The R_0-Parameter for Optical Communication Using Photon Counting

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We show that even under ideal conditions (no thermal noise or dark current, continuously-variable intensity lasers, perfectly accurate photon counters), R_0 is, at most, one nat per photon in optical systems which use photon counting techniques. Since even under less ideal circumstances channel capacity is infinite, this is a surprising and in some ways disappointing result.

I. Introduction

In a recent report (Ref. 1), it was shown that the R_0-parameter for the noiseless optical channel with pulse/no pulse modulation is exactly one nat per photon. Since R_0 is widely believed to measure the maximum rate at which "practical" reliable communication is possible, and since 1 nat/photon is disappointingly small,[1] it is natural to ask what R_0 would be if more general modulation schemes were used. In this article we will show that even with infinitely variable amplitude-modulation and perfectly accurate photon counters, R_0 remains equal to one nat per photon. This result supports Pierce's (Ref. 2) judgement that, "practically, the rates we can attain by photon counting will be limited by how elaborate codes we can instrument rather than by thermal photons."

In Section II, we will give a definition of R_0 which applies to any memoryless time-discrete channel whose output alphabet is finite or countable. Also, we will give a simple upper bound on R_0 which applies to any such channel. Then, in Section III, we will use this bound as applied to the specific channel model appropriate for noiseless optical communication to show that $R_0 \leqslant 1$ nat per photon. Finally we will argue that, in fact, $R_0 = 1$ nat per photon.

[1] Disappointing (and very surprising) since the channel capacity, i.e., the maximum rate at which reliable communication is possible (questions of practicality aside) is infinite! See (Ref. 1) or (Ref. 2) for a proof of this fact.

II. The R_0-Parameter for a General Channel

Consider a time-discrete memoryless channel with input alphabet A and output alphabet B. We assume B is finite or countable. For $x \epsilon A, y \epsilon B$, we denote the probability that y will be received, given that x is transmitted, by $p(y|x)$.

For each pair of input letters x_1, x_2 we define the Bhattacharyya distance between them as

$$d_B(x_1, x_2) = -\log \sum_{y \epsilon B} \sqrt{p(y|x_1)p(y|x_2)}\,. \qquad (1)$$

If now X is a random variable taking values in the set A, and if X_1, X_2 are independent random variables, both with the same distribution as X, we define

$$R_0(X) = -\log E(\exp - d_B(X_1, X_2))\,. \qquad (2)$$

Finally, the quantity R_0 is defined as:

$$R_o = \sup_X R_0(X), \tag{3}$$

the supremum in Eq. (3) being taken over all possible probability distributions on the set A (see Ref. 1, p. 68 and Ref. 3).

The quantity R_o has dimensions nats[1] per channel use, and is second only to channel capacity itself as a measure of the channel's capabilities. In particular, it is widely believed to be the rate beyond which the implementation of reliable communication systems become extremely difficult (Ref. 4).

We conclude this section with a simple and useful upper bound on R_0. Since the function $f(t) = e^{-t}$ is convex, it follows from Jensen's inequality (Ref. 1, Appendix B) that $E(\exp - d) \geqslant \exp - E(d)$, and hence from Eq. (2) that

$$R_0(X) \leqslant E(d_B(X_1, X_2)) \tag{4}$$

$$R_0 \leqslant \sup_X E(d_B(X_1, X_2)) \tag{5}$$

III. R_0 = 1 Nat/Photon for Optical Channels

We assume that our optical communication system works as follows. The time interval during which communication takes place is divided into many small intervals ("slots") of duration t_0 seconds each. The transmitter is a semiconductor laser which is pulsed during each slot. The intensity of the pulse in the i-th slot is denoted by x_i; this means that the expected number of photons emitted is x_i. The intensity can be any nonnegative real number, but the actual number of photons emitted is, of course, an integer. Because of the Poisson statistics governing photon emissions, the probability that exactly k photons will be emitted by the laser during the i-th slot is $e^{-x_i} x_i^k / k!$ The receiver is a photon counter, which we assume correctly reports the exact number of photons emitted during each slot.

Thus described, the optical channel fits the model of the previous section. The input alphabet A is the set of nonnegative real numbers; the output alphabet B is the set of nonnegative integers; and if $x \epsilon A$ is transmitted, the probability that $k \epsilon B$ is received is

$$p(k|x) = e^{-x} \frac{x^k}{k!} \tag{6}$$

[1] Throughout, all logarithms will be natural.

The first step in computing R_0 for this channel is the computation of the Bhattacharyya distance $d_B(X_1, X_2)$. According to Eqs. (1) and (6)

$$\begin{aligned} e^{-d_B(x_1, x_2)} &= \sum_{k=0}^{\infty} \sqrt{p(k|x_1) p(k|_2)} \\ &= e^{-(x_1 + x_2)/2} \sum_{x=0}^{\infty} \frac{1}{k!} \sqrt{x_1 x_2}^{\,k} \\ &= e^{-(x_1 + x_2)/2} \, e^{\sqrt{x_1 x_2}} \\ &= e^{-(\sqrt{x_1} - \sqrt{x_2})^2/2} \end{aligned} \tag{7}$$

Hence

$$d_B(x_1, x_2) = (\sqrt{x_1} - \sqrt{x_2})^2/2 \tag{8}$$

Note also that if we take only the term $k = 0$ in the sum in Eq. (7), we get the estimate

$$d_B(x_1, x_2) \leqslant (x_1 + x_2)/2 \tag{9}$$

It thus follows immediately from the bound Eq. (4) that

$$R_0(X) \leqslant E(X) \tag{10}$$

In words, Eq. (10) says that if the average laser intensity is β photons per slot, then the R_0-parameter is at most β nats per slot. In units of nats per photon, then, we have

$$R_0 \leqslant 1 \text{ nat/photon} \tag{11}$$

Note that the bound Eq. (11) was derived under very generous assumptions about the kind of signalling equipment available (infinitely variable laser intensity, perfectly accurate photon counters). We shall now show that, in fact, R_0 = 1 nat/photon, even if only two laser intensities ("on" and "off") are available, and if we replace the ultrasensitive photon counter with a simple photon detector, which emits a 1 if it is struck by one or more photons in a given time slot, and a 0 if it is not. An indirect proof that R_0 = 1 in this situation was given in Ref. 1. Here we will give a different proof, using q-ary pulse position modulation.

The idea is to select a fixed positive integer q, and to divide the transmission interval into consecutive blocks of q slots each. In each such block, the laser is pulsed exactly once, so that there are exactly q basic patterns in the signalling alphabet. For example, with $q = 4$, if we denote "no pulse" by 0 and "pulse" by 1, these patterns are 1000,0100,0010,0001. There are, however, $q + 1$ possibilities for the received pattern, because the laser may emit no photons when it is pulsed. The probability that a given transmitted pattern will be received in error is just the probability that the laser will emit no photons during a single pulse: $e^{-\lambda}$, if the laser's intensity is λ.

Thus, the appropriate channel model for this situation has input alphabet (illustrated for $q = 4$) $A = \{1000,0100,0010,0001\}$ and output alphabet $B = \{0000,1000,0100,0010,0001\}$. The transition probabilities are

$$
\begin{aligned}
p(y|x) &= 1 - e^{-\lambda} && \text{if } x = y \\
&= e^{-\lambda} && \text{if } y = 0000 \\
&= 0 && \text{otherwise}
\end{aligned}
$$

From this it easily follows that the Bhattacharyya distances are given by

$$
\begin{aligned}
d_B(x_1, x_2) &= 0 \text{ if } x_1 = x_2 \\
&= \lambda \text{ if } x_1 \neq x_2
\end{aligned}
\tag{12}
$$

Hence by Eq. (4), we have

$$R_0(X) \leqslant \lambda \cdot P\{X_1 \neq X_2\} \tag{13}$$

the units in Eq. (13) being nats per channel use. It is easy to see that $P\{X_1 \neq X_2\} \leqslant (q - 1)/q$, with equality if and only if X is equally likely to be any of the q channel input symbols. Thus,

$$R_0(X) \leqslant \lambda \frac{q-1}{q} \quad \text{(nats per channel use)} \tag{14}$$

or, since each channel use requires λ photons on the average,

$$R_0 \leqslant \frac{q-1}{q} \text{ nats per photon.} \tag{15}$$

On the other hand, if X is uniformly distributed on the input alphabet, a simple calculation gives

$$R_0(X) = -\frac{1}{\lambda} \log\left(\frac{1}{q} + \frac{q-1}{q}(e^{-\lambda})\right) \text{ nats/photon.} \tag{16}$$

The limit of Eq. (16) as $\lambda \to 0$ is easily seen to be $(q - 1)/q$, and so we conclude that for q-ary pulse-position modulation,

$$R_0 = \frac{q-1}{q} \text{ nats/photon.} \tag{17}$$

(For completeness, we remark that a similar calculation shows that the capacity for $q - PPM$ is

$$R_0 = \log q \text{ nats/photon.)} \tag{18}$$

Equation (17) shows that by taking q sufficiently large, R_0 can be made as close to 1 as desired. This fact, combined with Eq. (11), shows that $R_0 = 1$, as claimed.

We conclude with two remarks. First, a close examination of our calculations shows that the only possible input distributions that approach $R_0 = 1$ have both average and peak intensity very close to zero. This suggests that efficient coding schemes will have the same property.

Second, note that the bound Eq. (9) applies even if we allow pulsing at different frequencies, since Eq. (9) merely reflects the ambiguity at the receiver if no photons are received. Thus, the bound $R_0 \leqslant 1$ holds even for frequency-modulated direct-detection systems. Of course if we use different frequencies the number of nats per photon is no longer proportional to the number of nats per joule, which is of course the basic unit here. However, a multifrequency system operating at R nats per photon would consume more energy per transmitted nat than a monochromatic system using the lowest frequency of the multifrequency system operating at R nats per photon. And since $R_0 = 1$ for both systems, we would expect monochromatic systems to be more efficient.

References

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