Spectral Shaping Without Subcarriers

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For proper operation of the phase lock loop which tracks a carrier, it is important to minimize the spectral energy at frequencies near the carrier. A traditional method is to modulate the data onto a subcarrier in such a way that there is little energy near DC. The resulting signal is then used to modulate the carrier. The problem with such a scheme is that the total bandwidth is much larger than necessary to transmit the data. This paper proposes and analyzes a simpler scheme that increases the data bandwidth by a very small fraction, yet reduces the energy near DC to nearly zero.

I. Introduction

We will do our analysis at baseband and begin with a statistic which will allow us to estimate the energy of a process between the frequencies $-B$ and $+B$.

For a stationary process $X(t)$ with spectral density $S_X(f)$, define a new process by

$$Y_B(t) = \frac{1}{T}\int_0^T X(t-\tau)\,d\tau$$

where $T = 1/2B$. Then the spectral density of Y is

$$S_Y(f) = S_X(f)\left(\frac{\sin \pi f T}{\pi f T}\right)^2 \tag{1}$$

and the power in Y is

$$E\{Y^2(t)\} = \int_{-\infty}^{\infty} S_Y(f)\,df = \int_{-\infty}^{\infty} S_X(f)\left[\frac{\sin(\pi f T)}{\pi f T}\right]^2 df \tag{2}$$

Now

$$\left[\frac{\sin(\pi f T)}{\pi f T}\right]^2 \geqslant \begin{cases} \left(\frac{2}{\pi}\right)^2, \text{for } |f| \leqslant \frac{1}{2T} = B \\ 0, \text{otherwise} \end{cases} \tag{3}$$

So Eq. (2) implies that

$$E\{Y^2(t)\} \geqslant \left(\frac{2}{\pi}\right)^2 \int_{-B}^{B} S_X(f)\,df$$

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or

$$\int_{-B}^{B} S_X(f)\,df \leqslant \left(\frac{\pi}{2}\right)^2 E\{Y^2(t)\} \qquad (4)$$

Thus, the second moment of $Y(t)$ gives an estimate of the amount of energy in X between frequencies $-B$ and B.

For an application of this statistic, consider the process $X(t)$ which is +1 or -1 on each interval $[nT_0, (n+1)T_0]$. Assume the values on different intervals are independent and have probability 1/2.

Then

$$E\{Y^2(t)\} = E\left\{\frac{1}{T}\int_0^t X(t)\,dt\right\}^2$$

$$= E\left\{\frac{T_0}{T}\sum_{n=1}^{T/T_0} X((n-1)T_0)\right\}^2$$

$$= \left(\frac{T_0}{T}\right)^2 \frac{T}{T_0} = \frac{T_0}{T} = 2BT_0 \qquad (5)$$

and the bound is

$$\int_{-B}^{B} S_X(f)\,df \leqslant \left(\frac{\pi}{2}\right)^2 2BT_0 \qquad (6)$$

Of course, for the process

$$S_X(f) = \left(\frac{\sin(\pi T_0 f)}{\pi f}\right)^2$$

and for small B the energy between $-B$ and B is $2BT_0$. The factor $(\pi/2)^2$ indicates the looseness of the bound.

The signal design problem is to encode the data into a signal $X(t)$ such that $E\{Y^2(t)\}$ is small.

II. Proposed Solution

The proposed solution is to expand the data stream by inserting a redundant bit every L^{th} bit, the value of the bit being chosen to bring the total number of +1's and -1's into balance.

More precisely (see Figs. 1 and 2):

Let X_n be a sequence of ±1's, defined below

Define $X(t) = X_n$ for $t \in [(n-1)T_0, nT_0]$

$$\text{Define } C_n = \sum_{m=1}^{n} X_m \qquad (7)$$

Let L be an even integer

Then X_n is defined as follows: When n is not a multiple of L, X_n is a data bit (±1). When n is a multiple of L, then

$$X_n = -\operatorname{sgn}[C_{n-1}]$$

(Since L is even, $n - 1$ is odd. Then, from its definition, C_{n-1} must be odd and cannot be zero.)

The derivation of a bound on the power between $-B$ and B is given below, resulting in Eq. (15). For non-redundant data (flat random data) the amount of power is $2T_0B$, so the factor $[T_0B\,(3\pi^2/8)\,L^2]$ indicates what the gain has been when a redundancy of $1/L$ has been inserted. In particular, when T_0 = 1/30 MHz and B = 1 kHz, if the value of L is 30, then the factor is 1/8 or a gain of 9 dB. If L = 10, then the gain is 18.5 dB.

III. Analysis

It is clear from the definitions that, when T/T_0 is an integer,

$$Y(nT_0) = \left(C_n - C_{n-T/T_0}\right)\frac{T_0}{T}$$

Therefore, the second moments of $\{C_n\}$ must be studied. We will assume n so large that the stationary distributions have been obtained so that

$$E\{C_n^2\} = E\left\{C_{n-T/T_0}^2\right\}$$

In the case that the data bits are independent it can be shown that

$$E\left\{C_n\, C_{n-T/T_0}\right\} \geqslant 0$$

From this we have

$$E\{Y^2(nT_0)\} \leqslant \frac{2T_0^2}{T^2} E\{C_n^2\} \qquad (8)$$

To analyze C_n, let

$$Z_k = \sum_{n=kL+1}^{kL+L-1} X_n$$

That is, Z_k is the sum of $L - 1$ consecutive data bits. For most of the analysis we will assume only that the odd moments of Z_k are 0, but for the best result we must also assume that the X_n contributing to Z_k are mutually independent.

From the definition of X_n we have

$$X_{(k+1)L} = - \operatorname{sgn} [C_{kL} + Z_k] \tag{9}$$

and

$$C_{(k+1)L} = C_{kL} + Z_k - \operatorname{sgn} [C_{kL} + Z_k]$$

Multiplying through by $\operatorname{sgn} [C_{kL} + Z_k]$ gives

$$C_{(k+1)L} \operatorname{sgn} [C_{kl} + Z_k] = |C_{kL} + Z_k| - 1 \tag{10}$$

Since subtracting 1 from a positive odd integer cannot change the sign, the left side of Eq. (10) must be non-negative, and we have

$$|C_{(k+1)L}| + 1 = |C_{kL} + Z_k| \tag{11}$$

Next define

$$\mu_j = E\{Z_k^j\}$$

and

$$M_j = E\{|C_{kL}|^j\}$$

Then from Eq. (11) and the assumption that $\mu_k = 0$ for odd k, we get

$$\begin{aligned} M_2 + 2M_1 + 1 &= M_2 + \mu_2 \\ M_4 + 4M_3 + 6M_2 + 4M_1 + 1 &= M_4 + 6M_2\,\mu_2 + \mu_4 \end{aligned} \tag{12}$$

From these equations and the Swartz inequality for positive random variables $M_1 M_3 \geqslant M_2^2$, the following inequality can be derived:

$$\left[M_2 - \frac{3}{8}(\mu_2 - 1)^2\right]^2 \leqslant$$

$$(\mu_2 - 1)\left[\frac{\mu_4 - 4}{8} - \frac{\mu_2 - 1}{4} + \frac{9}{64}(\mu - 1)^3\right]$$

or

$$M_2 \leqslant \frac{3}{8}(\mu_2 - 1)^2$$

$$+ \frac{1}{8}\sqrt{(\mu_2 - 1\,[8\mu_4 - 16 - 16\mu_2 + 9(\mu_2 - 1)^3]} \tag{13}$$

When the data bits are independent, $\mu_2 = L - 2$, and Eq. (13) implies

$$M_2 \leqslant \frac{3}{4} L^2 \tag{14}$$

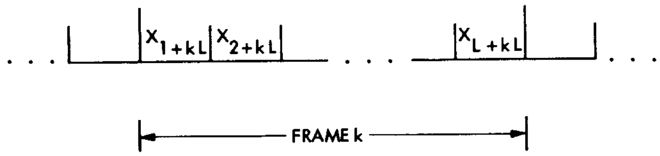
This combined with Eqs. (4) and (8) give

$$\int_{-B}^{B} S_X(f)df \leqslant \left(\frac{\pi}{2}\right)^2 \left(2T_0 B\right)^2 2\,\frac{3}{4} L^2$$

$$= \left(2T_0 B\right)^2 \left[\frac{3\pi^2}{8} L^2\right]$$

or

$$\int_{-B}^{B} S_X(f)df \leqslant 2T_0 B \left[T_0 B\,\frac{3\pi^2}{8}\,L^2\right] \tag{15}$$



$$M_t = \text{DATA BIT } (\pm 1)$$

$$X_{i+kL} = M_{i+k(L-1)}, \quad i=1, \ldots, L-1$$

$$X_{i+kL} = -\text{sgn}\left[\sum_{t=1}^{L+kL-1} X_t\right]$$

Fig. 1. Frame layout for data and redundant bit

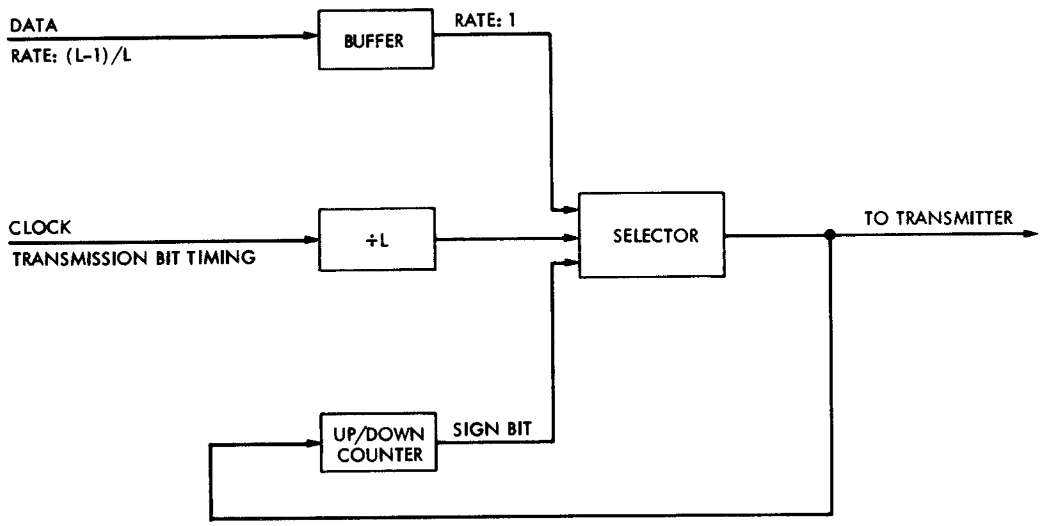


Fig. 2. Circuit for redundant bit computation and insertion