Walsh Transforms and Signal Detection

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This paper analyzes the detection of signals using Walsh power spectral estimates. In addition, a generalization of this method of estimation is analyzed. The conclusion is that Walsh transforms are not suitable tools for the detection of weak signals in noise.

I. Introduction

The economy of Walsh transform calculations has led to speculation on the feasibility of using Walsh power spectral estimates instead of Fourier spectral estimates in various estimation and detection problems such as in a signal presence indicator, precision signal power measurement and radio frequency spectrum monitoring for the DSN. At least one paper (Ref. 1) has appeared showing that there is a linear relation between the Walsh power spectrum and Fourier power spectrum when the signal is stationary.

However, in a practical problem, only a finite sample of a signal is available and only *estimates* of the spectra can be obtained. Simple examples can be constructed which show that there is not a one to one correspondence between Walsh and Fourier power spectral estimates obtained from finite signals. Hence, the Walsh power spectrum cannot be used to precisely reconstruct the Fourier power spectrum.

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The idea of using the Walsh spectrum can be salvaged by posing a power estimation problem in the context of stochastic noise theory. When random noise is present in a received signal, Walsh power estimates are random variables which have some statistical structure. By examining this structure, "optimal" estimates of the power of a signal in noise can be obtained using the Walsh power estimates. If the signal is exp (jwt), this this constitutes estimating a Fourier power component.

Another problem is the detection of weak signals in noise when there is some phase instability in the local oscillator. If the signal is sufficiently weak, its detection requires the gathering of data for a time period much greater than the period of stability of the oscillator. In this case a quadratic estimate can be constructed and accumulated over long time periods to provide a reliable estimate of signal power.

It will be seen that Walsh power estimates are not useful for these problems.

II. An Estimation Problem

The following assumptions are made:

(1) The received signal is a complex signal, $X(t)$, consisting of a signal, $\alpha \cdot S(t)$, plus broadband noise, $N(t)$.

(2) The signal is sampled at time intervals, $k \cdot t_0$, to obtain samples

$$\left.\begin{aligned} x(k) &= X(t_0 \cdot k) = \alpha \cdot S(k \cdot t_0) + N(k \cdot t_0) \\ x(k) &= \alpha s(k) + n(k) \end{aligned}\right\} \quad (1)$$

(3) The sequence, $[n(k)]$, is stationary, complex, Gaussian with mean zero,

$$\left.\begin{aligned} E\,[n(k)n(m)] &= 0 \\ E\,[\overline{n(k)}n(k+m)] &= 0, \text{ for } m \neq 0 \\ &= 1, \text{ for } m = 0 \end{aligned}\right\} \quad (2)$$

(4) The signal amplitude, α, is less than 1 and the signal function, $s(k)$, has more or less constant energy of 1 over segments of length N; that is,

$$\sum_{k=1}^{N} |s(k+m)|^2 \sim 1$$

The last normalization is the appropriate one for the Fourier power estimation problem. Also, with this normalization it is easy to characterize the detection problem in terms of α. For α near 1, a matched filter detector on N points is marginal for detection of the signal; for α much less than 1, the accumulation of data over many segments of length N is necessary. It is assumed that phase stability is such that longer matched filters are not possible, and this leads to the need for quadratic estimates. The relation between signal-to-noise ratio (SNR) and α is SNR = α^2/N.

The relevant class of estimation processes is the following: A received signal sequence of length $N \cdot M$ is segmented into M segments of length N. On each segment a unitary transformation (such as Fourier or Walsh) is applied and the absolute value squared of each component is averaged over the M segments. These N numbers are to be used to estimate α^2. The signal detection decision is obtained by comparing this estimate to a threshold.

III. Derivation of an Estimator

Let the unitary transformation have matrix $U = (u_{km})$ and define

$$x_U(m) = \sum_{k=1}^{N} u_{mk} x(k) = \sum_{k=1}^{N} u_{mk} [\alpha s(k) + n(k)]$$

$$= \alpha \cdot s_U(m) + n_U(m)$$

and

$$|x_U(m)|^2 = |n_U(m)|^2 + \alpha^2 \cdot |s_U(m)|^2 + \alpha\,[n_U(m)\overline{s_U(m)} + \overline{n_U(m)}s_U(m)] \quad (3)$$

Unitary transformations preserve all of the statistical properties of the Gaussian noise sequence. It can therefore be shown that the random variables

$$z(m) = |x_U(m)|^2 - \alpha^2\,|s_U(m)|^2 - 1 \quad (4)$$

are mean zero, and pairwise orthogonal. Their variances are

$$\sigma^2(z(m)) = 1 + 2\alpha^2\,|s_U(m)|^2$$

Next, make the calculations of Eq. (4) for the kth interval for k from 1 to M and average over the set of intervals

$$Y(m) = \frac{1}{M}\sum_{k=1}^{M} z^{(k)}(m) = \frac{1}{M}\sum_{k=1}^{M} [|x_U^k(m)|^2 - 1 - \alpha^2\,|s_U^k(m)|^2]$$

$$= \left\langle |x_U(m)|^2 \right\rangle - 1 - \alpha^2 \left\langle |s_U(m)|^2 \right\rangle \quad (5)$$

The $Y(k)$ are mean zero, orthogonal and have variance

$$\sigma^2(Y(m)) = \frac{1}{M}\left[1 + 2\alpha^2 \left\langle |s_U(m)|^2 \right\rangle\right]$$

The case where powerful methods of signal detection are needed is the case in which α is very small and indeed so small that all of the $\alpha^2\,|s_U(m)|^2$ are much smaller than 1. In this case

$$\sigma^2(Y(m)) \sim \frac{1}{M}$$

For large M, the $Y(k)$ are approximately normally distributed, and the maximum likelihood estimate of α^2 is that value of α^2 in Eq. (5) which minimizes

$$E = \frac{1}{2} \sum_{m=1}^{N} \left[\frac{Y^2(m)}{\sigma^2(Y(m))} + \ln\left[\sigma^2\left(Y(m)\right)\right] \right] \qquad (6)$$

With the above approximation of the variance, this is equivalent to minimizing

$$E' = \sum_{m} \left[\left\langle |x_U(m)|^2 \right\rangle - 1 - \alpha^2 \left\langle |s_U(m)|^2 \right\rangle \right]^2 \qquad (7)$$

Let $S_U(m) = \left\langle |s_U(m)|^2 \right\rangle$ and $W(m) = \left\langle |x_U(m)|^2 \right\rangle$. Then the value of α^2 which minimizes E' is

$$\widehat{\alpha}^2 = \frac{\sum_{m=1}^{N} S_U(m)[W(m) - 1]}{\sum_{m=1}^{N} S_U^2(m)} \qquad (8)$$

The numbers $W(m)$ are the accumulated spectral power estimates, and $S(m)$ are signal structure numbers relative to the particular transform.

IV. Properties of the Estimator

Again, assuming M is large so that the $W(\ell)$ can be assumed to be normally distributed, we have expectation and variance of $\widehat{\alpha}^2$:

$$E(\widehat{\alpha}^2) = \alpha^2$$

$$\sigma^2(\widehat{\alpha}^2) = \frac{\sum_{m=1}^{N} S_U^2(m) \frac{1}{M}}{\left[\sum_{m=1}^{N} S_U^2(m)\right]^2} = \frac{1}{M \sum_{m=1}^{N} S_U^2(m)} \qquad (9)$$

If there is no signal present ($\alpha = 0$), the estimator has expected value 0 but has the same variance as Eq. (9).

It can therefore be seen that the ratio α^2 to the standard deviation,

$$R = \alpha^2 \sqrt{M \sum_{m=1}^{N} S_U^2(m)} \qquad (10)$$

is a measure of the effectiveness of the estimator. If R is much less than 1, the estimator is ineffective in signal detection, and if R is much larger than 1, the estimator is a reliable indicator of the presence or absence of the signal. In terms of signal-to-noise ratio, SNR, this can be restated as: If

$$\mathrm{SNR} \ll \left[N \cdot \sqrt{M \cdot \Sigma S_U^2(m)}\right]^{-1}$$

the estimator is ineffective, while if

$$\mathrm{SNR} \gg \left[N \cdot \sqrt{M\, \Sigma S_U^2(m)}\right]^{-1} \qquad (10a)$$

the estimator is effective.

Since

$$\sum_{m=1}^{N} S_U^2(m)$$

plays a key role in the effectiveness of the detection process, it will be called the efficiency factor and designated $E(U, s)$. For a given signal-to-noise ratio, SNR, and given efficiency, $E(U, s)$, the number of segments, M, of data that must be accumulated for reliable detection of the signal, s, satisfies

$$M > 1/(\mathrm{SNR} \cdot N)^2 \cdot E(U, s) \qquad (10b)$$

V. Efficiency of Unitary Transformations

When the unitary transformation, U, is the discrete Fourier transform, the numbers $S_U(m)$ are the averaged power spectrum of the desired signal. In the case where U is the Walsh transform, the $S_U(m)$ will be called the Walsh spectral power, and more generally $S_U(m)$ will be called the spectral power relative to U. Recalling that the signal, s, was normalized to unit energy over N samples, we have

$$\sum_{m=1}^{N} S_U(m) \sim 1$$

From this it can be seen that the maximum of $E(U, s)$ is 1.

This occurs when $S_U(m_0) = 1$ for some m_0 and $S_U(m) = 0$ for $m \neq m_0$. In this case, Eq. (10b) gives

$$M > 1/(\mathrm{SNR} \cdot N)^2$$

for reliable detection.

The opposite extreme occurs when $S_U(m) = 1/N$ for all m. Then $E(U, s) = 1/N$ and

$$M > N/(\mathrm{SNR} \cdot N)^2 = 1/N\,(\mathrm{SNR})^2$$

for reliable detection.

An intermediate case is when $S_U(m) = 1/L$ for L choices of m and is 0 for all other m. In this case $E(U, s) = 1/L$ and

$$M > L/(\mathrm{SNR} \cdot N)^2 = \frac{L}{N} \frac{1}{N\,(\mathrm{SNR})^2}$$

This example shows that there is a relation between efficiency and bandwidth, since if the L choices of m are contiguous and U is the discrete Fourier transform, then the signal has bandwidth $L/N\,t_0$.

VI. Walsh Transforms Applied to Sinewaves

Let $N = 2^n$ and let $(k_{n-1}, \ldots, k_0)$ be the binary representation for k, $0 \leqslant k < N$; that is,

$$k = \sum_{i=0}^{n-1} 2^i k_i$$

Then the Walsh transform matrix is

$$W_{k,\ell} = \frac{1}{\sqrt{N}} (-1)^{\Sigma k_i \ell_i} \tag{11a}$$

If the signal to be detected is a complex sinewave of frequency f/t_0, then

$$s(k) = \exp_f(k) = \frac{1}{\sqrt{N}} e^{j(2\pi f k + \varphi)} \tag{12}$$

When the Walsh transform of the signal is taken, the result is

$$s_W(\ell) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j(2\pi f k + \varphi)} (-1)^{\Sigma k_i \ell_i}$$

$$= \frac{1}{N} e^{j\varphi} \sum_{k=0}^{N-1} e^{\pi j_i \Sigma k_i (2f2^i + \ell_i)}$$

$$s_W(\ell) = e^{j\varphi} \prod_{i=0}^{n-1} \left[\frac{1 + e^{\pi j(2f2^i + \ell_i)}}{2} \right] \tag{13a}$$

The power spectrum is

$$S_w(\ell) = |s_w(\ell)|^2 = \prod_{i=0}^{n-1} \cos^2 \left(\pi \left(2^i f + \frac{\ell_i}{2}\right)\right) \tag{14a}$$

and the sum of squares of the power spectrum terms is

$$\sum_{\ell=0}^{N-1} S_W^2(\ell) = \sum_{\ell=0}^{N-1} \prod_{i=0}^{n-1} \cos^4 \left(\pi \left(2^i f + \frac{\ell_i}{2}\right)\right) \tag{15a}$$

$$= \prod_{i=0}^{n-1} \left[\cos^4(\pi 2^i f) + \sin^4(\pi 2^i f)\right]$$

$$E(W, \exp_f) = \prod_{i=0}^{n-1} \left[\frac{3}{4} + \frac{1}{4} \cos(4\pi 2^i f)\right]$$

For any particular choice of f, the efficiency $E(W, \exp_f)$ can be computed from Eq. (15a). In particular, if $f = 1/3$, the expression can be evaluated and is

$$E(W, \exp_{1/3}) = \left(\frac{5}{8}\right)^n = N^{-\log_2 (8/5)} \sim N^{-2/3} \tag{16}$$

For $N = 1024$, the efficiency is only about 0.01, and becomes worse for larger N. Thus, the Walsh transform is very poor for detecting a sinewave with frequency $(1/3)t_0$.

For a sampled data system, the frequency, f lies in the range $[-½, ½]$. If Eq. (15a) is integrated over this range, the average efficiency is obtained:

$$\mathrm{Ave}_f \{E(W, \exp_f)\} = \left(\frac{3}{4}\right)^n = N^{-\log_2 (4/3)} \sim N^{-0.4} \tag{17}$$

Since the efficiency is never negative, this implies that the efficiency is less than $2 \times N^{-\log_2(4/3)}$ for at least half of the values of f in $[-½, ½]$. Therefore, for reliable detection, M grows like $½N^{0.4}$ for at least half of the values of f. For $n = 1024$, the efficiency is less than 0.06 for more than half the frequencies. Thus, Walsh transforms are poor for detecting most sinewaves.

The same calculations for the Fourier transform are

$$F_{k,\ell} = \frac{1}{\sqrt{N}} e^{-2\pi jk\ell/N} \tag{11b}$$

$$s_F(\ell) = \frac{1}{N} e^{j\varphi} \sum_{k=0}^{N-1} e^{2\pi jk(f-(\ell/N))} \tag{13b}$$

$$\sum_{\ell=0}^{N-1} S_F^2(\ell) =$$

$$\frac{1}{N^4} \sum_{\ell,k_1k_2k_3k_4=0}^{N-1} \exp 2\pi j\left(f-\frac{\ell}{N}\right)\left(k_1+k_2-k_3-k_4\right) \tag{15b}$$

With a little algebraic manipulation, this last expression is

$$E(F, \exp_f) = \frac{1}{3}\left(\frac{2N^2+1}{N^2}\right) + \frac{1}{3}\left(\frac{N^2-1}{N^2}\right)\cos(2\pi fN) \tag{15b$'$}$$

The smallest value of this efficiency is

$$\min_f E(F, \exp_f) = \frac{1}{3}\left(1+\frac{2}{N^2}\right) \sim \frac{1}{3}$$

VII. Conclusions

We have seen from the previous analysis that Walsh transforms are very inefficient in the detection of sinewaves. This analysis can be extended to show that they are inefficient whenever the signal to be detected is narrowband (when the energy is confined to a bandwidth which is a small multiple of $1/N$). Therefore, Walsh transforms are not useful in systems which must detect such signals.

Reference

1. Robinson, G. S., "Logical Convolution and Discrete Walsh and Fourier Power Spectra," *IEEE Transactions on Audio and Electroacoustics,* Vol. AU-20, No. 4, October 1972.