A First-Principles Derivation of Doppler Noise Expected From Solar Wind Density Fluctuations

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The level of Doppler noise (DN) expected from solar wind (SW) density fluctuations (DF) is derived beginning with the expression for refractive index variations. The calculation takes account of up- and downlink paths and of the method actually used to produce the DN values. The usual assumptions that the DF are frozen in, that the large-scale radial variation can be separated from the DF, that the DF power spectrum is a power law with "outer scale" k_o, and that the DF are homogeneous on scales less than $2c\Delta t$, Δt = sample time, are made. The result agrees quite well with the observations of DN by Berman (Refs. 4, 6, and 7). Corrections for the finite number of points used in the actual algorithm are discussed.

I. Introduction

The present discussion is meant to give a clearer understanding of the relationship of Doppler noise (DN) to density fluctuations (DF) in the solar wind (SW) by providing a detailed first-principles derivation. The analysis is similar to that of Refs. 1, 2, and 3, but is adapted specifically to the round-trip case and to the actual method used to obtain DN. This analysis will also clarify the usefulness of DN for radio science (Ref. 4).

II. Basic Principles of Doppler Noise

The phase variations measured by DN are produced by refractive index variations along the path which occur because the index of a plasma is proportional to the electron density N:

$$\mu^2 = 1 - \frac{e^2 N}{\pi m f^2} = 1 - \frac{AN}{f^2} \tag{1}$$

where f is the radio frequency and A = 80.6 in MKS units. Local variations of N on time scales $\gtrsim$ 1 hour are nearly 100% in the SW. The total change in phase for a reference point in traveling to a spacecraft and back is

$$\Phi_0 = \int_{\text{raypath}} \frac{f\,ds}{v_\phi} = \frac{f}{c}\int \frac{ds}{[1 - (A/f^2)(N + \delta N)]^{1/2}} \cong \frac{f}{c}\int ds \left[1 + \frac{AN}{2f^2} + \frac{A\delta N}{2f^2}\right] \tag{2}$$

where c is the speed of light. The approximation is very good for S-band propagating in the solar wind.

Because we are interested in phase fluctuations the result is not sensitive to the average value of N but only to the fluctuations δN. For this analysis we will assume that the spacecraft is at rest so there is no ordinary Doppler tone. This latter assumption is an accurate approximation of the mechanization of DN: predicts are subtracted from the observed Doppler and then a linear fit is done to 15 points to remove any remaining trends. Because of this DN will not be sensitive to slow ($\gtrsim 30\,\Delta t$, $\Delta t \equiv$ sample time) changes in the refractive index.

In summary, we are interested in the phase change

$$\delta\Phi = \frac{f}{c}\int_{\text{raypath}} \left(\frac{A}{2f^2}\right)\delta N\,ds \tag{3}$$

where δN is a function of time and position along the raypath. δN is defined so that $\delta\Phi$ has a mean of 0. At any instant one would measure a phase deviation $\delta\Phi(t)$. However, Doppler is not an instantaneous measurement but is accumulated (or averaged) over some sample time Δt. To work in the "accumulation picture" one uses the time derivative of Eq. (3) so that the phase deviation from t_1 to $(t_1 + m\Delta t)$ is

$$\Delta\Phi(t_1, m\Delta t) = \int_{t_1}^{t_1+m\Delta t} dt \int_{\text{raypath}} \left(\frac{A}{2fc}\right)\delta\dot{N}\,ds \tag{4}$$

where the dot denotes the time derivative of δN. Doppler is produced by differencing two adjacent phase accumulations and dividing by the sample interval. Because we have assumed that there is no velocity-induced Doppler tone, the "Doppler" produced in this case is immediately the material for producing Doppler noise (mean-squared phase fluctuation).

$$DN(t_1, \Delta t, M) = \frac{1}{M}\sum_{m=0}^{M-1}\left[\frac{\Delta\Phi(t_1, (m+1)\Delta t) - \Delta\Phi(t_1, m\Delta t)}{\Delta t}\right]^2 \text{Hz}^2 \tag{5}$$

where M is the number of points included in the linear fit. Note that what we are calling DN is the square of the experimental quantity reported by Refs. 4, 6, and 7.

If we carry out the prescription of Eq. (5) using Eq. (4), we have

$$DN(t_1, \Delta t, M) = \left(\frac{A}{2fc\Delta t}\right)^2 \frac{1}{M}\sum_{m=0}^{M-1}\left[\int_{t_1+m\Delta t}^{t_1+(m+1)\Delta t} dt \int_{\text{raypath}} \delta\dot{N}\,ds\right]^2 \text{Hz}^2 \tag{6}$$

An obvious simplification results if the order of integration is interchanged so that δN is evaluated at the two endpoint times. The interchange is allowed as long as the DF are homogeneous on scales $<2c\Delta t$, and the round-trip light time is not so large as the time scale on which the magnitude of δN changes (~1 day in the SW). The first requirement is a statement of the Nyquist theorem that data sampled at Δt contain no information about frequencies $> 1/2^*\Delta t$. Recall that DN is also not sensitive to long-term trends $> 2M\Delta t$.

We use the above assumption to carry out the time integral in Eq. (6). We also explicitly exhibit the up- and downlink parts of the raypath. Time is measured at the receiver and the geometry is shown in Fig. 1. The result is

$$
\begin{aligned}
DN(t_1, \Delta t, M) = \left(\frac{A}{2fc\Delta t}\right)^2 \frac{1}{M} \sum_{m=0}^{M-1} \Bigg\{ \int_0^L dz \Big[& \delta N\,[\mathbf{r}, t_1 + (m+1)\,\Delta t - (2L - z)/c] \\
& - \delta N\,[\mathbf{r}, t_1 + m\Delta t - (2L - z)/c] + \delta N\,[\mathbf{r}, t_1 + (m+1)\,\Delta t - z/c] \\
& - \delta N\,[\mathbf{r}, t_1 + m\Delta t - z/c] \Big] \Bigg\}^2 \ \mathrm{Hz}^2
\end{aligned} \tag{7}
$$

where $\mathbf{r}$ is the heliocentric position vector.

To proceed with the evaluation of DN we must have a way of characterizing the sixteen products represented in Eq. (7). We approximate the sum over m by an integral over t. We assume that the time and space integrals may be interchanged. This is valid if the correlation scale of the medium is $\ll L$, and for the "frozen-in" assumption which will be used later. We show these operations in just the first term of Eq. (7).

$$
\begin{aligned}
DN(t_1, \Delta t, M) = \left(\frac{A}{2fc\Delta t}\right)^2 \int_0^L dz \int_0^L dz' \frac{1}{T_M} \int_0^{T_M} \Big\{ & \delta N\,[\mathbf{r}, t_1 + t + \Delta t - (2L - z)/c] \\
& \delta N\,[\mathbf{r}', t_1 + t + \Delta t - (2L - z')/c] \Big\}\, dt
\end{aligned}
$$

where $T_M = M\Delta t$ and the two terms of the product are distinguished by primed and unprimed coordinates. In the limit $T_M \to \infty$ the t-integral gives the autocorrelation function of δN, $F(\Delta r, \tau)$, which depends only on the separation in time and/or space of the DF. The limit $T_M \to \infty$ requires that $T_M \gg T_o$, where T_o is the correlation time of the medium. For M = 15–18 and $\Delta t \leqslant 60$ sec as in the DN algorithm this condition is not accurately fulfilled in the SW ($T_o \gtrsim 1$ hr, for the larger changes, $\delta N/N \sim 1$). Therefore, we will do an approximate analysis consisting of two terms: the first, the result for $T_M \to \infty$; the second, a correction (involving the autocorrelation function as an approximation for the mean-squared density change) for the fact that $T_M \leqslant T_o$. The autocorrelation function is symmetric, so we have only 8 terms:

$$
\begin{aligned}
DN(t_1, \Delta t, M) \cong \left(\frac{A}{2fc\Delta t}\right)^2 2\,F_1(t_1) \int_0^L dz \int_0^L dz' \Bigg\{ 2 \Big[& F\,[\mathbf{r} - \mathbf{r}', (z - z')/c] \\
& + F\,[\mathbf{r} - \mathbf{r}', (2L - z - z')/c] \Big] - \Big[F\,[\mathbf{r} - \mathbf{r}', -\Delta t + (z - z')/c] \\
& + F\,[\mathbf{r} - \mathbf{r}', -\Delta t + (2L - z - z')/c] + F\,[\mathbf{r} - \mathbf{r}', \Delta t + (z - z')/c] \\
& + F\,[\mathbf{r} - \mathbf{r}', \Delta t + (2L - z - z')/c] \Big] \Bigg\}
\end{aligned}
$$

$$- \left(\frac{A}{2fc\Delta t}\right)^2 2F_1(t_1)\int_0^L dz \int_0^L dz' \int_{T_M}^{\infty} \frac{dt}{T_M} \Bigg\{ 2\,[F\,[\mathbf{r} - \mathbf{r}', t + (z - z')c]\,]$$

$$\ldots \Bigg\} \; Hz^2 \qquad (8)$$

where the second term contains all the F's of the first with the arguments modified by the addition of t. The factor $F_1(t_1)$ is to account for the fact that the general level of activity may vary on a time scale longer than T_o (i.e., if we use a yearly average value for $F(\Delta r, \tau)$, the daily value of $DN(\Delta t, T_M)$ could easily be a factor of 2 higher or lower than we would expect). The terms with L arise because we are considering round-trip measurements. The Δt's occur because of the phase accumulation time. Reference 3, in doing a one-way analysis of range data, uses only the first term of Eq. (8).

So far there has been no use made of any particular properties of the SW. The effect of any fluctuating plasma, whose statistical properties satisfy the restrictions discussed above, on round-trip Doppler data is given by Eq. (8).

III. Doppler Noise From Solar Wind Density Fluctuation

We now proceed to investigate Eq. (8) for the case of SWDF. First, we note that the SW has a large-scale radial variation which should be separated from the autocorrelation function, just as the long-term time variations were. We take

$$F[\mathbf{r} - \mathbf{r}', \tau] = b(r)\, G[\mathbf{r} - \mathbf{r}', \tau] \qquad (9)$$

where b(r) gives the radial variation of the mean-squared density,

$$b(r) = \delta N_1^2 (A_1/r)^{4+2\gamma} \qquad (10)$$

where δN_1 is the *total* RMS DF at radius A_1 and γ allows for radial variations different from r^{-2} (see Section IV).

Time and position for the SWDF are usually related by the "frozen-in" assumption because the speeds at which disturbances propagate are much less than the bulk velocity. Thus, $G(\mathbf{r},\tau) = G(\mathbf{r} - \mathbf{v}\tau)$, where $\mathbf{v}$ is the bulk SW velocity. For simplicity we will assume that $\mathbf{v}$ is not a function of $|\mathbf{r}|$ and that it is only in the radial direction. We take the raypath to be in the x-z plane (see Fig. 1) so that $\mathbf{v} = (v_x, o, v_z)$; note that the components of $\mathbf{v}$ are functions of position along the raypath. Equation (8) may be rewritten as

$$\begin{aligned} DN(t_1, \Delta t, M) = B^2 \int_0^L dz\, b(r) \int_z^{z-L} (-d\ell) \Bigg\{ 2 \Big[& G\,[\mathbf{R} - \mathbf{v}\ell/c] \\ & + G\,[\mathbf{R} - \mathbf{v}(2L - 2z + \ell)/c] \Big] - \Big[G\,[\mathbf{R} - \mathbf{v}(-\Delta t + \ell/c)] \\ & + G\,[\mathbf{R} - \mathbf{v}(-\Delta t + (2L - 2z + \ell)/c)] + G\,[\mathbf{R} - \mathbf{v}(\Delta t + \ell/c)] \\ & + G\,[\mathbf{R} - \mathbf{v}(\Delta t + (2L - 2z + \ell)/c)] \Big] \Bigg\} \end{aligned}$$

$$- B^2 \int_0^L dz\, b(r) \int_z^{z-L} (-d\ell) \int_{T_M}^{\infty} \frac{dt}{T_M} \left\{ 2 \left[G\, [\mathbf{R} - \mathbf{v}(t + \ell/c)] \right. \right.$$

$$\left. + \ldots \right\} Hz^2 \tag{11}$$

where $\ell \equiv z - z'$, $\mathbf{R} \equiv (o, o, \ell)'$ $\mathbf{B}^2 = 2F_1(t_1)(A/2fc\Delta t)^2$, and the second term contains all the G's of the first with the arguments modified by t.

The well-studied characteristic of SWDF is not the autocorrelation but its Fourier transform, the power spectrum. The relationship between the two is

$$g\,(\mathbf{k}) = \int_{-\infty}^{\infty} G\,(\mathbf{R})\, e^{-i\mathbf{k}\cdot\mathbf{R}}\, d^3 R \qquad \text{(a)}$$

$$\text{(12)}$$

$$G\,(\mathbf{R}) = \int_{-\infty}^{\infty} g(\mathbf{k})\, e^{+i\mathbf{k}\cdot\mathbf{R}}\, \frac{d^3 k}{(2\pi)^3} \qquad \text{(b)}$$

where **k** is the wavenumber vector of the DF. Experimentally it is found that the spectrum can be well represented by

$$g\,(k) = \frac{k_o^{\beta-3}}{[k_o^2 + a_x^2 k_x^2 + a_y^2 k_y^2 + a_z^2 k_z^2]^{\beta/2}} \tag{13}$$

where k_o is the "outer scale" of the DF, $k_o = 2\pi/\ell_o$, $\ell_o \cong 10^6$ km, the a's allow for the possibility that the spectrum is anisotropic (there is no good experimental evidence that they differ from 1, but they are easy to keep in the computation), and the spectral index $\beta = 3.5 - 4.0$ (see Section IV). We use Eq. (12b) and Eq. (11) to obtain

$$DN\,(t_1, \Delta t, M) = 2B^2 \int_0^L dz \int_{-\infty}^{\infty} \frac{d^3 k}{(2\pi)^3}\, g(k) \int_z^{z-L} (-d\ell)\, e^{i\ell(k_z - \mathbf{k}\cdot\mathbf{v}/c)}$$

$$\left\{ [1 + e^{-i2\mathbf{k}\cdot\mathbf{v}(L-z)/c}]\, [1 - e^{-i\mathbf{k}\cdot\mathbf{v}\Delta t}] \right.$$

$$\left. - \int_{T_M}^{\infty} \frac{dt}{T_M}\, e^{-i\mathbf{k}\cdot\mathbf{v}t}\, [1 + e^{-i2\mathbf{k}\cdot\mathbf{v}\,(L-z)/c}]\, [1 - e^{-i\mathbf{k}\cdot\mathbf{v}\Delta t}] \right\} Hz^2 \tag{14}$$

where use has been made of the fact that G is real, so we need consider terms of only one sign in Δt.

In the SW at heliocentric distances greater than a few solar radii the distance scale is much greater than the scale of the DF. The limits on the ℓ-integral can then be approximated as $\pm\infty$, so it gives a delta function $2\pi\delta(k_z - \mathbf{k}\cdot\mathbf{v}/c)$. The delta function can be used to do the k_z-integral with $k_z \rightarrow k_x v_x/c\,(1 - v_z/c) \equiv k_x v_x V/c$ and $\mathbf{k}\cdot\mathbf{v} = k_x v_x V$, giving

$$DN\,(t_1, \Delta t, T_M) = 2\,B^2 \int_0^L dz\, V\, b(r) \int_{-\infty}^{\infty} \frac{d^2 k}{(2\pi)^2}\, g(k_x, k_y, k_x v_x\, V/c)$$

$$\times\left\{\left[1+e^{-i2k_xv_xV(L-z)/c}\right]\left[1-e^{-ik_xv_x\Delta t}\right]\right.$$

$$-\int_{T_M}^{\infty}\frac{dt}{T_M}\,e^{-ik_xv_xVt}\left[1+e^{-i2k_xv_xV(L-z)/c}\right]*$$

$$\left.\left[1-e^{-ik_xv_x\Delta t}\right]\right\}Hz^2 \tag{15}$$

where k_y drops out of the exponentials because $v_y \equiv 0$.

For $g(\mathbf{k})$ of the form of Eq. (13) the k_y-integral can be carried out to give

$$DN(t_1,\Delta t,T_M) = 2B^2\int_0^L dz\, b(r)\,\frac{V k_o^{\beta-3}\sqrt{\pi}\,\Gamma(\beta-1/2)}{a_y(2\pi)^2\,\Gamma(\beta/2)}$$

$$\times\int_{-\infty}^{\infty}\frac{dk_x}{[k_o^2+k_x^2(a_x^2+a_z^2V^2)]^{(\beta-1)/2}}\left\{\left[1+e^{-i2k_xv_xV(L-z)/c}\right]*\right.$$

$$\left.\left[1-e^{-ik_xv_x\Delta t}\right]-\int_{T_M}^{\infty}\frac{dt}{T_M}\,e^{-ik_xv_xt}\,[\cdots]\,[\cdots]\right\}Hz^2 \tag{16}$$

where the second brackets are the same as the first.

The four parts of the first term of Eq. (16) can be integrated using

$$\int_{-\infty}^{\infty}\frac{dx\,e^{-ixy}}{(1+x^2)^{(2n+1)/2}} = 2\left(\frac{y}{2}\right)^n\frac{\sqrt{\pi}}{\Gamma(n+1/2)}K_n(y) \tag{17}$$

For Eq. (17) we have $\beta - 1 = 2n + 1$ or $n = (\beta - 2)/2$, and

$$x = k_x(a_x^2 + a_z^2V^2)^{1/2}/k_o \equiv k_x a/k_o$$

Before doing the k_x-integral of the second term, we must consider the effect of the t-integral. It is easily carried out to give a factor $[e^{-ik_xv_xVT_M}/(-ik_xv_xVT_M)]$. For $|k_x| \gtrsim k_M \equiv 2\pi/2v_xT_M$ the exponential term varies rapidly so that there is little contribution to the integral. This effect is important if $2v_xT_M < \ell_o$ ($k_M > k_o$), because the k_x-integral of this (correction) term is not cut off before reaching the sloping part of the spectrum. To repeat: when $2v_xT_M < \ell_o$, the mean-squared phase fluctuation will depend on the averaging time $M\Delta t$. We approximate the second term of Eq. (16) by finite limits on the k_x-integral and have

$$DN(t_1,\Delta t,T_M) = 2B^2\int_0^L\frac{dz\,b(r)V}{a\,a_y\pi\Gamma(\beta/2)\,2^{n+1}}\left\{\left[2^{n-1}\Gamma(n)\right.\right.$$

$$- \left(\frac{k_o v_x V \Delta t}{a}\right)^n K_n(\ )\Bigg] + \Bigg[\left(\frac{2k_o v_x V(L-z)}{ac}\right)^n K_n(\ )$$

$$- \left(\frac{k_o v_x V(\Delta t + 2(L-z)/c)}{a}\right)^n K_n(\ )\Bigg]\Bigg\} + 4B^2 \int_0^L dz\, b(r) *$$

$$\left[\frac{Vk_o^{\beta-3} \sqrt{\pi}\,\Gamma\left((\beta-1)/2\right)}{a_y (2\pi)^2\, \Gamma(\beta/2)}\right] \Bigg\{ \int_0^{k_M} \frac{dk_x \left[1 + e^{-i2k_x v_x V(L-z)/c}\right] *}{[k_o^2 + a^2 k_x^2]^{(\beta-1)/2}}$$

$$\left[1 - e^{-ik_x v_x \Delta t}\right]\Bigg\}\ \mathrm{Hz}^2 \tag{18}$$

where Γ is the gamma function, K_n is the modified Bessel function of order n, and the arguments of the K's are the same as the respective factors in front of them. This result is similar to that of Ref. 3, but contains all the terms required to describe two-way Doppler data. The analysis of the second term requires some care and is deferred until the basic result for long averaging times ($T_M \to \infty$, $k_M \to 0$) is completed.

We are now faced with the task of integrating Eq. (18) along the raypath. First, we assume that only b(r) and v_x (the velocity across the raypath) depend on z. A much greater simplification occurs when the order of magnitudes of the arguments are considered. We take typical numerical values $k_o = 2\pi \times 10^{-6}$ km^{-1}, $v_x = 400$ km/sec, $\Delta t = 60$ sec, $L = 3 \times 10^8$ km, $a = a_y = V = 1$ and find $k_o v_x \Delta t = 0.15$, $2k_o v_x L/c = 5.0$, $L/c\Delta t = 17$. Thus, except for $z = L$, $\Delta t \ll (L - z)/c$; and for $z = L$ the coefficient b(r) will make the contribution small. In this limit we may treat the two differences in Eq. (18) as derivatives. The only z dependence in the first term is the variation of v_x along the raypath.

We proceed to simplify the geometric expressions in order to integrate along the raypath. From Fig. 1 we have

$$r^2 = (z - q\mu_1)^2 + q^2 = q^2(\mu^2 + 1)$$

$$v_x = v_o/(1 + \mu^2)^{m/2}$$

$$dz = q\,d\mu \tag{19}$$

where v_o is the radial velocity of the SW and m slightly different from 1 allows for nonradial velocities. We ignore the variation of v_z along the raypath since it always occurs as $v_z/c < 10^{-3}$. We will also suppose that the a's are constant along the raypath as there is no evidence to the contrary. For $k_o(r)$ we chose $k_o(r) = k_1(q/A_1)^{\ell}(1 + \mu^2)^{\ell/2}$ to find the effects of an outer scale which changes with heliocentric distance. Recalling the form for b(r) from Eq. (10) and using Bessel function relations to reduce the derivatives introduced, we have

$$DN_1(t_1, \Delta t, T_M) = 2B^2 \left[\frac{V\delta N_1^2 (A_1/q)^{4+2\gamma} qk_1 (v_o \Delta t)^2}{\pi a_y\, a\, \Gamma(\beta/2) 2^{n+1}}\right] \int_{-\mu_1}^{\mu_2} d\mu \Bigg\{\Bigg[2^{n-3}\,\Gamma(n-1)\frac{V^2}{a^2} *$$

$$(q/A_1)^{\ell}(1 + \mu^2)^{-2-\gamma-m+\ell/2}\Bigg] + \left[\frac{c(1 + \mu^2)^{-2-\gamma-\ell/2}}{2q(k_1 v_o \Delta t)^2 (q/A_1)^{\ell}}\right] *$$

$$\frac{d}{d\mu}\left[\left(\frac{2k_1\,(q/A_1)^{\ell}\,v_o V q\,(\mu_2-\mu)}{ac(1+\mu^2)^{(m-\ell)/2}}\right)^n K_n\,(\;)\right]\Bigg\}\ \mathrm{Hz}^2 \tag{20}$$

Because of the large power of μ in the denominator (~5), so long as $\mu \gtrsim 3$, i.e., $q \lesssim 0.3$ AU, the limits on the integral can be considered as $\pm\infty$ for the first term, and the second term will be less than the first. The second term is integrated by parts, and the only contribution is for $\mu = \mu_1$. The result of the integration along the raypath is

$$DN_1\,(t_1, \Delta t, T_M) = F_1\,(t_1)\left(\frac{A}{2fc\Delta t}\right)^2 \left[\frac{\delta N_1^2\,(A_1/q)^{4+2\gamma}\,k_1 q\,(v_o\Delta t)^2\,V^3}{\pi a_y\,a^3\Gamma\,(\beta/2)}\right] *$$

$$\left\{\left[\frac{\sqrt{\pi}\,\Gamma(n-1)\,(q/A_1)^{\ell}\,\Gamma\left((3+2\gamma+2m-\ell)/2\right)}{2\Gamma\left((4+2\gamma+2m-\ell)/2\right)}\right]\right.$$

$$\left. + \left[\frac{a^2c\,(1+\mu_1^2)^{-2-\gamma-\ell/2}}{2^n\,qk_1v_o^2\,\Delta t\,V^2\,(q/A_1)^{\ell}}\right]\left[\left(\frac{2k_1\,(q/A_1)^{\ell}\,v_o\,VL}{ac(1+\mu_1^2)^{(m-\ell)/2}}\right)^n K_n\,(\;)\right]\right\}\ \mathrm{Hz}^2 \tag{21}$$

where $n = (\beta - 2)/2$. For a long averaging time $T_M \to \infty$, this gives the mean-squared Doppler noise in terms of the mean-squared density fluctuation at 1 AU (A_1) appropriate to the time of observation $F_1(t_1)\delta N_1^2$, the outer scale size k_1, the radial solar wind velocity v_o, and the sampling time Δt. Note that to first order the sampling time drops out completely. This result will be discussed further in Section IV.

We now return to the correction term for finite averaging times in Eq. (18). The important point is that $k_M > k_o$, so we divide all the k's by k_M. This shows the basic $(k_M^{\beta-2})^{-1} \propto \Delta t^{\beta-2}$ dependence that we wish to evaluate. We note that $k_M v_x \Delta t \ll 1$, so the second term in the integral can be expanded. The overall correction term is

$$DN_2\,(t_1, \Delta t, T_M) = 4B^2\int_0^L \frac{dz\,b(r)\,V\sqrt{\pi}\,k_o^{\beta-3}}{(2\pi)^2\,a\,a_y\;\Gamma(\beta/2)\,k_M^{\beta-2}}\int_0^1 \frac{dx\,\left[-(\pi x/Ma)^2 + (i\pi x/Ma)\,e^{-ixp}\right]}{[x_o^2+x^2]^{(\beta-1)/2}}\ \mathrm{Hz}^2 \tag{22}$$

where $x \equiv ak_x/k_M$, $x_o \equiv k_o/k_M$, $p = \pi(L - z)/aMc\Delta t$, and we have made use of the fact that the final result must be real.

The x-integral in Eq. (22) can be evaluated approximately, though we note that it is mainly a numerical factor (there is z-dependence in the second term) of order 1. Finally, we use Eqs. (10) and (19), the expression for $k_o(r)$ following Eq. (19), and the approximation $\mu_2, \mu_1 \to \infty$ to arrive at the correction term

$$DN_2\,(t_1, \Delta t, T_M) \cong F_1\,(t_1)\left(\frac{A}{2fc\Delta t}\right)^2\left[\frac{\delta N_1^2\,(A_1/q)^{4+2\gamma}\,k_1 q\,(v_o\Delta t)^2\,V^3}{\pi a_y\,a^3\Gamma\,(\beta/2)}\right] *$$

$$\left\{\left[\frac{2a}{V^2}\left(\frac{M}{\pi}\right)^{\beta-3}\frac{(k_1v_o\Delta t)^{\beta-4}\,(q/A_1)^{\ell(\beta-3)}\,\Gamma(n_2-1/2)}{(\beta-3)\,\Gamma\,(n_2)}\right] *\right.$$

$$\left.\left[\frac{\sin p_o\mu_2}{(1+x_o^2)^{(\beta-3)/2}} - p_o\mu_2\,(\cos p_o\mu_2)\,\ln\left(1+\sqrt{1+x_o^2}\right)\right]\right\}\ \mathrm{Hz}^2 \tag{23}$$

where $n_2 \equiv 2 + \gamma - \ell(\beta - 3)/2 + m(\beta - 2)/2$, $p_o \equiv \pi q/aMc\Delta t$, and all other terms are as before. Note that the coefficient in front of the braces is the same as in Eq. (21). It should be emphasized that this form of the correction term applies *only* for $2Mv_o\Delta t < \ell_o$, or $\Delta t \lesssim 60$ sec, for M = 15 - 18. The result shows the Δt-dependence $\Delta t^{\beta-4}$ that DN can be expected to have when the averaging time $M\Delta t$ is less than the correlation time ℓ_o/v_o.

IV. Discussion and Conclusions

The sum of Eqs. (21) and (23) gives the mean-squared Doppler noise (DN) in Hz^2 expected from solar wind (SW) density fluctuations (DF). Because the DF are functions of both time and position, considering them as a factor times the local average density $\delta N(R,t) = \epsilon(r,t)N(r)$ would not be useful as the important information would be contained in the correlation function of ϵ. Thus, it is difficult theoretically to relate DN to the total columnar content. If a proportionality exists (Refs. 6, 7) further analysis of the structure of SWDF will be needed to find the physical basis of the proportionality. However, it will be shown below that the theory developed above provides a completely adequate explanation of DN. It also shows that DN can be useful for studying the spectrum and amplitude of SWDF, and perhaps also the SW velocity near the sun.

We proceed to a numerical evaluation of Eqs. (21) and (23). First, we set $a = a_y = V = 1$, as they are not known to differ much from 1. For the SW parameters we take $\delta N_1 = 5\ cm^{-3}$, $v_o = 400$ km/sec, $k_1 = 2\pi \times 10^{-6}\ km^{-1}$. A number of observations of DF (Refs. 5, 6) suggest $\gamma \cong -0.2$; however, for now it will be kept as a parameter. We will use the geometric parameters $q = 0.1A_1$, $L = 2A_1$, and recall the restriction $\mu_1 \gtrsim 3$. For the sampling parameters we take M = 15, $\Delta t = 60$ sec, f = 2.2 GHz. Since the Γ functions are not very sensitive, we use $\beta = 4$, $\gamma = 0$, $\ell = 0$, m = 1 in them. We then find

$$
\begin{aligned}
DN(t_1, \Delta t, T_M) \cong\ & F_1(t_1)\left(\frac{4.47 \times 10^{-3}\ 10^{2\gamma}\,(\delta N_1/5)^2\,(v_o/400)^2}{(q/0.1\ A_1)^{3+2\gamma}}\right) * \\
& \left\{ 0.589\,(0.1)^{\ell}\,(q/0.1\ A_1)^{\ell} + \left(\frac{2.64 \times 10^{-4}\,(q/0.1\ A_1)^2}{(\Delta t/60)^2}\right) * \right. \\
& \Big[(0.503\,(q/0.1\ A_1)\,(L/2\ A_1))\ K_1(\ )\Big] \\
& \left. - \left(\frac{5.08}{(7.2 \times 10^4)^{4-\beta}\,(\Delta t/60)^{4-\beta}}\right)\left[\sin\left(\frac{5\pi}{9(\Delta t/60)}\right) - \left(\frac{5\pi}{9(\Delta t/60)}\right)\cos\left(\frac{5\pi}{9(\Delta t/60)}\right)\right]\right\} Hz^2
\end{aligned}
\tag{24}
$$

where critical dependences on q, Δt, β, γ, and ℓ have been retained, and $A_1 = 1$ AU. We see that, except for the largest values of q ($\cong 0.3\ A_1$), the second term is not important. The value of the last bracket is +0.56 for $\Delta t = 60$ sec and -0.50 for $\Delta t = 10$ sec.

What do observations tell us about the remaining free parameters β, γ, ℓ, in Eq. (24)? Many observations of SWDF (Refs. 2-6) find that $3.4 \lesssim \beta \lesssim 4.0$. It will be shown below that DN observations seem to further restrict this range. Reference 5 finds that SWDF power spectra decline as $q^{-2.6\pm0.4}$ ($\gamma = -0.2 \pm 0.2$) for Viking S-X Doppler data from August to December 1976. Reference 7 finds DN $\propto q^{-2.6}$ for a longer span of Viking data. Reference 6 interprets this dependence of DN as $\gamma = +0.3$, $\ell = 1$. Such an interpretation is compatible with Eq. (24); however, several points should be made in this regard. First, the uncertainty of γ (unstated by Ref. 6) is as large as the value, and systematic effects will tend to lead to the observed slow decline; so any specification of parameters must be viewed with caution. Second, Ref. 3 finds $\gamma \cong 0$ for the decline of the average density. Third, there is no support in the literature for a strong dependence of the *outer scale* on heliocentric distance. Fourth, the acceleration of the SW near the sun has been ignored in this analysis and will tend to produce the observed slow falloff (DN $\propto (\delta Nxv)^2$). Finally, the amplitude of Eq. (24) would not agree with observations for $\gamma = +0.3$, $\ell = 1$. Thus, for the remainder of the discussion we take $\ell = \gamma = 0$.

The fact that Ref. 4 is able to determine the dependence of DN on sample time indicates that $\beta < 4$. Reference 4 finds DN $\propto \Delta t^{-0.6}$ (for the squared quantity used in this article), implying $\beta \cong 3.4$. A numerical evaluation of Eq. (24) shows that for such a low value of β, the third term is much less than the first so that no sample time effect would be seen. On the other hand, for $\beta \gtrsim 3.8$, the third "correction" term would be larger than the first. DN observations then lead to $3.5 < \beta \leqslant 3.7$. These are fairly tight limits on an important parameter of SWDF and are in good accord with many other measurements (Refs. 2–6).

As we have now fixed all the parameters in Eq. (24), we need to check that it gives reasonable results for DN. With the parameters chosen and for 60-sec sample times only the first term is important, so we have

$$\mathrm{DN}\,(t_1, \Delta t, T_M) \cong F_1(t_1) \left(\frac{4.47 \times 10^{-3}\, (\delta N_1/5)^2\, (v_0/400)^2}{(q/0.1 A_1)^3} \right) (0.589)\ \mathrm{Hz}^2$$

$$= 2.6 \times 10^{-3} \left(\frac{F_1(t_1)\, (\delta N_1/5)^2\, (v_0/400)^2}{(q/0.1 A_1)^3} \right) \mathrm{Hz}^2 \qquad (25)$$

Figure 2 is taken from Ref. 7, and points evaluated from Eq. (25) are plotted with the data and the model of Ref. 7. The agreement is quite good considering that nominal values of δN_1, v_0, $\gamma(=0)$, $F_1(t_1) = 1$ were used. It seems likely that with large amounts of data these parameters could be refined by fits to DN. Thus, DN could be a useful radio science tool putting limits on $(\delta N_1 \times v_0)$, β, and γ, and the residuals from such a model would provide estimates of $F_1(t_1)$ for correlation with solar features.

The relationship between solar wind density fluctuations and Doppler noise has been derived. It is shown that with nominal, well-known values for SW parameters the agreement between theory and observation is quite good.

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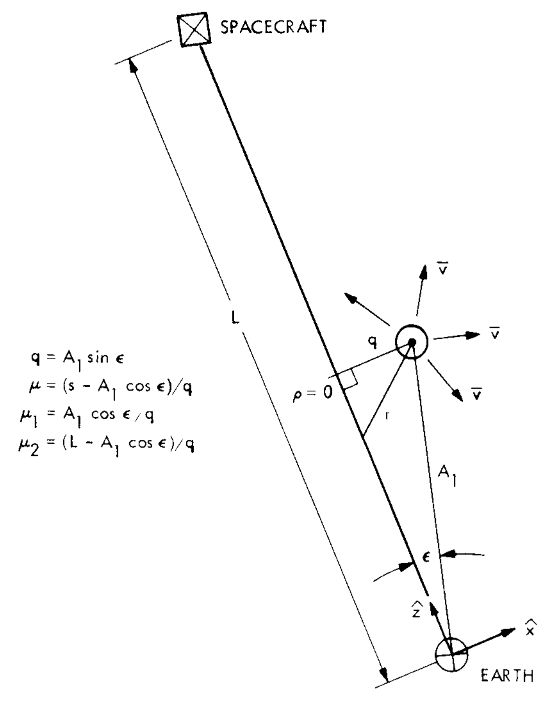

Fig. 1. Earth — spacecraft geometry

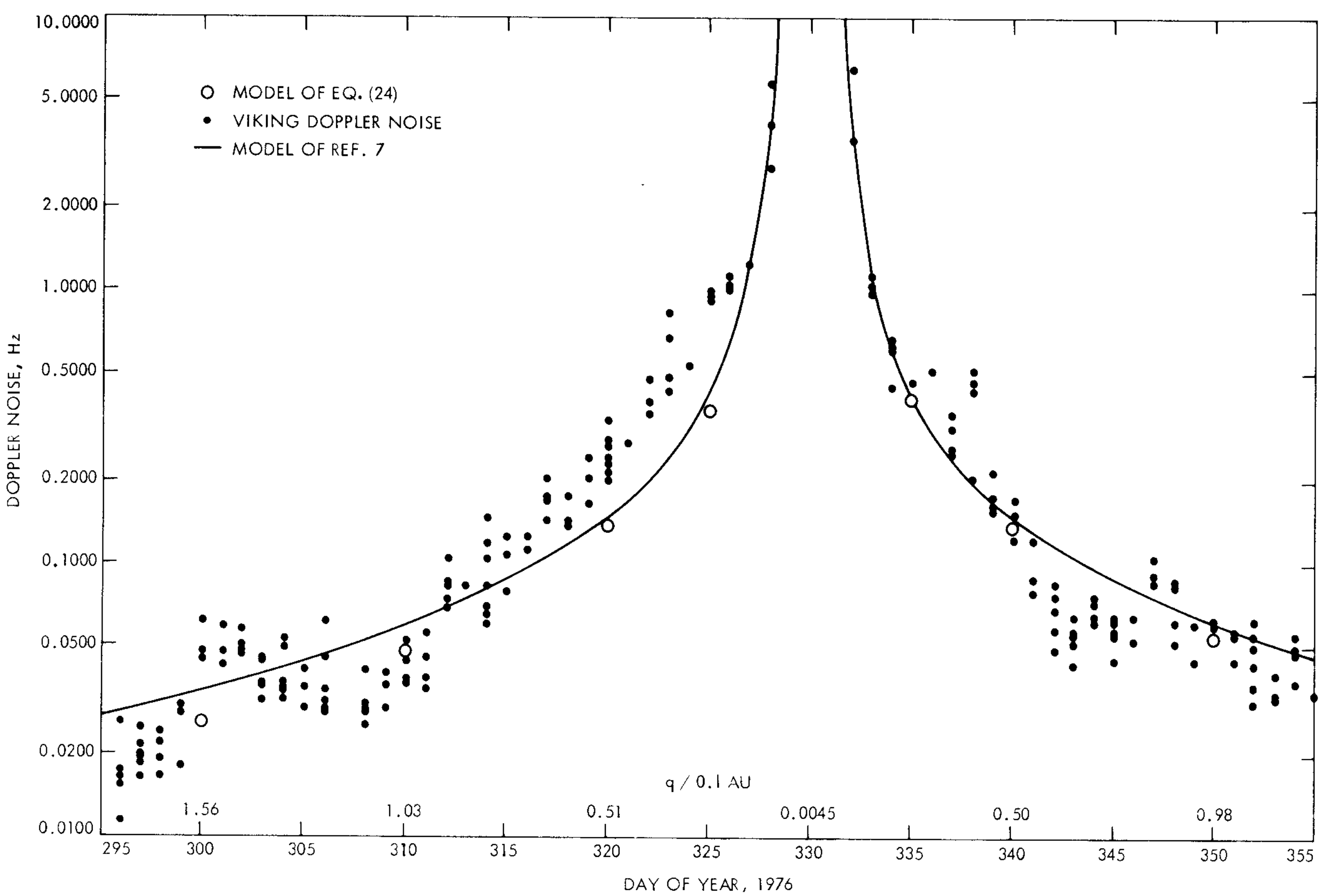

Fig. 2. Viking Doppler noise and the ISEDC model vs. day of year (295 to 355) (from Ref. 7)