An Asymptotic Analysis of a General Class of Signal Detection Algorithms

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For applications to the problem of radio frequency interference identification, or in the search for extraterrestrial intelligence, it is important to have a basic understanding of signal detection algorithms. In this article we give a general technique for assessing the asymptotic sensitivity of a broad class of signal detection algorithms. In these algorithms, the decision is based on the value of $X_1 + X_2 + \cdots + X_n$*, where the* X_i*'s are obtained by sampling and preliminary processing of a physical process.*

I. Introduction

Consider the following class of signal-detection rules. A physical process, which is either pure noise or contains a weak signal, is under observation. By sampling and preliminary calculation, n real numbers $X_1, X_2, \cdots, X_n$ are obtained from the process. We suppose that the X's can be modeled as independent, identically distributed random variables, and that their common distribution depends on a nonnegative real parameter ϵ, which we may think of as the "signal-to-noise ratio" of the process. If $\epsilon = 0$, no signal is present, but if one is, ϵ is positive.

The decision as to whether a signal is present or not is made by comparing $S_n/n = (X_1 + \cdots + X_n)/n$ to a threshold t_n:

$$\left.\begin{array}{ll} S_n/n \leqslant t_n: & \text{report signal absent} \\ S_n/n > t_n: & \text{report signal present} \end{array}\right\} \qquad (1)$$

Such a threshold test can fail in two distinct ways: it can report a signal present when it is not, or it can report no signal present when one actually is. These error events are commonly called *false alarm* and *miss*, and their respective probabilities are denoted by P_F and P_M. In different applications these two kinds of errors might not be equally severe, so let us choose arbitrary but fixed numbers α and β between 0 and 1 and try to design a detection system for which $P_F \leqslant \alpha$, $P_M \leqslant \beta$. For a given value of n, it will in general be possible to design such a system only if the signal-to-noise ratio ϵ is sufficiently large. In Section II we will prove a theorem giving an asymptotic expression for ϵ_n, the smallest such ϵ, as a function of n. This expression is typically of the form

$$\epsilon_n \sim \frac{K}{\sqrt{n}} \left[Q^{-1}(\alpha) + Q^{-1}(\beta) \right] \qquad (2)$$

where Q^{-1} denotes the inverse to the complementary Gaussian distribution function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt \qquad (3)$$

and K is a constant which depends on the common distribution of the X's.

In Section III we apply our general theorem to several special cases, in which the random variables X are derived from the real and imaginary components of a complex Gaussian signal. By computing the constants K in the asymptotic expansions (cf. Eq. (2)), we are able to compare their sensitivities. For example, we will show that binary quantization of the data prior to the computation of S_n/n costs about 0.94 dB in signal sensitivity.

II. Main Theorem

The following theorem will yield the asymptotic expansion of Eq. (2) above. In it the distribution function $F_\epsilon(x)$ is the common distribution of the random variables X when the signal-to-noise ratio is ϵ. Thus, in particular, $F_0(x)$ is the distribution when no signal is present.

THEOREM. For $0 \leqslant \epsilon \leqslant \epsilon_0$, let $F_\epsilon(x)$ be a distribution function with mean $\mu(\epsilon)$, variance $\sigma^2(\epsilon)$, and third moment $\tau(\epsilon) = \int |x - \mu(\epsilon)|^3 \, dF_\epsilon(x)$, such that

$$\mu(\epsilon) \text{ is continuous, and } \mu(\epsilon) > \mu(0) \text{ for } \epsilon > 0 \tag{4}$$

$$\sigma(\epsilon) \text{ is continuous and } \sigma(0) > 0 \tag{5}$$

$$\tau(\epsilon) \text{ is bounded for } 0 \leqslant \epsilon \leqslant \epsilon_0 \text{: } \tau(\epsilon) \leqslant T \tag{6}$$

For a given value of ϵ, and $n \geqslant 1$, let $X_1, \cdots, X_n$ be independent random variables with the distribution function $F_\epsilon(x)$, and put $S_n = X_1 + \cdots + X_n$. Define

$$p_+(a, \epsilon) = P\{S_n > na\}$$

$$p_-(a, \epsilon) = P\{S_n \leqslant na\}$$

Let α, β be positive with $\alpha + \beta < 1$. For $n \geqslant 1$, take

$$a_n = \inf \{a: p_+(a, 0) \leqslant \alpha\}$$

Since $p_+(a, 0)$ is a monotonic function of a, it follows that

$$p_+(a_n, 0) = \alpha$$

If n is sufficiently large, there are values of ϵ on $(0, \epsilon_0)$ for which $p_-(a_n, \epsilon) \leqslant \beta$. If we define

$$\epsilon_n = \inf \{\epsilon: p_-(a_n, \epsilon) \leqslant \beta\} \tag{7}$$

then

$$\mu(\epsilon_n) - \mu(0) = \frac{1}{\sqrt{n}} \sigma(0) \left[Q^{-1}(\alpha) + Q^{-1}(\beta)\right] + o(n^{-1/2}) \tag{8}$$

as $n \to \infty$, where $Q(x)$ is given by Eq. (3), above. In particular, if $\mu'(0)$ exists and is positive,

$$\epsilon_n = \frac{1}{\sqrt{n}} \frac{\sigma(0)}{\mu'(0)} \left[Q^{-1}(\alpha) + Q^{-1}(\beta)\right] + o(n^{-1/2}) \tag{9}$$

PROOF. We apply the Berry-Esseen refinements of the central limit theorem. By Theorem 1 on p. 542 of Ref. 1, Vol. 2,

$$\left| p_+(a, \epsilon) - Q\left[\sqrt{n} \frac{a - \mu(\epsilon)}{\sigma(\epsilon)}\right] \right| \leqslant \frac{3T}{\sigma(\epsilon)^3 \sqrt{n}}$$

By the continuity of $\sigma(\epsilon)$ at 0, $\sigma(\epsilon)$ is bounded away from 0 in some interval $0 \leqslant \epsilon \leqslant \epsilon_0'$. Suppose hereafter that $\epsilon \leqslant \epsilon_0'$. Then $3T/\sigma(\epsilon)^3 < B$, some positive constant, and

$$\left| p_+(a, \epsilon) - Q\left[\sqrt{n} \frac{a - \mu(\epsilon)}{\sigma(\epsilon)}\right] \right| \leqslant \frac{B}{\sqrt{n}}, \quad 0 \leqslant \epsilon \leqslant \epsilon_0' \tag{10}$$

In terms of the distribution function $P(x) = 1 - Q(x)$, this becomes

$$\left| p_-(a, \epsilon) - P\left[\sqrt{n} \frac{a - \mu(\epsilon)}{\sigma(\epsilon)}\right] \right| \leqslant \frac{B}{\sqrt{n}}, \quad 0 \leqslant \epsilon \leqslant \epsilon_0' \tag{11}$$

It follows from Eq. (10) and the fact that $p_+(a_n, 0) = \alpha$, that

$$Q\left[\sqrt{n} \frac{a_n - \mu(0)}{\sigma(0)}\right] = \alpha + \frac{\theta_1 B}{\sqrt{n}}$$

where $|\theta_1| \leqslant 1$. Let C be any constant such that $C > |Q^{-1\prime}(\alpha)|$. Then by the mean value theorem, for n sufficiently large,

$$\left| Q^{-1}\left(\alpha + \frac{\theta_1 B}{\sqrt{n}}\right) - Q^{-1}(\alpha) \right| \leqslant \frac{BC}{\sqrt{n}}$$

Hence

$$\sqrt{n} \frac{a_n - \mu(0)}{\sigma(0)} = Q^{-1}\left(\alpha + \frac{\theta_1 B}{\sqrt{n}}\right) = Q^{-1}(\alpha) + 0\left(\frac{1}{\sqrt{n}}\right)$$

and so

$$a_n = \mu(0) + \frac{\sigma(0) Q^{-1}(\alpha)}{\sqrt{n}} + 0\left(\frac{1}{n}\right) \tag{12}$$

The argument of the function P in Eq. (11) for $a = a_n$ is thus

$$\frac{\sqrt{n}}{\sigma(\epsilon)}\left[\mu(0) - \mu(\epsilon) + \frac{\sigma(0) Q^{-1}(\alpha)}{\sqrt{n}} + 0\left(\frac{1}{n}\right)\right]$$

$$= \frac{\sigma(0)}{\sigma(\epsilon)}\left[Q^{-1}(\alpha) - \frac{\sqrt{n}(\mu(\epsilon) - \mu(0))}{\sigma(0)} + 0\left(\frac{1}{\sqrt{n}}\right)\right]$$

By hypothesis (4), $\mu(\epsilon) - \mu(0) > 0$. Hence for fixed small $\epsilon > 0$ on $(0, \epsilon_0')$, the above expression approaches $-\infty$ as n increases. Thus the function P in Eq. (11) approaches 0 as $n \to \infty$, and so $p_-(a_n, \epsilon) \to 0$. This shows that for n sufficiently large, ϵ_n exists, and furthermore that $\epsilon_n \to 0$ as $n \to \infty$. By the definition (7) of ϵ_n, in every neighborhood of ϵ_n there are values of ϵ for which $p_-(a_n, \epsilon) \leqslant \beta$, and other values with $p_-(a_n, \epsilon) > \beta$. Thus there exists a sequence $\epsilon_{n,m}$ such that

$$\lim_{m \to \infty} \epsilon_{n,m} = \epsilon_n$$

$$\lim_{m \to \infty} p_-(a_n, \epsilon_{n,m}) = \beta$$

From Eq. (11),

$$\left| p_-(a_n, \epsilon_{n,m}) - P\left[\sqrt{n}\, \frac{a_n - \mu(\epsilon_{n,m})}{\sigma(\epsilon_{n,m})}\right]\right| \leqslant \frac{B}{\sqrt{n}}$$

Taking limits as $m \to \infty$, and using hypotheses (5) and (6) on the continuity of μ and σ, we conclude

$$P\left[\sqrt{n}\, \frac{a_n - \mu(\epsilon_n)}{\sigma(\epsilon_n)}\right] = \beta + \frac{\theta_2 B}{\sqrt{n}}$$

where $|\theta_2| \leqslant 1$. Arguing as above,

$$\sqrt{n}\, \frac{a_n - \mu(\epsilon_n)}{\sigma(\epsilon_n)} = P^{-1}(\beta) + 0(n^{-1/2})$$

and using the relationship $P^{-1}(x) = -Q^{-1}(x)$, we conclude

$$a_n = \mu(\epsilon_n) - \frac{\sigma(\epsilon_n)}{\sqrt{n}} Q^{-1}(\beta) + 0(n^{-1}) \tag{13}$$

Combining Eqs. (12) and (13), we have

$$\mu(\epsilon_n) - \mu(0) = \frac{1}{\sqrt{n}}[\sigma(0) Q^{-1}(\alpha) + \sigma(\epsilon_n) Q^{-1}(\beta)] + 0(n^{-1})$$

Since by Eq. (5) σ is continuous at $\epsilon = 0$, and $\epsilon_n \to 0$, then $\sigma(\epsilon_n) = \sigma(0) + o(1)$, and Eq. (8) follows. Finally if $\mu'(0)$ exists, then $\mu(\epsilon_n) - \mu(0) = \epsilon_n \mu'(0) + o(\epsilon_n)$, and Eq. (9) follows. QED.

In Section III we will apply formula (9) to compare competing signal-detection algorithms. Since the term $[Q^{-1}(\alpha) + Q^{-1}(\beta)]$ does not depend on the distribution $F(x)$, it will only be necessary to compute the values of $\sigma(0)/\mu'(0)$. But it is instructive to see how the term $Q^{-1}(\alpha) + Q^{-1}(\beta)$ behaves as a function of α and β, so here is a short table of $Q^{-1}(\alpha)$:

α	$Q^{-1}(\alpha)$
0.1	1.28
0.01	2.33
10^{-3}	3.09
10^{-4}	3.72
10^{-5}	4.26
10^{-6}	4.75
10^{-8}	5.61
10^{-10}	6.36

From this table we can compute the price paid in sensitivity for stringent error requirements. For example, if we can tolerate $\alpha = 0.01$, $\beta = 0.001$, our test will be about 1-dB more sensitive than if $\alpha = 0.001$, $\beta = 0.0001$.

III. Examples

In each of the following examples the random variables X_i will be derived from samples of a complex Gaussian random process. We denote the real (in phase) components of these samples by $I_1, I_2, \cdots, I_n$, and the imaginary (quadrature) components by $Q_1, Q_2, \cdots, Q_n$. The I's and Q's are assumed to be independent, identically distributed normal random variables with mean zero and variance $1 + \epsilon$. This situation would occur if the observed process was the sum of a variance 1 normal process (noise with average power 1) and a variance ϵ-normal process (a stochastic signal with average power ϵ).

In fact, all of our examples will be of detection algorithms based on the sample powers[1] $Z_i = I_i^2 + Q_i^2$. These random variables are distributed exponentially, and in fact for $\epsilon = 0$ we have

$$P\{Z \leq x\} = \int_0^{\sqrt{x}} re^{-r^2/2}\, dr = 1 - e^{-x/2} \tag{14}$$

A. Case 1

Test based on the random variables $X_i = Z_i^a$, where $a > 0$. Here an easy manipulation of Eq. (14) yields, for any $a > 0$, and $\epsilon = 0$,

$$E(X^b) = 2^{ab} \int_0^{\infty} y^{ab} e^{-y}\, dy$$

$$= 2^{ab}\, \Gamma(ab + 1)$$

where Γ denotes the gamma function. Thus $\mu(\epsilon) = E((1+\epsilon)^a X) = 2^a(1+\epsilon)^a \Gamma(a+1)$, and so $\mu(0) = 2^a \Gamma(a+1)$, $\mu'(0) = a2^a\Gamma(a+1)$. To compute $\sigma^2(0)$, we use the formula $\sigma^2(0) = E(X^2) - (E(X))^2 = 2^{2a}\Gamma(2a+1) - (2^a\Gamma(a+1))^2$. Thus in the asymptotic expansion of Eq. (9) we have

$$K(a) = \frac{\sigma(0)}{\mu'(0)} = \frac{1}{a}\left\{\frac{\Gamma(2a+1)}{\Gamma(a+1)^2} - 1\right\}^{1/2}$$

A short table of $K(a)$ follows:

a	$K(a)$ Numerically	$K(a)$ Exactly
0^+	1.283	$\pi/\sqrt{6}$
1/2	1.045	$2((4-\pi)/\pi)^{1/2}$
1	1.000	
2	1.118	$\sqrt{5}/2$
3	1.453	$\sqrt{19}/3$

From this we see that the best test of this kind is for $a = 1$ (a fact which follows from the Neyman-Pearson theorem); if the test is based on the amplitude $X = \sqrt{I^2 + Q^2}$, the loss is $10 \log_{10}(1.045) = 0.19$ dB. Notice also that in the limit as $a \to 0$, $X = Z^a = e^{a \log z} = 1 + a \log Z + 0(a^2)$, and so the performance must be the same as if we took $X = \log Z$. We conclude that the loss in this case is $10 \log_{10}(\pi/\sqrt{6}) = 1.08$ dB.

[1] There is no loss in considering the powers instead of the components separately, by the Neyman-Pearson criterion Ref. 2, Sec. 13.2.

B. Case 2 (Binary Quantization)

Here the test is based on the random variables

$$X_i = \begin{cases} 0 & \text{if } Z_i \leq T \\ 1 & \text{if } Z_i > T \end{cases}$$

where the threshold T is a fixed positive real number. According to Eq. (14), the distribution of the X's is given by

$$P\{X = 1\} = P\{Z \geq T\}$$

$$= \int_{\sqrt{T}}^{\infty} re^{-r^2/2}\, dr = e^{-T/2}$$

Hence, if the signal to noise ratio is ϵ,

$$\mu(\epsilon) = P\{(1+\epsilon)Z > T\}$$

$$= \exp\left[-\frac{T}{2(1+\epsilon)}\right]$$

Hence, $\mu(0) = e^{-T/2}$, $\mu'(0) = 1/2\, Te^{-T/2}$. Since X is a Bernoulli random variable, $\sigma(0) = (\mu(0)(1-\mu(0)))^{1/2} = (e^{-T/2}(1-e^{-T/2}))^{1/2}$. Hence in Eq. (9) the value of $\sigma(0)/\mu'(0)$ is

$$K = \frac{(e^x - 1)^{1/2}}{x}, \quad x = T/2 \tag{15}$$

Of course one wishes to choose x so that K will be minimized. Numerically one calculates that the minimum of Eq. (15) occurs for $T = 3.18725$ and is $K_{min} = 1.24263$. Thus the loss in sensitivity due to binary quantization is $10 \log_{10}(1.24263) = 0.94$ dB.

C. Case 3

In this case we let b be a fixed positive integer, and assume that n is a large multiple of b, say $n = n_0 b$. The test is based on the random variables

$$X_i = \max(Z_{(i-1)b+1}, Z_{(i-1)b+2}, \cdots, Z_{ib}) \tag{16}$$

i.e., X_i is the maximum of the i-th block of length b. Since by Eq. (14), the distribution function of Z for $\epsilon = 0$ is $F_1(x) = 1 - e^{-x/2}$, it follows (see Ref. 3, Sec. 13) that the distribution of each X_i is given by

$$F_b(x) = (F_1(x))^b = (1 - e^{-x/2})^b$$

Hence the density for the X's is

$$p_b(x) = \frac{d}{dx} F_b(x)$$

$$= \frac{b}{2} e^{-x/2} (1 - e^{-x/2})^{b-1}, \quad x \geqslant 0$$

The r-th moment of X is then given by

$$\mu_r(b) = \frac{b}{2} \int_0^\infty x^r e^{-x/2} (1 - e^{-x/2})^{b-1}\, dx \qquad (17)$$

After expanding the term $(1 - e^{-x/2})^{b-1}$ by the binomial theorem, integrating term-by-term, and summing, one obtains finally

$$\mu_r(b) = r!\, 2^r \sum_{k=1}^{b} (-1)^{k+1} \binom{b}{k} k^{-r} \qquad (18)$$

The sum in Eq. (18) was considered in Ref. 4; in particular it was shown there that

$$\mu_1(b) = 2 \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{b}\right)$$

$$\mu_2(b) = 4 \left[\left(1 + \frac{1}{2} + \cdots + \frac{1}{b}\right)^2 + \left(1 + \frac{1}{2^2} + \cdots + \frac{1}{b^2}\right)\right]$$

Hence if $\epsilon = 0$, the mean of X is

$$\mu = 2 \left(1 + \frac{1}{2} + \cdots + \frac{1}{b}\right)$$

and the variance is

$$\sigma^2 = 4 \left(1 + \frac{1}{2^2} + \cdots + \frac{1}{b^2}\right)$$

For general ϵ, $\mu(\epsilon) = E[\max(1+\epsilon) Z_1, \cdots, (1+\epsilon) Z_n] = (1+\epsilon)\mu(0)$; and so $\mu'(0) = \mu(0)$. Thus in Eq. (9) the term $\sigma(0)/\mu'(0)$ is

$$\frac{\sigma(0)}{\mu'(0)} = \frac{\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{b^2}\right)^{1/2}}{\left(1 + \frac{1}{2} + \cdots + \frac{1}{b}\right)}$$

However in Eq. (9) the number n should be replaced by $n_0 = n/b$, and so the formula for ϵ_n becomes

$$\epsilon_n \sim \frac{1}{\sqrt{n}} \left[\sqrt{b}\, \frac{\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{b^2}\right)^{1/2}}{\left(1 + \frac{1}{2} + \cdots + \frac{1}{b}\right)}\right]$$

$$\cdot \left(Q^{-1}(\alpha) + Q^{-1}(\beta)\right)$$

and so the sensitivity is measured by the quantity

$$K_b = \sqrt{b}\, \frac{\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{b^2}\right)^{1/2}}{\left(1 + \frac{1}{2} + \cdots + \frac{1}{b}\right)}$$

A table of this quantity follows:

b	K_b	Loss in dB
1	1.000	0
2	1.054	0.23
3	1.102	0.42
4	1.145	0.59
5	1.185	0.74
6	1.221	0.87
7	1.255	0.99
8	1.286	1.09
16	1.489	1.73
32	1.771	2.48
64	2.153	3.33
128	2.664	4.26
256	3.348	5.25
512	4.256	6.29
1024	5.464	7.38

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