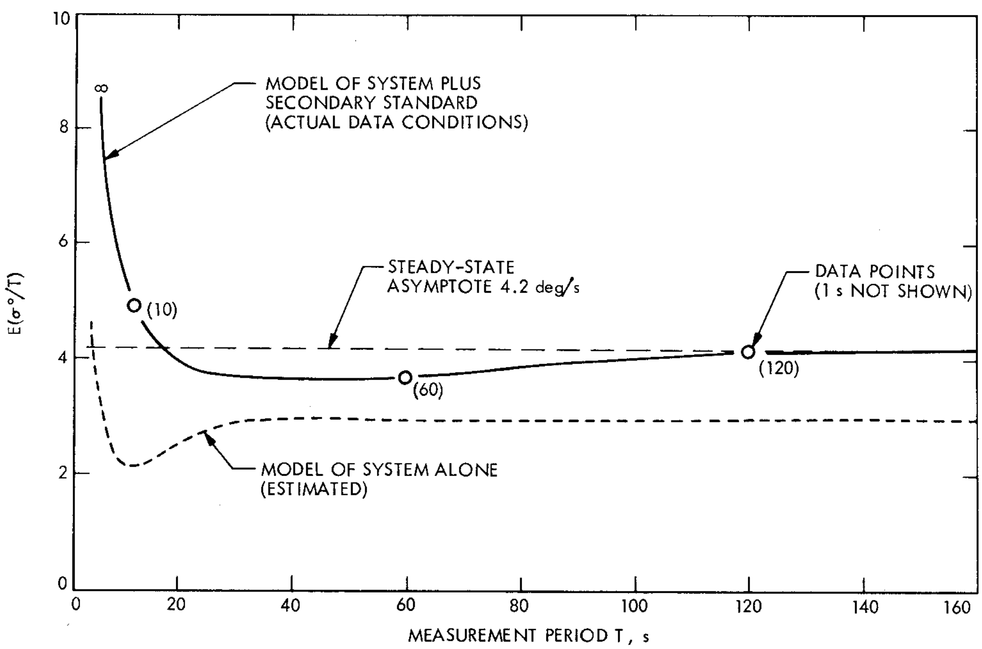

Fig. 2. Allan variance model, DSS 11 data, non-coherent mode

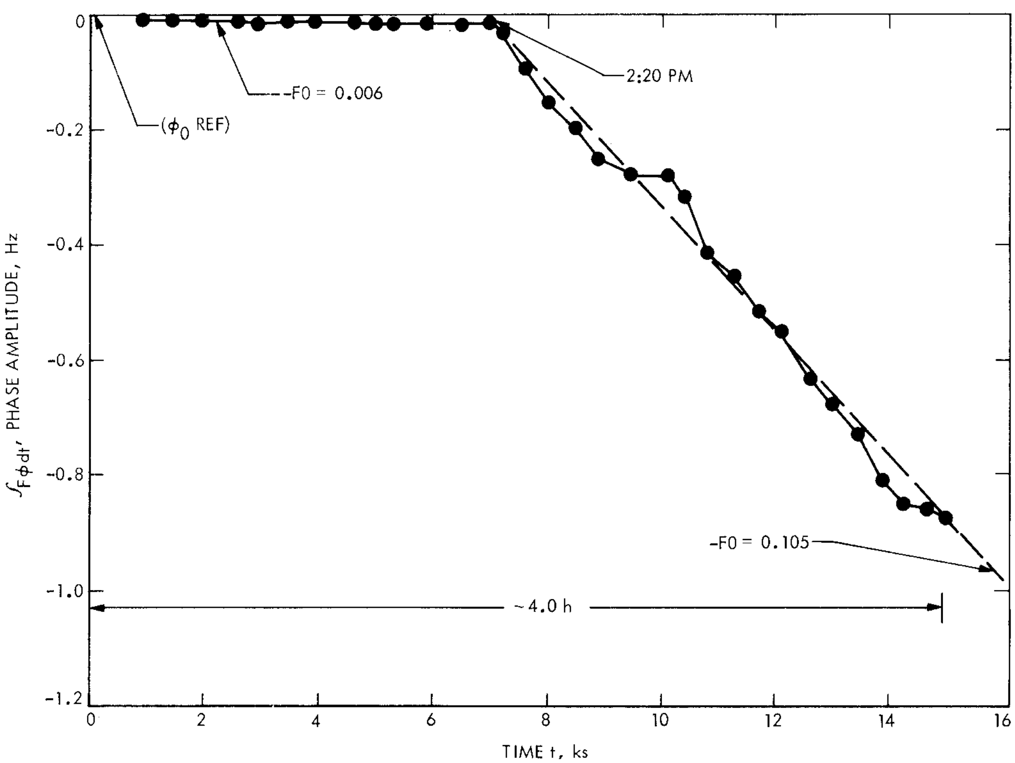

Fig. 3. Phase waveform of DSS 11 $F\phi$ break point, Day 17