Use of A Priori Statistics to Minimize Acquisition Time for RFI Immune Spread Spectrum Systems[1]

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A theory is given which allows one to obtain the optimum acquisition sweep strategy of a PN code despreader when the a priori probability density function is not uniform. This theory has application to psuedo noise spread spectrum systems which could be utilized in the DSN to combat Radio Frequency Interference (RFI). In a sample case, when the a priori probability density function is Gaussian, the acquisition time is reduced by about 41% compared to a "uniform sweep" approach.

I. Introduction

The acquisition circuitry of a despreader (a PN code acquisition and tracking system) is commonly designed so that complete passes are made across the code range uncertainty, as shown in Fig. 1a, during the initial search for the code epoch. This search, which is commonly implemented by retarding one half a chip at a time, then integrating and comparing to a threshold (Fig. 2), continues until the signal is acquired. This scheme is efficient when the a priori location of the signal in the uncertainty region has a uniform probability density function. However, when the a priori density function is peaked, it is more likely to find the signal in the peaked region than elsewhere, so the full sweep approach may not be the best one.

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This article is concerned with a method that allows one to determine the optimum sweep pattern to minimize the acquisition time, while achieving a required probability of signal detection, for a given a priori probability density of the signal location. The calculation is carried out for a Gaussian a priori signal location probability density function as illustrated in Fig. 1b. However, the approach is general so that it can be applied to any given a priori signal location probability density function.

The basis of this method relies on the fact that any meaningful statistics (Ref. 1, for example) of acquisition time, which is the time required to search the code until acquisition, depends directly upon the number of chips (code symbols) to be searched. Therefore searching where there is the greatest likelihood that the signal will be found first reduces the total number of positions to be searched.

II. Probability of Detection After N Sweeps

Consider a symmetric search centered at the mean of a symmetric, unimodal, a priori probability density function as shown in Fig. 1c for the case of $N = 3$ sweeps. Denote Q_N as the probability of acquiring at the end of N sweeps. Typically Q_N would be 0.9 or 0.5, for example. Let $L_1, L_2, L_3 \ldots, L_N$ denote the lengths (in number of cells) of the code range uncertainty to be searched during the N sweeps, and assume that $L_N \geqslant L_{N-1} \geqslant \ldots \geqslant L_1$. Let $p(x)$ be the a priori probability density of the location of the signal. Further, let S_i denote the event that the signal is not detected in any one of the first i sweeps over regions with lengths $L_1, L_2, \ldots, L_i$. Also, we shall use the notation S_0 to denote the event that the signal is not detected with zero sweeps, which is of course a sure event. It is clear that the conditional probability density of the signal location x, given that no sweep has yet been made, is equal simply to the a priori density $p(x)$, i.e.,

$$p(x|S_0) = p(x) \tag{1}$$

This density is illustrated in Fig. 3a. Suppose no signal is detected during the first sweep over L_1, i.e., suppose the event S_1 has occurred. The conditional density $p(x|S_1)$ is equal to, by use of Baye's rule,

$$p(x|S_1) = \frac{p(S_1|x)p(x)}{P(S_1)} \tag{2}$$

In (2) the conditional (or a posteriori) probability density $p(S_1|x)$, is clearly given by

$$p(S_1|x) = \begin{cases} 1 - P_d & \text{if } x \in L_1 \\ 1 & \text{if } x \notin L_1 \end{cases} \tag{3}$$

where P_d is the probability of detection given the signal is there. The notations $x \in L_1$ and $x \notin L_1$ denote the fact that the location of the signal is within the set L_1 or not in L_1, and $P(S_1)$ is the probability of the event S_1:

$$P(S_1) = 1 - P_d P(L_1) \tag{4}$$

where $P(L_1)$ denotes the probability that the signal location x is within the set L_1:

$$P(L_1) = \int_{-L_1/2}^{L_1/2} p(x)dx \tag{5}$$

Substituting (3), (4) into (2) we thus have

$$p(x|S_1) = \begin{cases} \dfrac{(1 - P_d)p(x)}{1 - P_d P(L_1)} & \text{if } x \in L_1 \\[2ex] \dfrac{P(x)}{1 - P_d P(L_1)} & \text{if } x \notin L_1 \end{cases} \tag{6}$$

This conditional density is illustrated in Fig. 3b. It is easy to show that

$$\int_{-\infty}^{\infty} P(x|S_1)dx = \frac{(1 - P_d)P(L_1)}{1 - P_d P(L_1)} + \frac{1 - P(L_1)}{1 - P_d P(L_1)} = 1 \tag{7}$$

The joint probability density of x and the event S_1 is thus given by (from Eqs. 2 and 6):

$$p(x, S_1) = p(x|S_1)P(S_1) = \begin{cases} (1 - P_d)p(x) & \text{if } x \in L_1 \\ p(x) & \text{if } x \notin L_1 \end{cases} \tag{8}$$

since $P(S_1) = 1 - P_d P(L_1)$. The joint probability of S_1 and $x \in L_1$ is given by

$$P(L_1, S_1) = \int_{-L_1/2}^{L_1/2} p(x, S_1)dx = (1 - P_d)P(L_1) \tag{9}$$

Since $L_2 \geqslant L_1$, the joint probability of S_1 and $x \in (L_2 - L_1)$ is given by (from Eq. 8)

$$\begin{aligned} p(L_2 - L_1, S_1) &= \int_{-L_2/2}^{-L_1/2} P(x, S_1)dx + \int_{L_1/2}^{L_2/2} P(x, S_1)dx \\ &= P(L_2 - L_1) = P(L_2) - P(L_1) \end{aligned} \tag{10}$$

Further, let P_i, $i = 1, 2, \ldots, N$ denote, respectively, the probabilities that the signal is acquired during the i^{th} sweep, but not in the first, second, $\ldots$, $(i-1)^{\text{th}}$ sweeps. If we can compute these P_i's, then Q_N is clearly equal to the sum

$$Q_N = P_1 + P_2 + P_3 + \ldots + P_N \tag{11}$$

It is clear that the probability of acquiring in the first sweep is given by

$$P_1 = P_d P(L_1) \tag{12}$$

The probability P_2 by definition, then, is equal to the joint probability

$$\begin{aligned} P_2 &= \text{Prob [acq. in second sweep, fail to acq. in first sweep]} \\ &= P_d P(L_2 - L_1, S_1) + P_d P(L_1, S_1) \\ &= P_d [P(L_2) - P(L_1)] + P_d(1 - P_d)P(L_1) \end{aligned} \tag{13}$$

In computing (13) we have used Eqs. (9) and (10). Following identical arguments one obtains the joint probability density

$$p(x, S_2) = \begin{cases} (1 - P_d)^2 p(x) & \text{if } x \in L_1 \\ (1 - P_d)P(x) & \text{if } x \in (L_2 - L_1) \\ p(x) & \text{if } x \notin L_2 \end{cases} \tag{14}$$

It is then easy to compute the a posteriori probability density function after two sweeps:

$$p(x|S_2) = \begin{cases} \dfrac{(1 - P_d)^2 p(x)}{P(S_2)} & x \in L_1 \\ \dfrac{(1 - P_d)p(x)}{P(S_2)} & x \notin L_1, x \in L_2 \\ \dfrac{p(x)}{P(S_2)} & x \notin L_2 \end{cases} \tag{15}$$

This density function is shown in Fig. 3c. From (14) one obtains, for $L_3 \geqslant L_2 \geqslant L_1$,

$$\begin{aligned} P_3 &= P_d P(L_3 - L_2, S_2) + P_d P(L_2 - L_1, S_2) + P_d P(L_1, S_2) \\ &= P_d [P(L_3) - P(L_2)] + P_d(1 - P_d) [P(L_2) - P(L_1)] \\ &\quad + P_d(1 - P_d)^2 P(L_1) \end{aligned} \tag{16}$$

In general, P_i can be written for $i = 1, 2, \ldots, N$ as:

$$P_i = P_d \sum_{n=1}^{i} (1 - P_d)^{i-n} [P(L_n) - P(L_{n-1})] \tag{17}$$

with $P(L_0) \equiv 0$. When (17) is substituted into (11) we obtain, after some simple algebra:

$$Q_N = \sum_{i=1}^{N} P_i = P_d (1 - P_d)^N \sum_{k=1}^{N} \frac{P(L_k)}{(1 - P_d)^k} \tag{18}$$

The fact that we can write P_i in the form of Eq. (17) is intuitively clear. We can argue through the case of P_3, as given in (16). The region of search during the third sweep can be divided into three nonoverlapping regions = $L_3 - L_2$, $L_2 - L_1$, and L_1. By the time the third sweep is initiated, the region L_1 has already been searched twice. Hence the joint probability that signal is detected in this region during the third sweep but not during the first two sweeps is $P_d(1 - P_d)^2 P(L_1)$. Similarly, the probability that signal is detected in the region $L_2 - L_1$ during the third sweep, but not during the first two sweeps, is $P_d(1 - P_d)$ $[P(L_2) - P(L_1)]$, since this region has only been searched once before the initiation of the third sweep. The region $L_3 - L_2$ has not yet been searched, and thus the probability of detecting the signal in this region is simply $P_d[P(L_3) - P(L_2)]$. P_3 is the probability of detecting the signal in any one of these three nonoverlapping regions during the third sweep, and is thus equal to the sum of these three terms, which is precisely the result of (15).

III. Optimum Symmetric Search Strategy

Suppose we design a search algorithm with search lengths $L_1, L_2, \ldots, L_N$ in the first N sweeps, and suppose the resulting probability of signal acquisition in these N sweeps is Q_N, which is computed in Section II as a function of the a priori probability density of the signal location, the number of sweeps N, and the search lengths. Denote the acquisition time that is required for the probability of signal acquisition to reach Q_N by T_{Q_N}. For example, $T_{0.9}$ is the time required to arrive at a Q_N of 0.9. The basis of the following optimization procedure is that, regardless the actual value of Q_N, the acquisition time T_{Q_N} is proportional to the number of chips searched, which is proportional to the total search sweep length

$$\sum_{i=1}^{N} L_i$$

Therefore our problem becomes: Determine the optimum search lengths $L_1, L_2, \ldots L_N$ so that Q_N equals the desired acquisition probability and

$$L_T = \sum_{i=1}^{N} L_i$$

is minimized, thereby minimizing our acquisition time. Our method of solution is to use the LaGrange Multiplier method. Let

$$F = P_d \sum_{k=1}^{N} P(L_k)(1 - P_d)^{N-k} - \lambda \sum_{k=1}^{N} L_k \tag{19}$$

where λ is the unknown LaGrange Multiplier. Up to this point the theory is quite general, the only requirement being that the a priori density be unimodal and symmetric and that $P(L_k)$ be differentiable. Since this problem was motivated by the desire to improve acquisition time for the spread spectrum receivers and since a reasonable estimate for the a priori location of the signal is Gaussian, we shall illustrate the theory by assuming that the a priori density function is Gaussian. Now with this assumption we have

$$P(L_i) = 2 \int_0^{L_i/2} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-t^2}{2\sigma^2}\right) dt = \mathrm{erf}\left(\frac{L_i}{2\sqrt{2}\sigma}\right) \tag{20}$$

Differentiating, we obtain

$$\frac{\partial F}{\partial L_i} = -\lambda + P_d(1 - P_d)^{N-i} \frac{\partial P(L_i)}{\partial L_i} = 0 \tag{21}$$

Since

$$\frac{\partial P(L_i)}{\partial L_i} = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{L_i^2}{8\sigma^2}\right) \tag{22}$$

we can solve (21) to obtain

$$l_i = 2\sqrt{2} \sqrt{\ln\left[\frac{1}{\lambda'(1 - P_d)^i}\right]} \tag{23}$$

where

$$\lambda' = \frac{\lambda\sqrt{2\pi}\sigma}{P_d(1 - P_d)^N} \tag{24}$$

$$l_i = \frac{L_i}{\sigma} \tag{25}$$

Substituting for l_i (the normalized chip search numbers) back into the equation for Q_N allows us in principle to solve for λ':

$$Q_N = 2P_d \sum_{k=1}^{N} (1 - P_d)^{N-k} \times \int_0^{\sqrt{2}\sqrt{\ln[1/\lambda'(1-P_d)^k]}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt \tag{26}$$

This equation appears nearly impossible to solve analytically; however it can be solved simply on a digital computer by trial and error, picking λ' so that Q_N equals the desired probability. This must be done for all N; of course in practice a few values of N will be sufficient. Before we discuss the results of the computer solutions let us consider the improvement over an unoptimized sweep.

IV. Uniform A Priori Density Sweep Strategy

The usual strategy for sweeping to obtain acquisition is to start at the end of the uncertainty region, where the range delay is minimum, and then retard the range in increments of typically one-half chip. By sweeping from the minimum delay to the maximum delay the chances of acquiring a multipath signal are diminished. If the probability of detection, given that the received code and the reference code are aligned, is given by P_d and if the a priori probability density function is uniformly distributed, then the cumulative probability of acquisition is as shown in Fig. 4. If, for example, a probability of 0.5 is chosen as the desired probability of acquisition, then the curve could be read off of the abscissa and the associated time, denoted by $T_{0.5}$, would be the time it takes to acquire with a probability of 0.5.

A measure of the improvement of the optimized scheme over the uniform sweep scheme can be measured as follows. Denote T_Q^U as the time to acquire with a probability of Q using the uniform sweep approach. Next, let T_Q^O denote the time to acquire with the optimized sweep. Then the improvement factor of the optimized sweep over the uniform sweep is given by

$$r_Q = \frac{T_Q^U}{T_Q^O} \tag{27}$$

The acquisition time is then $T_Q^U r_Q^{-1} = T_Q^O$. Clearly $r_Q \geqslant 1$ since unity is achieved with the uniform sweep strategy, and therefore the method never increases acquisition time. Our result is for $Q = 0.5$ as a representative case.

V. Numerical Results

As discussed in Section III, to obtain the optimum sweep lengths we have to first solve for λ' from (26), from which the optimum sweep lengths are given, for $i = 1, 2, \ldots, N$, by

$$L_i = 2\sqrt{2}\,\sigma\sqrt{\ln\frac{1}{\lambda'} - i\ln(1 - P_d)} \tag{28}$$

As evident from (28), the solution of λ' must satisfy the condition:

$$\ln\frac{1}{\lambda'} - \ln(1 - P_d) \geqslant 0 \tag{29}$$

in order for L_i to be meaningful. Thus it is convenient to define the left-hand side of (29) to be of the form

$$\ln\frac{1}{\lambda'} - \ln(1 - P_d) \equiv e^x \tag{30}$$

so that instead of solving (26) for λ' we can solve for the root of the following equation:

$$f(x) = P_d \sum_{k=1}^{N} (1 - P_d)^{N-k} \times \operatorname{erf}\sqrt{e^x - (k-1)\ln(1 - P_d)} - Q_N \tag{31}$$

which can be obtained numerically by the Newton's iterative solution, computing successive iterations according to

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \tag{32}$$

where the derivative of $f(x)$ can be evaluated from (31) to be

$$f'(x) = P_d \sum_{k=1}^{N} (1 - P_d)^{N-k} \frac{e^{x - [e^x - (k-1)\ln(1-P_d)]}}{\sqrt{\pi[e^x - (k-1)\ln(1 - P_d)]}} \tag{33}$$

Before we illustrate the procedure by numerical example a discussion on the choice of N is in order. First we note that in order for a given N to give finite optimum sweep lengths, N must satisfy

$$Q_N \leqslant P_d \sum_{k=1}^{N} (1 - P_d)^{N-k} \tag{34}$$

since the right-hand side of (34) is the probability of signal acquisition with infinite sweep lengths (i.e., for the case $x = \infty$). Next, an upper bound on N can also be given by noting the fact that the smallest value of x is $-\infty$, so that N must satisfy, from (31),

$$Q_N \geqslant P_d \sum_{k=1}^{N} (1 - P_d)^{N-k} \operatorname{erf}\sqrt{(k-1)\ln\frac{1}{1 - P_d}} \tag{35}$$

Equations (34) and (35) give lower and upper bounds of N, for given values of P_d and Q_N, within which optimum solutions of sweep lengths are feasible. The following examples illustrate the results obtained by this optimum search technique as compared to the uniform search algorithm.

Example 1

Suppose $P_d = 0.25$ and the desired $Q_N = 0.5$. The admissible number of sweeps N for this case is 3 and 4. The sweep lengths in terms of σ (standard deviation of the signal location probability density) are shown in Table 1. Also shown in Table 1 is the ratio between the uniform search acquisition time and the acquisition time using this optimum procedure. The search region in the uniform sweeps is assumed to be $\pm 3\sigma$. The four-sweep search gives an improvement ratio of 1.73 over the uniform sweep search algorithm, i.e., a reduction acquisition time of 41%.

Example 2

Suppose $P_d = 0.5$ and $Q_N = 0.9$. It is found that only $N = 4$ or $N = 5$ are admissible since solutions do not exist for all other N. Table 2 gives the sweep lengths and the corresponding improvement ratios over uniform searches.

As illustrated by these two examples the improvement factor of this optimum search strategy over the uniform search varies according to P_d, Q_N and N. Nevertheless, the improvement factor is always $\geqslant 1$.

VI. Conclusions

We have presented a method that can be used to optimize (minimize) the acquisition time for a PN-type spread spectrum system when the a priori probability density function is not uniform. Specifically we have calculated, for an assumed a priori Gaussian density function, that, when the 0.5 probability acquisition time was used as a measure, the acquisition time was reduced by 40% for a cell detection probability of 0.25 when three sweeps were used. For the same parameters and with four sweeps the acquisition time was reduced by 41%. When the acquisition time probability was set to 0.9 instead of 0.5 the reduction was 25% of the uniform sweep time.

This technique has application to the DSN if psuedo-noise spread spectrum systems are utilized to combat RFI.

References

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2. M. Holmes, Private Communication, Oct. 1977.
3. C. Gumacos "Analysis of an Optimum Sync Search Procedure," *IEEE Trans. on Communications*, pp. 89–99, March 1963.

Table 1. $P_d = 0.25$, $Q_N = 0.5$

	Opt. search with 3 sweeps	Opt. search with 4 sweeps
L_i/σ	2.54	1.19
	2.96	1.93
	3.33	2.46
		2.89
Improvement ratio over uniform search	1.66	1.73

Table 2. $P_d = 0.5$, $Q_N = 0.9$

	Opt. search with 4 sweeps	Opt. search with 5 sweeps
L_i/σ	2.63	0.56
	3.53	2.42
	4.24	3.38
	4.85	4.12
		4.74
Improvement ratio over uniform search	1.34	1.34

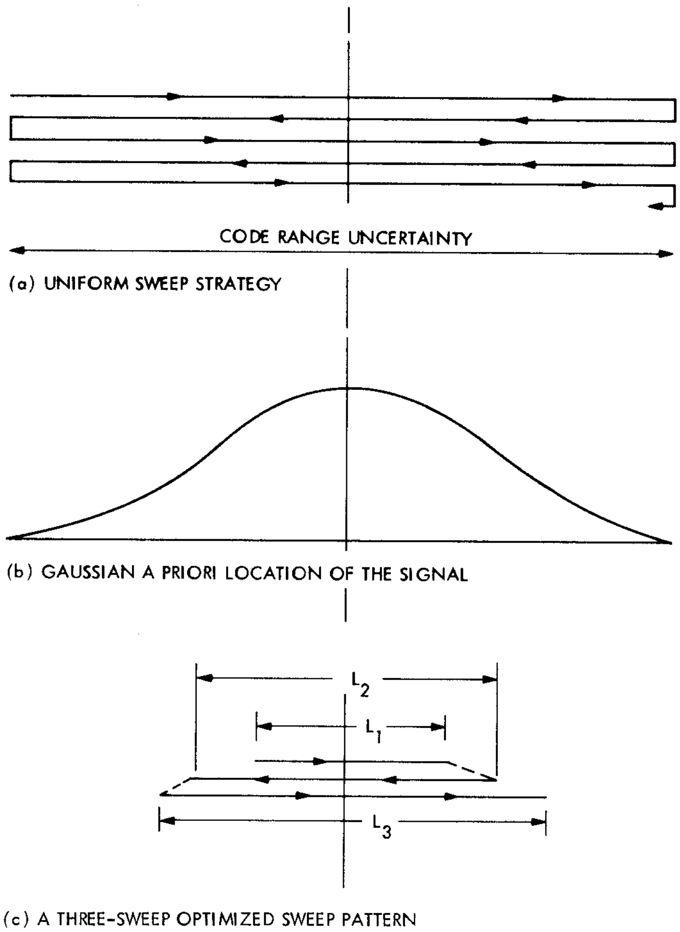

Fig. 1. A priori signal location and search strategies

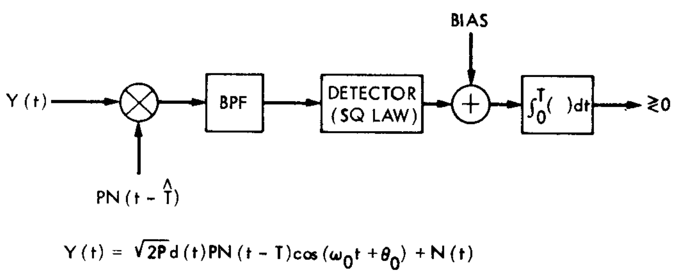

Fig. 2. Typical simplified fixed dwell time acquisition system

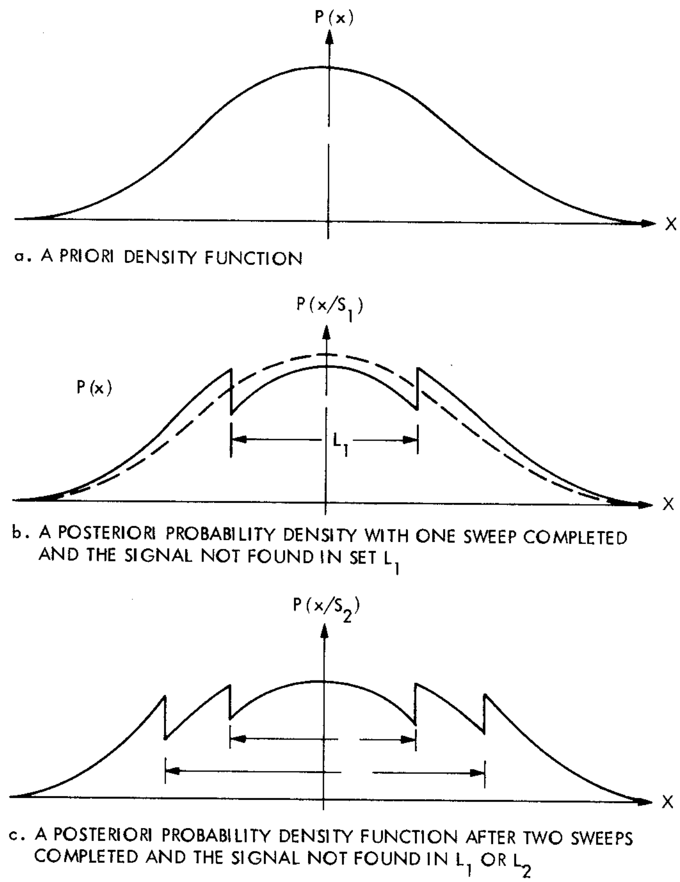

Fig. 3. Evolution of the a priori density function

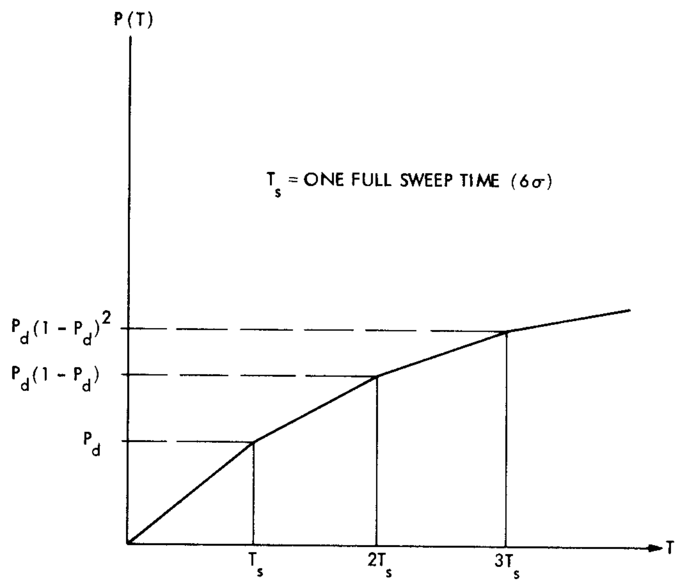

Fig. 4. Cumulative probability of acquiring in time *T* for the uniform sweep scheme