An Improvement in Fourier Transform Accuracy

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Most Fourier transform algorithms, when seeking a single frequency at the sample-time midpoint, simply choose the largest component of the set. The error is at least ±0.5 component, and often larger. This article describes an addition to such algorithms that reduces the variance error by three-to-one (nominal). The addition achieves this by averaging all *components within a predetermined "window," selected as a function of frequency rate, and it is quite easy to mechanize within the existing algorithm.*

I. Introduction

The Fourier transform, particularly the fast algorithm or "FFT", is used in numerous data-reduction applications in the Deep Space Network (DSN). In many of these applications, the transform acts as a kind of "digital filter," yielding a single largest component representing an input sinusoid. The usual input signal is initially contaminated by additive noise, which the transform "spreads out" into a low-order spectrum, attenuated "around" the representative component. The spectrum in use is the "discrete" form of the "finite" Fourier transform, finite implying a bounded time sample, and "discrete," a finite number of sample points.

When the sinusoid is not precisely synchronized with the sample-point period (usual case), or when the frequency is time-variant, or both (normal tracking conditions), the signal transform itself is a spectrum. Under normal conditions, this spectrum contains from four to ten or more significant components. If accuracy is important, the frequency chosen must be associated with a given "time-tag" within the time sample.

Normally, the first-order frequency excursion, or "ramp" assumption is acceptable for DSN applications using sample periods of (up to) a few seconds.

The present process is to simply select the "largest" component as the estimate, associating this with the sample-time center.

This article describes a supplementary averaging procedure that gives promise of a significant improvement in accuracy over the "largest component" method. If implemented, it would require an addition to the existing algorithms, but the addition would be neither particularly lengthly nor difficult to mechanize. It would require one additional a priori input: an estimate of the ramp-rate, or "$\dot{F}$", as present on the sample. This could be obtained from RF doppler, as modified by any additional corrective predicts.

The "$\dot{F}$" values encountered in tracking normally vary from about 1.0 Hz/sec (or less) up to 20 Hz/sec (or more). A nominal value of 5 Hz/sec was chosen for the model to follow.

II. The Finite Fourier Transform of a "Frequency Ramp"

The continuous form of the finite Fourier transform, or power spectral density, is (if noiseless):

$$\left.\begin{aligned} P(f) &= \left| \int_0^{\tau} e^{-j2\pi ft} F(t)\,dt \right|^2 \\ &\text{where, for the case in question:} \\ F(t) &= \sin\left(2\pi f_0 t + \pi \dot{f} t^2 + \phi_0\right) \end{aligned}\right\} \quad (1)$$

This is the envelope of the components obtained by the discrete expression. With the discrete algorithms, components are spaced "$1/\tau$" apart, across a range "$N/2\tau$"; N the discrete sample point-count. "f_0" is the frequency at the start of the sample (sample length "τ"), and $\dot{f}$ is the "ramp-rate", Hz/sec, during τ. $\dot{f}$ is considered constant, thus implying neglect of higher-order terms, if any. A model program to simulate Eq. (1) in discrete form was coded for this study. At first glance, Eq. (1) appears reasonably straightforward; it would seem logical to simply evaluate the envelope (given $\dot{f}$), then "slide it across" sample data to obtain an estimate of f_0 at $t = 0$, or some similar "best fit" result. This was, in fact, the initial approach.

However, the envelope of Eq. (1) proved very cumbersome to calculate, and this approach was discarded in favor of the simpler routine to follow.

III. Frequency-Time Estimate by Component Averaging

Refer to Fig. 1; this is a typical discrete transform (Fig. 1) that simulates Eq. (1) by the model program when:

$$\tau = 1 \text{ sec}$$

$$f_0 = 22 \text{ Hz}$$

$$\dot{f} = 5 \text{ Hz/sec}$$

The transform base is 50 Hz long, with components 1.0 Hz apart. $N = 100$ points. These are the "standard conditions" of this study. The "bulk" of the study is an investigation of the transform of Fig. 1 under various signal-to-noise ratios (SNR). The purpose was to evaluate the accuracy with which the center frequency of the spectrum could be related to the center of the time sample, $\tau/2$, or 0.5 sec. The true frequency at 0.5 sec. was 24.5 Hz in all cases.

The "inherent" error in the above case, strong signal, by the "largest component" method, is 0.5 Hz; the true center frequency is midway between two components, one of which is chosen. In general, this strong signal error will have a standard deviation of 0.35 Hz, the well-known "$1/\sqrt{12}$" constant.

However, as the noise increases (artificially "injected" in the model program), the "largest component" became more and more likely to appear at locations other than "adjacent to" the center. Refer to Figs. 2, 3 and 4 for typical results. In one run, at -5 dB SNR, this component actually appeared at *2.0 Hz*, a value that would have led to data discard.

The random-displacement naturally led to increased "sigma", or standard deviation. Model results for this, based on 12 samples at each of three SNR, are shown on Fig. 5 (outer set), with estimated extrapolations. These describe the error, using present techniques, for the stated (Fig. 1) conditions, plus-noise, as obtained by the model program.

Even though the envelope of Eq. (1) is unreasonably complex to express in its total form, it *is* easy to show that it is not only symmetric above the center $f(\tau/2)$, but that it attenuates ("dwindles") rapidly outside of the waveform frequency excursion due to $\dot{f}$. The details are omitted.

These characteristics of the envelope led directly to the concept of *component averaging*, which relies on the signal power of *all* signal components present, rather than upon a "single representative." The process is two-fold:

(1) Using an estimate of the envelope width, or "spread" obtained from an a priori measure of $\dot{f}$, a sequential set of component power summations is taken, within a "window" of this width, across the entire spectrum. The region of *greatest total power* is assumed to contain the center frequency.

(2) The second step involves averaging the power-frequency components in the "window." Even though envelope symmetry does not imply component symmetry, when several components are in the "window," the heights (power) of the corresponding "pairs" (closest frequency "fit" on either side) is very nearly a linear function of their absolute offset from center; they "average out."

Said in another way, the envelope derivative on opposite sides of center is approximately of the same slope and linear over short corresponding absolute intervals. Such a derivative is simply the argument of Eq. (1). Investigation showed little discernible error over intervals of "$1/\tau$" in n.

This and several "intuitive" observations from the model all suggested that a very good estimate of the desired center frequency could be obtained by simply averaging the (normalized) power-frequency products of the components within the "maximum power window", the second step of the procedure as stated.

These steps can be described notationally for inclusion in existing algorithms. Let:

N = total number of components in discrete spectrum (about 100 to 1000)

τ = sample length, sec

$\dot{f}$ = ramp-rate (a priori), Hz/sec

Then:

$$SR = \text{search range}$$

$$= INT\,[2\dot{f}\tau]$$

The digit "2" is not critical. It is chosen here from machine observations of the extent of significant "side lobes" in the transform assisted with various "$\dot{f}$".

FW = center frequency "window" chosen for averaging, low-side index

$$= K \text{ of } MAX \text{ of } \left\{ \sum_{m=K}^{K+SR} C_m \right\} \tag{2}$$

where

$$K = 1, 2, \cdots N - SR$$

$$C_m = \text{(power) size of } m^{\text{th}} \text{ Fourier component}$$

(Step 1)

And, finally:

$$\hat{F}\left[\frac{\tau}{2}\right] = \frac{\displaystyle\sum_{n=FW}^{FW+SR} F_n C_n}{\displaystyle\sum_{M=FW}^{FW+SR} C_n} \tag{3}$$

where

F_n = frequency of component C_n

(Step 2)

The model statistical results, using the above, with $SR = 8$, are shown in collected form on Fig. 5. The accuracy advantage under stated conditions is evident. Improvement up to nearly 3 to 1 was accomplished in the mid-SNR region, and the strong-signal residual was barely evident. Signal-to-noise is that within a 100-Hz bandwidth.

IV. Conclusions

The component summation addition to existing Fourier transform algorithms in DSN applications appears to have considerable potential as a minor tool for accuracy improvement. The data on Fig. 5 demonstrates its behavior with respect to the present process, using the model and levels described herein.

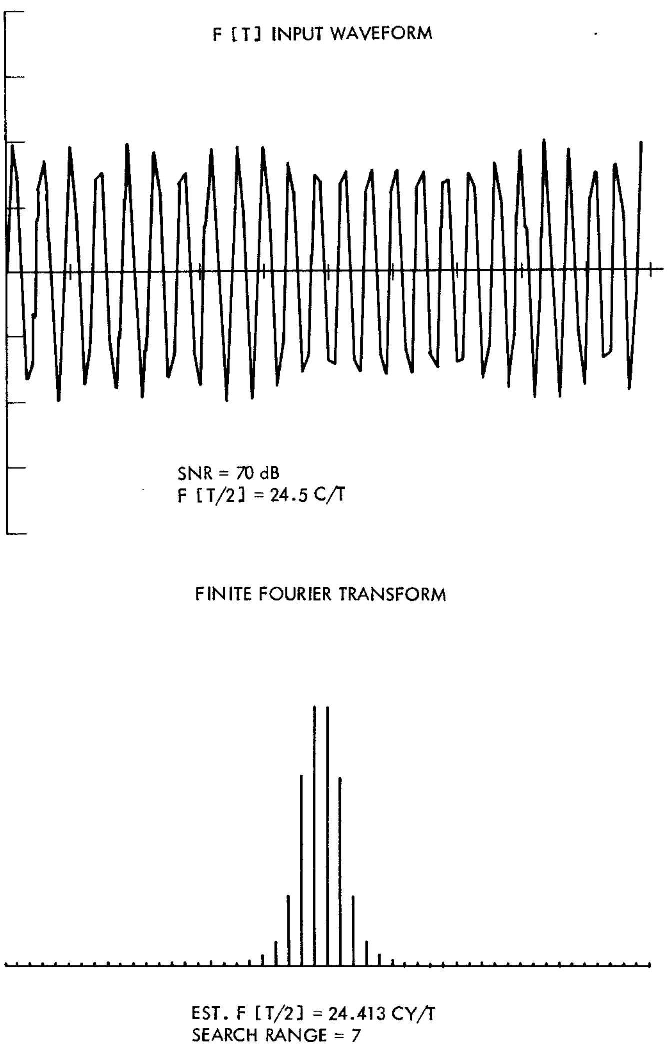

Fig. 1. Transform of ramped sinusoid, noise-free

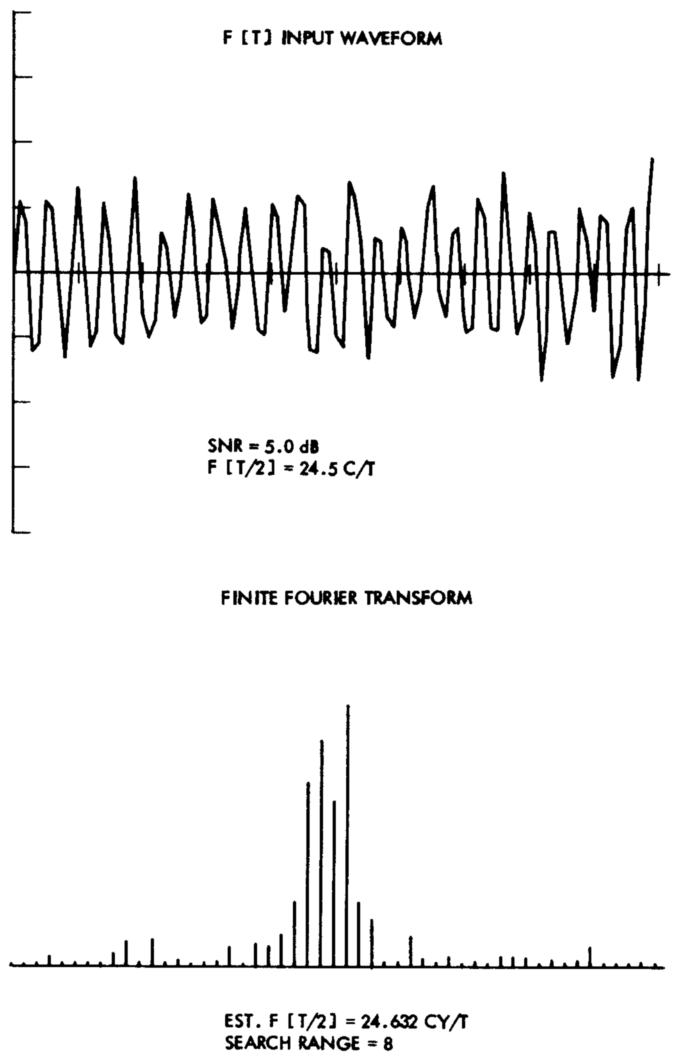

Fig. 2. Transform of ramped sinusoid, SNR = +5 dB (typical) in 100 Hz

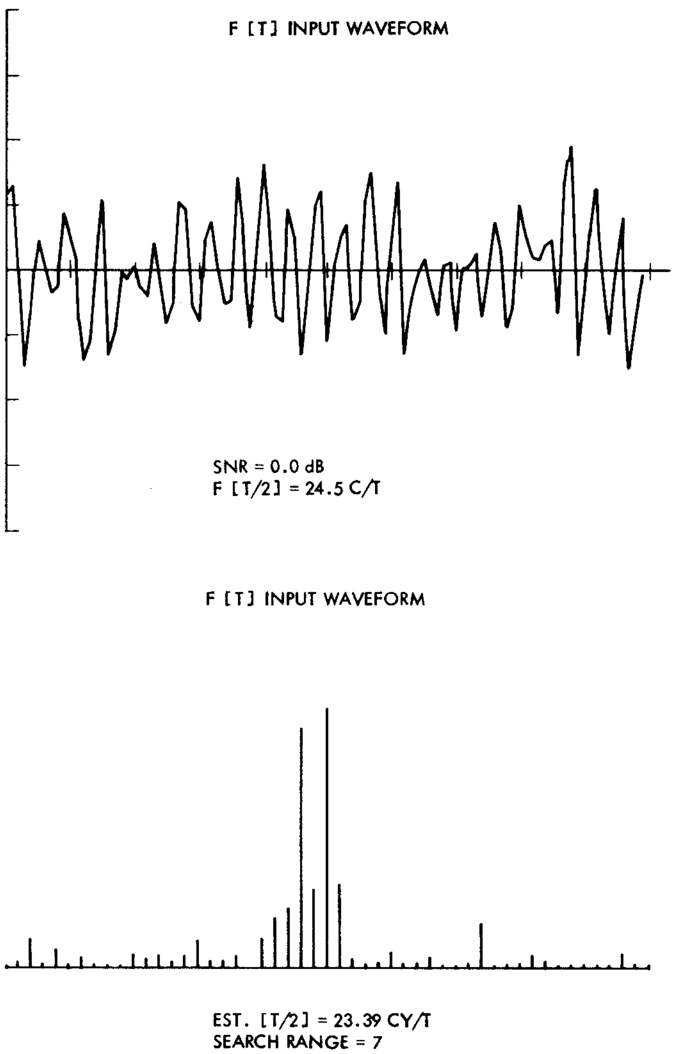

Fig. 3. Transform of ramped sinusoid, SNR = 0 dB (typical) in 100 Hz

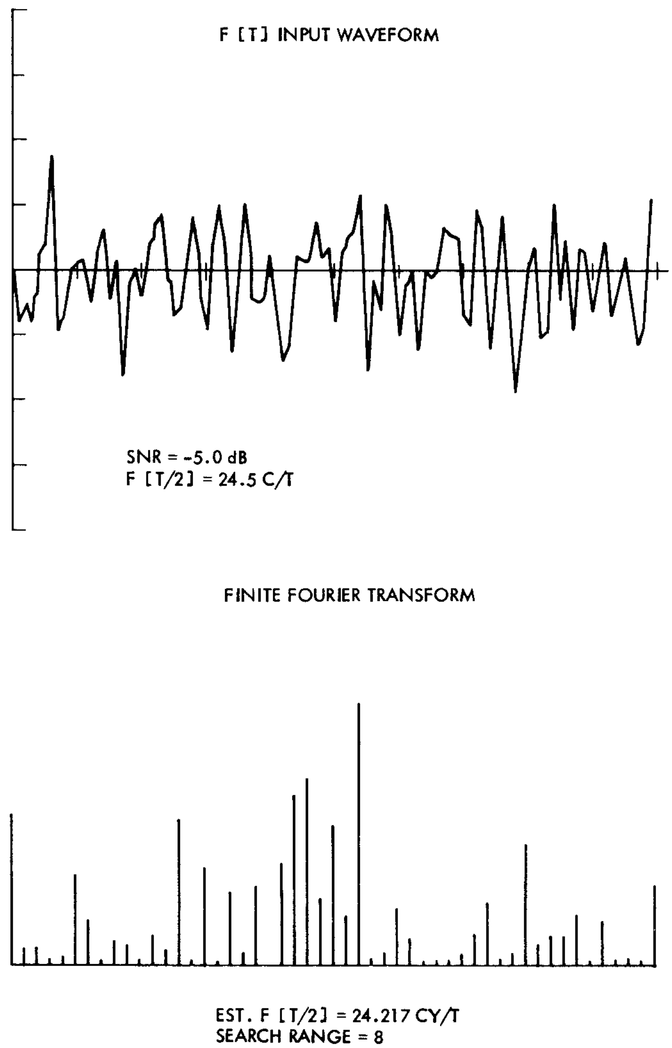

Fig. 4. Transform of ramped sinusoid, SNR = −5 dB (typical) in 100 Hz

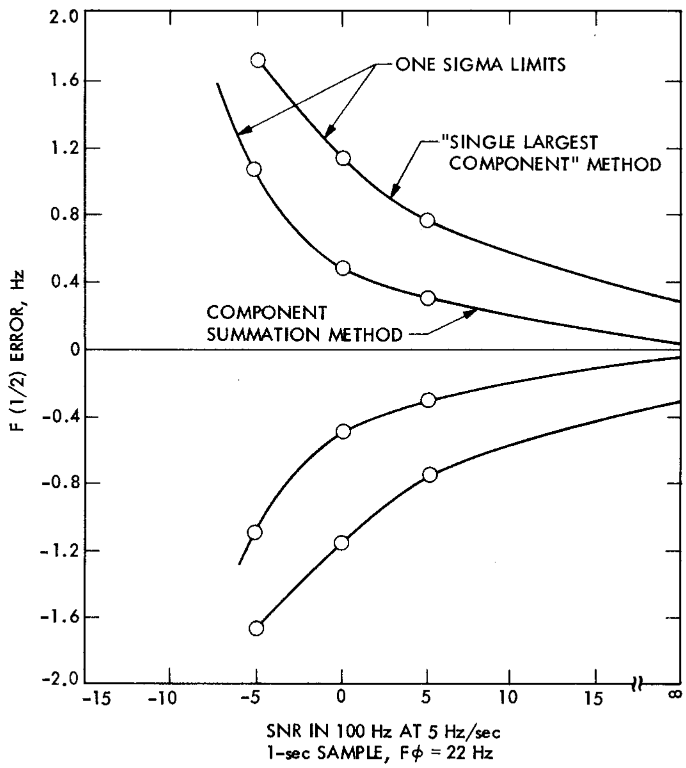

Fig. 5. Statistical comparison, Fourier transform accuracy, two methods