Some Effects of Quantization on a Noiseless Phase-Locked Loop

C. A. Greenhall
Communications Systems Research Section

If the VCO of a phase-locked receiver is to be replaced by a digitally programmed synthesizer, the phase error signal must be sampled and quantized. We investigate effects of quantizing after the loop filter (frequency quantization) or before (phase error quantization). Constant Doppler or Doppler rate noiseless inputs are assumed. The main result gives the phase jitter due to frequency quantization for a Doppler-rate input. By itself, however, frequency quantization is impractical because it makes the loop dynamic range too small.

I. Introduction

This article contributes to the effort to develop a partly-digital receiver for the DSN. In this connection, it has been suggested that the voltage-controlled oscillator (VCO) of the carrier tracking loop be replaced by a number-controlled oscillator (NCO), an example of which is the Digiphase® Frequency Synthesizer. This synthesizer operates between 40 and 51 MHz; the frequency decades down to 10^{-6} Hz can be programmed synchronously during a specified portion of the 10 μs clock period. Thus, the filtered, digitized phase error signal, plus a bias derived from the Doppler prediction, would be used to program the synthesizer, whose output would be multiplied up to the local oscillator frequency (RF − 1st IF). Some effort (Ref. 1), later abandoned, has already been expended toward development of a third-order tracking loop with this synthesizer as NCO.

Two advantages of this arrangement are predicted to be: (1) the phase jitter produced by the synthesizer (and partly tracked out by the loop) should be less than that produced by an analog VCO because the synthesizer phase is controlled by a wideband loop that tends to track out slow phase variations; and (2) the synthesizer input, being a digital signal, would yield directly a convenient estimate of Doppler frequency. It would be unnecessary to extract Doppler data by mixing analog signals from the receiver and the exciter. The success of this approach depends on the truth of (1), for if the synthesizer exhibits slow, unbounded phase variations, then the integrated digital Doppler estimate might vary unacceptably from the true local oscillator phase. This question may be treated in a future article.

In order to program an NCO in the loop, the phase error must first be sampled and quantized. One might do this either before or after the loop filter. This article investigates both possibilities separately. Assumptions are (1) linearized second order loop, active form of loop filter, with special attention paid to the 1-Hz bandwidth setting of the DSN Block IV Receiver; (2) strong signal, at least 40 dB above margin; (3) all noises absent, including oscillator jitter; (4) quadratic phase input $\theta = \theta_0 + \omega_0 t + 1/2\,\lambda t^2$; and (5) zero sampling interval; time is not quantized.

We show how the loop approaches lock from conditions close to lock, and we derive the steady state behavior of phase and frequency.

II. Loop Model

Figure 1 shows a block diagram of the mathematical model. The phase detector output is p and its quantized version is q. The unquantized oscillator frequency is y and its quantized version is z. The phase detector gain αK_d includes the limiter output signal amplitude of the real receiver (the limiter suppression factor α is 1 in a strong-signal situation). To simulate the Block IV 1-Hz loop at S-band with a 40-51 MHz oscillator, we set $\tau_2 = 1.5s$, $\tau_1 = 33919s$, $K_d = 5$ volt/rad, $g = 4$, $K_v/2\pi = 96$ Hz/volt, and $M = 40$. The loop gain is $\alpha K = \alpha K_d g K_v M = 4.83 \times 10^5\ s^{-1}$ and the loop parameter defined by

$$r = \frac{\alpha K \tau_2^2}{\tau_1}$$

equals 32. This value also holds for the 10 Hz and 100 Hz bandwidth settings. For the 3, 30, and 300 Hz settings, the value of r is 18.55. At threshold, $r = 2$ for all settings.

The quantizing function $Q_a(x)$ rounds x to the nearest odd multiple of a. The distance between quantizer levels is $2a$. The even multiples of a are "decision levels." To make mathematical sense out of this model, we assume a small "dead zone" of width 2ϵ around each decision level. For example, if $x(t)$ is a signal that starts at a and decreases, then $Q_a(x)$, initially at a, flips to $-a$ only when x reaches $-\epsilon$. Then $Q_a(x)$ does not return to $+a$ until x returns to $+\epsilon$. Results will be obtained by letting ϵ tend to 0.

III. Oscillator Frequency Quantization

Let the phase quantizer be removed from the loop, whose equations are then

$$y = \frac{\alpha K_d g K_v}{2\pi\tau_1}\left(\tau_2\phi + \int \phi\, dt\right)$$

$$z = Q_{\Delta y}(y)$$

$$\phi = \theta - 2\pi M \int z\, dt$$

The effective quantization interval of the RF radian frequency $\omega = 2\pi M z$ is

$$2\Delta\omega = 4\pi M \Delta y$$

In terms of the dimensionless variables

$$t^* = \frac{t}{\tau_2},\quad \theta^* = \frac{r}{\tau_2 \Delta\omega}\theta,\quad \phi^* = \frac{r}{\tau_2\Delta\omega}\phi$$

$$y^* = \frac{r}{\Delta y}y,\quad z^* = \frac{r}{\Delta y}z$$

the equations become

$$y^* = r\left(\phi^* + \int \phi^*\, dt^*\right) \tag{1}$$

$$z^* = Q_r(y^*) \tag{2}$$

$$\phi^* = \theta^* - \int z^*\, dt^* \tag{3}$$

At the risk of confusion, we shall drop the asterisks from the dimensionless variables for a while; when results are stated we shall make clear what variables we are talking about. Equations (1) and (3) really mean

$$y(t) - y(t_0) = r\left(\phi(t) - \phi(t_0) + \int_{t_0}^{t} \phi(s)\, ds\right)$$

$$\phi(t) - \phi(t_0) = \theta(t) - \theta(t_0) - \int_{t_0}^{t} z(s)\, ds$$

for any t_0 and t. Initial conditions for Eqs. (1) – (3) are specified by $\phi(t_0)$ and $y(t_0)$.

We shall assume that the quantizer Q_r has a dead area $[-\epsilon, \epsilon]$.

A. Constant Doppler Input

Let $\theta = \theta_0 + \omega_0 t$, where $|\omega_0| < r$. We show the loop pull-in behavior for sufficiently small $\phi(0)$ and $y(0)$. Without loss of generality we can assume $-\epsilon < y(0) < 2r$ and z is initially $+r$. For $t > 0$,

$$\phi(t) = \phi(0) + (\omega_o - r)t$$

$$y(t) = y(0) + r\left\{[\omega_o - r + \phi(0)]\, t + \frac{1}{2}(\omega_o - r)t^2\right\}$$

Let $\phi(0) < r - \omega_o$. Then y decreases until it reaches $-\epsilon$ at some time t_1. This makes z flip to $-r$. For $t > t_1$,

$$\phi(t) = \phi(t_1) + (\omega_o + r)(t - t_1)$$

$$y(t) = -\epsilon + r\left\{[\omega_o + r + \phi(t_1)]\,(t - t_1) + \frac{1}{2}(\omega_o + r)(t - t_1)^2\right\}$$

If $y(0)$ is small enough, then $\phi(t_1)$ is not too negative, so that we can assume $\omega_o + r + \phi(t_1) > 0$. Thus, y increases, and soon reaches $+\epsilon$. Then z flips back to $+r$, and so on. We see that y judders back and forth inside the dead zone $[-\epsilon, \epsilon]$, and z flips violently between $-r$ and $+r$. Meanwhile, what happens to the phase error ϕ? It satisfies the integral equation

$$\phi + \int \phi\, dt = \frac{y}{r} \tag{4}$$

For small ϵ, we can pretend $y = 0$ for $t > t_1$. Then the solution of Eq. (4) is $\phi(t) = \phi(t_1)\, e^{t_1 - t}$. We could, of course, compute the additional ϕ-disturbance due to the juddering of y.

If $\omega_o > r$ and $\phi(0)$, $y(0)$ are small, then whether $z(0)$ is r or $-r$, y floats upward, trying to make $\omega_o - z$ small. Eventually, the loop reaches a condition in which $y = 2kr$ (k is an integer), where $2kr$ is the nearest decision level to ω_o, and ϕ decays exponentially to 0. (We ignore the case where ω_o/r is an odd integer.)

We state this result in "real-world" units: *For a constant Doppler input $\theta = \theta_o + \omega_o t$ to the frequency-quantized loop, the unquantized frequency y eventually sticks at the nearest decision level $2k\Delta y$ to ω_o, the quantized frequency z flips rapidly between $(2k - 1)\Delta y$ and $(2k + 1)\Delta y$, and the phase error ϕ decays to 0 exponentially with time constant τ_2. The static phase error is* 0.

B. Constant Doppler Rate Input

We return to the use of dimensionless variables (unstarred). Let $\theta = \theta_o + \omega_o t + 1/2\,\lambda t^2$, $\lambda > 0$. The unquantized loop has a static phase error (SPE); the steady-state solution of Eqs. (1) and (3), with $z = y$, is $\phi = \lambda/r$, $y = \omega_o + \lambda t$. We are going to find a periodic solution of the quantized system; the period is necessarily $2r/\lambda$, the time it takes the Doppler to traverse one quantization interval. The idea is that for small λ the frequency y is normally stuck at a decision level of the quantizer and the phase error has decayed exponentially to 0. As $\dot{\theta} = \omega_o + \lambda t$ increases, though, there comes a time when y can no longer remain at that level; it must eventually float up to the next level in order to stay close to $\dot{\theta}$. During this *floating period*, ϕ is allowed to wander off. In fact, it exceeds the nominal SPE. During the subsequent *sticking period*, when y is stuck at the next decision level, ϕ again decays exponentially to 0. The sum of the lengths of the floating and sticking periods is $2r/\lambda$.

So, let us assume that at time 0, y is stuck at 0 (really, within $[-\epsilon, \epsilon]$). Let $\phi(0) = \phi_o$. For small $t > 0$, while z remains constant,

$$\phi(t) = \phi_o + (\omega_o - z)t + \frac{1}{2}\lambda t^2 \tag{5}$$

$$y(t) = r\left[(\omega_o + \phi_o - z)t + \frac{1}{2}(\lambda + \omega_o - z)t^2 + \frac{1}{6}\lambda t^3\right] \tag{6}$$

If $|\omega_o + \phi_o| < r$, then the coefficient of t in Eq. (6) has sign opposite to z, so y remains stuck at 0. In order to make the time origin the start of a floating period, we assume

$$\omega_o + \phi_o = r, \qquad \phi_o \leq \lambda \tag{7}$$

Then z immediately becomes $+r$ and stays there. For $t > 0$,

$$\phi(t) = \phi_o - \phi_o t + \frac{1}{2}\lambda t^2 \tag{8}$$

$$y(t) = r\left[\frac{1}{2}(\lambda - \phi_o)t^2 + \frac{1}{6}\lambda t^3\right] \tag{9}$$

The floating period ends when y reaches $2r$, the next decision level, where z must choose between r and $3r$. This happens at time t_1, the positive solution of

$$\frac{1}{6}\lambda t_1^3 + \frac{1}{2}(\lambda - \phi_o)\, t_1^2 - 2 = 0$$

For $t = t_1 + u$, u small and positive, while z is constant,

$$\phi(t_1 + u) = \phi(t_1) + (r - \phi_o + \lambda t_1 - z)u + \frac{1}{2}\lambda u^2 \tag{10}$$

$$y(t_1 + u) = r\{2 + [\phi(t_1) - \phi_o + \lambda t_1 + r - z]\, u + O(u^2)\} \tag{11}$$

where $\phi(t_1)$ is computed from Eq. (8). Assume that

$$\phi(t_1) - \phi_o + \lambda t_1 < 2r \tag{12}$$

(The left side equals $(\lambda - \phi_o)t_1 + 1/2\ \lambda t_1^2$, and hence is positive.) Then the coefficient of u in Eq. (11) has opposite sign to $z - 2r$. Therefore, y is stuck at $2r$. For $t \geq t_1$, ϕ satisfies

$$\phi(t) = \phi(t_1)e^{t_1 - t}$$

and the "microscopic" behavior of the system near t is determined from

$$\phi(t + u) = \phi(t) + (\omega_o + \lambda t - z)\,u + \frac{1}{2}\lambda u^2$$

$$y(t + u) = r\left\{2 + [\omega_o + \phi(t) + \lambda t - z]\,u + \frac{1}{2}(\lambda + \omega_o + \lambda t - z)u^2 + \frac{1}{6}\lambda u^3\right\}$$

where $z = 2r \pm r$. The sticking period ends and a new floating period begins when $\omega_o + \phi(t) + \lambda t$ reaches $3r$. Comparing with Eqs. (5) and (6), we see that the situation is just like the one at time 0 (except that now $y = 2r$) provided $\lambda t = 2r$, $\phi(t) = \phi_o$, that is to say, $\phi(2r/\lambda) = \phi_o$. Of course, this makes sense only if $t_1 \leq 2r/\lambda$. We are led to the following system of equations:

$$\frac{1}{6}\lambda t_1^3 + \frac{1}{2}(\lambda - \phi_o)t_1^2 = 2 \tag{13}$$

$$\phi_1 = \phi_o(1 - t_1) + \frac{1}{2}\lambda t_1^2 \tag{14}$$

$$\phi_o = \phi_1 \exp\left(t_1 - \frac{2r}{\lambda}\right) \tag{15}$$

A solution ϕ_o, ϕ_1, t_1 is called *admissible* if

$$0 < \phi_o \leq \lambda \tag{16}$$

$$0 < t_1 \leq \frac{2r}{\lambda} \tag{17}$$

$$\phi_1 - \phi_0 \leq 2r - \lambda t_1 \tag{18}$$

A special case should be exposed first. Equality in Eq. (17) for a solution is equivalent to equality in Eq. (18), and means that the sticking period has zero length. Working backwards by setting $t_1 = 2r/\lambda$ and $\phi_1 = \phi_o$, we solve Eq. (14) for ϕ_o and Eq. (13) for λ. We find that for $r \geq 4/3$ and $\lambda = \lambda_m$, where

$$\lambda_m = \frac{r^2}{2}\left[1 + \left(1 - \frac{4}{3r}\right)^{1/2}\right] \tag{19}$$

there is the admissible solution

$$\phi_o = r, \quad \phi_1 = r, \quad t_1 = 2r/\lambda_m$$

For other values of λ, Eqs. (13) – (15) can be solved numerically by iterating ϕ_o (Newton's method is used to solve Eq. (13) for t_1). Convergence is slow, but Steffensen's acceleration method (Ref. 2) yields the solution within about 8 decimal places using only 6 iterations of Eqs. (13) – (15).[1] For $\lambda < r/10$ we can take $\phi_o \approx 0$ and solve for t_1, ϕ_1 immediately. We have not proved existence or uniqueness of the admissible solution; indeed, some of the results are empirical.

Eqs. (13) – (15) were solved for $r = 32$, 18.55, 10, 5, and 2, and for $10^{-3}r \leq \lambda \leq r^2$. For $\lambda > \lambda_m$, λ_m given by Eq. (19), the solution became inadmissible. Evidently, for $\lambda \geq \lambda_m$ we enter a situation in which there is no sticking period. We have not looked at this yet.

For any admissible solution, the time average of $\phi(t)$ over a period $2r/\lambda$ is exactly the SPE λ/r. The maximum of ϕ is ϕ_1, and the minimum is

$$\phi_{\min} = \phi_o - \frac{\phi_o^2}{2\lambda} > 0$$

obtained by minimizing Eq. (8). Thus ϕ does fluctuate about the nominal SPE of λ/r. The peak-to-peak variation

$$V_\phi = \phi_1 - \phi_{\min}$$

is plotted vs. λ/r in Fig. 2 with r as a parameter (the variables are starred). These curves show that V_ϕ is an insensitive function of r and λ/r. For $r = 32$,

$$\max V_\phi = 1.72$$

[1] If one iterates t_1 instead of ϕ_o, then it is not necessary to solve a cubic equation. Unfortunately, this iteration is violently unstable at the admissible solution.

achieved for $\lambda/r = 0.6$ approximately. Figure 3 sketches $\phi(t)$ vs. t for $r = 32$, $\lambda/r = 0.5$ (the variables are starred). As λ decreases, V_ϕ decreases slowly. The width of the ϕ-pulse, however, gets narrow relative to the period $2r/\lambda$, so that the RMS phase jitter decreases faster than the peak-to-peak phase jitter.

We have actually been working with dimensionless variables. Here are some of the results in real-world units.

Let the loop with RF frequency quantization $2\Delta\omega$ have the input $\theta = \theta_o + \omega_o t + 1/2\ \lambda t^2$ where the Doppler rate λ satisfies

$$0 < \lambda < \frac{\Delta\omega}{\tau_2}\frac{r}{2}\left[1 + \left(1 - \frac{4}{3r}\right)^{1/2}\right] \tag{20}$$

The steady-state phase error fluctuates about the static phase error

$$\text{SPE} = \frac{\tau_2^2\lambda}{r} \tag{21}$$

periodically with period $2\Delta\omega/\lambda$. The peak-to-peak amplitude V_ϕ of this fluctuation satisfies

$$V_\phi \le 0.86\frac{\tau_2}{r}\,2\Delta\omega \qquad (r = 32) \tag{22}$$

$$V_\phi \le 0.83\frac{\tau_2}{r}\,2\Delta\omega \qquad (r = 18.55) \tag{23}$$

the maximum being achieved when

$$\lambda = 0.6\,\frac{\Delta\omega}{\tau_2} \tag{24}$$

approximately.

Assume a 1-Hz loop with a synthesizer programmed down to the 10^{-3} Hz decade. Then

$$2\Delta\nu = 0.001 \text{ Hz}$$

$$\frac{2\Delta\omega}{2\pi} = 0.04 \text{ Hz at S-band}$$

$$\max V_\phi = 0.0101 \text{ rad} = 0.58 \text{ deg}$$

achieved when

$$\lambda = 0.05 \text{ rad/s}^2 = 0.008 \text{ Hz/s}$$

$$\text{SPE} = 0.2 \text{ deg}$$

The maximum Doppler rate and SPE for which the present analysis is valid is obtained from Eqs. (20) and (21) as

$$\lambda_{max} = 0.422 \text{ Hz/s}$$

$$\text{SPE}_{max} = 10.7 \text{ deg}$$

The phase jitter V_ϕ is only 0.17 deg, a small ripple on the SPE.

At this point it appears that digitizing the phase error signal at the oscillator input is not practical. To achieve a 10^{-3} Hz quantization with a 12-bit A-D converter would mean a loop-controllable oscillator frequency range of only 4 Hz, whereas the Block IV VCO, in narrow mode, has a range of 960 Hz. It is probably necessary to digitize the phase error in front of the loop filter or even earlier.

IV. Phase Error Quantization

Returning to Fig. 1, we remove the frequency quantizer and install the phase quantizer. The loop equations are

$$p = \alpha K_d\,\phi, \qquad q = Q_{\Delta p}(p)$$

$$y = \frac{gK_v}{2\pi\tau_1}\left(\tau_2 q + \int q\,dt\right)$$

$$\phi = \theta - 2\pi M\int y\,dt$$

The effective quantization interval for ϕ is

$$2\Delta\phi = \frac{2\Delta p}{\alpha K_d}. \tag{25}$$

Appropriate dimensionless variables are now

$$t^* = \frac{t}{\tau_2}, \qquad \theta^* = \frac{\theta}{\Delta\phi}, \qquad \phi^* = \frac{\phi}{\Delta\phi}$$

$$q^* = \frac{q}{\Delta p}, \qquad y^* = \frac{2\pi M\tau_2}{\Delta\phi}\,y$$

in terms of which the loop equations become

$$q^* = Q_1(\phi^*), \tag{26}$$

$$y^* = r\left(q^* + \int q^* \, dt^*\right) \tag{27}$$

$$\phi^* = \theta^* - \int y^* \, dt^* \tag{28}$$

As before, we shall drop the asterisks from these variables during the analysis.

The quantization interval of q is 2. As before, we assume that Q_1 has a dead zone of width 2ϵ about each decision level $2k$. Implicit in this model is the decision to make 0 a decision level, not a quantization level. Eq. (27) says that if q jumps by ± 2 then y jumps by $\pm 2r$. If q is a constant, say $2k + 1$, on the interval $[t_1, t_2]$, then

$$y(t_2) - y(t_1) = r(t_2 - t_1)(2k + 1).$$

Let $\theta = \theta_o + \omega_o t + 1/2 \, \lambda t^2$, where we now allow λ to be 0. The unquantized loop has a SPE $\phi = \lambda/r$. In order for the quantized phase error q to be λ/r in some average sense, it must happen that ϕ gets stuck at the closest decision level $2k$ to λ/r, while q jumps between $2k + 1$ and $2k - 1$, bracketing λ/r. To show how this happens, it is convenient to define some more variables

$$\phi' = \phi - \frac{\lambda}{r}, \qquad q' = q - \frac{\lambda}{r},$$

$$y' = y - \omega_o - \lambda t$$

(we shall *not* drop the primes from these variables). They satisfy

$$q' = Q'(\phi') \tag{29}$$

$$y' = r\left(q' + \int q' dt\right) \tag{30}$$

$$\phi' = -\int y' dt \tag{31}$$

where[2]

$$Q'(\phi') = Q_1\left(\phi' + \frac{\lambda}{r}\right) - \frac{\lambda}{r}$$

is just an offset quantizer whose decision level m lying closest to the origin satisfies

$$-1 \leq m \leq 1, \qquad m = -\frac{\lambda}{r} \pmod 2.$$

Assume that λ/r is not an odd integer, so that m is unique and $-1 < m < 1$. We shall show merely that if $\phi'(0) - m$ and $y'(0)$ are small enough, then ϕ' eventually sticks at m. Without loss of generality we can assume $\phi'(0) > m - \epsilon$ and $q'(0) = m + 1$ (rather than $m - 1$). For $t > 0$,

$$y'(t) = y'(0) + r(m + 1)t \tag{32}$$

$$\phi'(t) = \phi'(0) - y'(0)t - \frac{1}{2} r(m + 1)t^2 \tag{33}$$

Let $y'(0) \geq 0$. Then ϕ' decreases, reaching m at time

$$t_1 = \frac{-y'(0) + \sqrt{D}}{r(m + 1)}$$

where

$$D = y'(0)^2 + 2r(m + 1)\left[\phi'(0) - m\right]$$

(we neglect ϵ). Now, q' flips to $m - 1$ and y' flips to

$$y'(0) + r(m + 1)t_1 - 2r = \sqrt{D} - 2r$$

If

$$D < 4r^2 \tag{34}$$

Then ϕ' immediately starts to increase again, so q' flips back to $m + 1$, and so on. Thus ϕ' is stuck at m.

If $y'(0) < 0$ then ϕ' initially increases. We can make sure that ϕ', given in Eq. (33), never reaches the next decision level $m + 2$ by requiring

$$D < 4r(m + 1) \tag{35}$$

In fact, Eq. (35) implies Eq. (34) if $r \geq 2$. Then ϕ' reaches m at time t_1 as before.

When ϕ' sticks at m, y' starts flipping between $\sqrt{D}$ and $\sqrt{D} - 2r$. By examining the microscopic behavior of ϕ' in the

[2] Because of the indefinite integrals, additive constants in Eqs. (30) and (31) can be deleted.

2ϵ-dead zone, one can show, using an analysis not given here, that the upper and lower limits y_+, y_- of y' change with time, and in fact

$$y_+ \to (m+1)r, \ y_- \to (m-1)r$$

exponentially with time constant 1. This result, though intuitively attractive, is on shakier ground than the other results here because it depends on the detailed behavior of the quantizer in its dead zones, which, after all, are but mathematical artifices introduced to make sense of the loop equations.

If λ/r happens to be an odd integer, then it appears that ϕ' wanders irresolutely between the two nearest decision points $\lambda/r - 1$ and $\lambda/r + 1$.

Returning through two changes of variables to real-world units, we state results: *Consider the input $\theta = \theta_o + \omega_o t + 1/2\, \lambda t^2$ to the loop with detector output quantization, where 0 is a decision level of the quantizer and $2\Delta\phi$, defined by Eq. (25), is the effective phase error quantization interval. Then the error ϕ sticks at the decision level $2k\Delta\phi$ that is nearest to the SPE of $\tau_2^2\lambda/r$. The RF frequency ω of the local oscillator flips rapidly between the two values*

$$\frac{d\theta}{dt} + \frac{r}{\tau_2}\left(2k\Delta\phi - \frac{\tau_2^2\lambda}{r}\right) \pm \frac{r\Delta\phi}{\tau_2}\ . \tag{36}$$

Thus, the peak-to-peak jitter in ω is

$$V_\omega = \frac{r}{\tau_2}\, 2\Delta\phi\ . \tag{37}$$

Evidently, Eq. (37) is the counterpart of Eq. (22) or (23).

In the 1-Hz loop let the maximum range $\pm K_d$ of the phase detector be quantized by an 8-bit A-D converter. Then for strong signals ($\alpha = 1$), we have

$$2\Delta\phi = 2^{-7}\ \text{rad} = 0.45\ \text{deg.},$$

$$V_y = \frac{V_\omega}{2\pi M} = 6.6 \times 10^{-4}\ \text{Hz}.$$

We have been assuming in this section that the oscillator frequency y is not quantized. Since an NCO is being used, its frequency is necessarily quantized. If, however, we make its quantization $2\ \Delta y$ small compared to its peak-to-peak jitter V_y, then the results of this section still hold. In the 1-Hz loop example, we must assume that the synthesizer is programmed at least as far down as the 10^{-4} Hz decade; this is feasible because the signal has been digitized earlier. Then the phase jitter caused by frequency quantization (Section III B) will be small compared to the phase error quantization $2\ \Delta\phi$. If the synthesizer is programmed down only to 10^{-3} Hz, then one must examine a more complicated model that includes both frequency and phase error quantization.

V. Conclusions

There is a rough rule-of-thumb relationship

$$\Delta\phi \approx \frac{\tau_2}{r}\ \Delta\omega \tag{38}$$

which holds when either the phase error ϕ or the RF frequency ω is quantized. If ω is quantized into pieces of width $2\Delta\omega$, then $2\Delta\phi$ from Eq. (38) gives an upper bound for the peak-to-peak phase jitter. Conversely, if ϕ is quantized into pieces of width $2\Delta\phi$ then $2\Delta\omega$ from Eq. (38) is the peak-to-peak frequency jitter. In either case, the variable being quantized tends to stick at a decision level of the quantizer, which acts as a bang-bang control element in the loop.

It is not practical to digitize the error signal at the input to the loop oscillator (frequency quantization), because the oscillator frequency range would be too small. The digitizing must be done earlier, in which case Eq. (38) still tells us how fine the programming of the digitally-controlled oscillator must be to maintain phase jitter below a certain level.

References

1. Johns, C. E., *Block IV Receiver Development*, JPL Technical Report 32-1526, Vol. X, Jet Propulsion Laboratory, Pasadena, Calif., 1972, pp. 175-178.

2. Henrici, P., *Elements of Numerical Analysis*, New York, 1964, pp. 90-96.

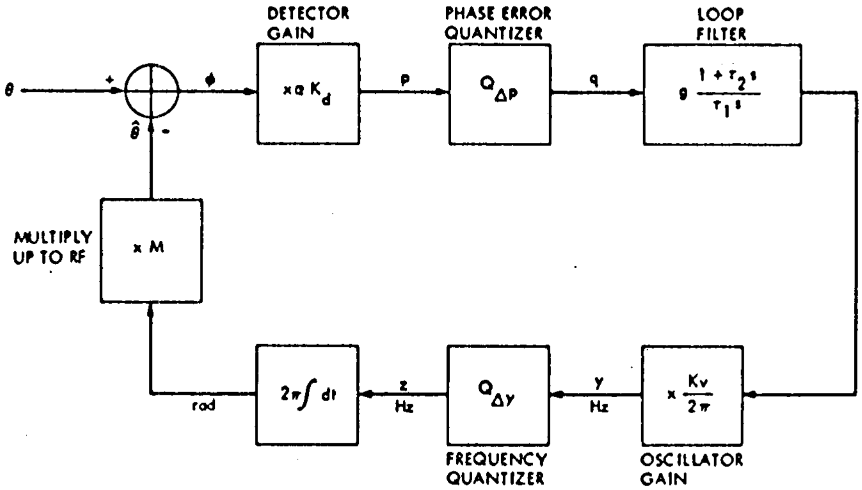

Fig. 1. Loop mathematical model

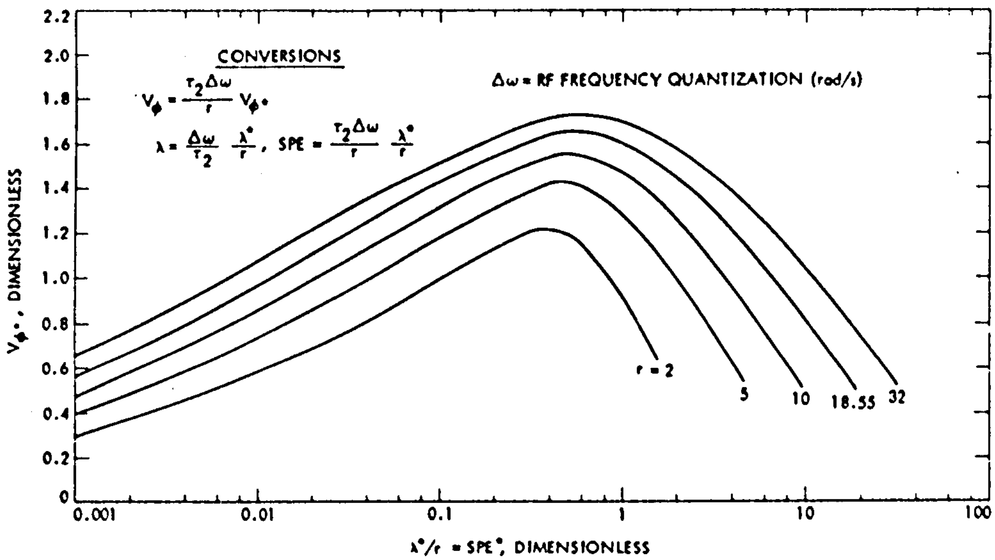

Fig. 2. Peak-to-peak phase variation for Doppler-rate input $\theta = 1/2\, \lambda t^2$ to a loop with frequency quantization

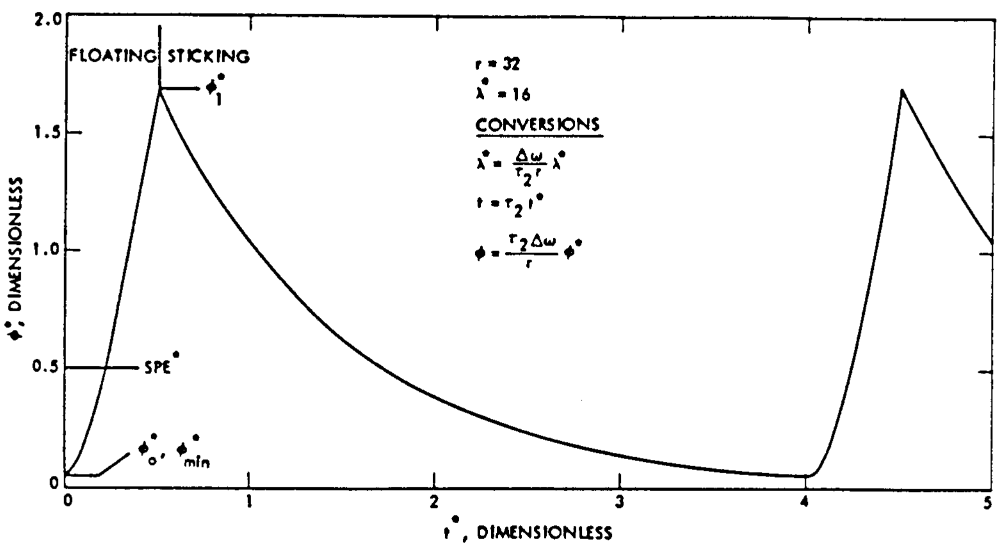

Fig. 3. Phase error vs. time for a Doppler-rate input to a loop with frequency quantization