Estimating the Angular Position of a Moving Deep Space Vehicle Using Two Rotating Tracking Stations

H. H. Tan
Communications Systems Research Section

The problem of estimating the angular position of a deep space vehicle moving at a constant velocity using two rotating tracking stations is considered. This article reports on an initial phase of analytical studies on the optimal attainable estimation performance and associated receiver design. Parametric dependence of the optimum attainable estimation performance is also studied.

I. Introduction

Consider the situation in Fig. 1 where at a reference time $t = 0$ there is a vehicle at point V and two stations at points S_1 and S_2 relative to Earth geocenter 0. The vehicle is assumed to be at a distance d_0 from the geocenter 0 at time $t = 0$. It is also assumed to be moving at a constant velocity v relative to geocenter 0 for all time in a direction which is at an angle α with respect to OV. The stations are both assumed to be rotating at a constant angular velocity ω about 0. At time $t = 0$, station S_1 is at an angle ϵ_0 and the vehicle at an angle γ_0 with respect to a star reference. The angle between the stations is denoted by η and the distances of the stations from geocenter by R_1 and R_2, as indicated in Fig. 1.

We assume that the vehicle is continuously transmitting a signal $s(t)$. Each station receives an additive noise-corrupted version of this transmitted signal. The received waveforms from both stations over a given time interval are then used to estimate the unknown vehicle angle γ_0. This article considers this estimation problem based on the assumptions that d_0, v, α, η, ω, and ϵ_0 are known. These assumptions are made so that the effect of the rotating stations and the moving target on angular position estimation can be studied. Future studies will take into account imprecise knowledge of these parameters.

The estimation problem is defined in Section II below along with a discussion of the minimum attainable mean square estimation error performance. Since we could not determine the optimal estimator, a suboptimal estimator is derived in Section III. The performance of this suboptimal estimator is examined relative to the optimal attainable performance derived in Section II. In Section IV we examine the dependence of the optimal attainable performance on angular position, station rotation, station distance from geocenter and observation time duration. The specific case of a sinusoidal ranging signal is considered and numerical computations of the optimum attainable angular estimation accuracy are performed for several parameter values. Conclusions are given in Section V.

II. The Estimation Problem

Let us first derive the equations for the received waveforms at each station. The vehicle is assumed to continuously transmit a signal $s(t)$. The received waveform at station S_1 is assumed to be

$$y_1(t) = s[t - \phi_1(t, \gamma_0)] + n_1(t) \tag{1}$$

where $n_1(t)$ is additive white Gaussian noise with power spectral density N_1 and $\phi_1(t, \gamma_0)$ is effectively the signal delay time. Similarly, the received waveform at station S_2 is assumed to be

$$y_2(t) = s[t - \phi_2(t, \gamma_0)] + n_2(t) \tag{2}$$

where $n_2(t)$ is additive white Gaussian noise with power spectral density N_2 and $\phi_2(t, \gamma_0)$ is the signal delay time to station S_2. The signal delay times $\phi_1(t, \gamma_0)$ and $\phi_2(t, \gamma_0)$ are both deterministic functions of the unknown vehicle angle γ_0. We assume that γ_0, $n_1(t)$ and $n_2(t)$ are mutually statistically independent. In Appendix A, the geometry of Fig. 1 is used to derive the following expressions for ϕ_1 and ϕ_2 (c = velocity of light):

$$\begin{aligned}
\phi_1(t,\gamma_0) = {} & [1 - (v/c)^2]^{-1} \, [-(v/c)^2 t - (vd_0 \cos\alpha)/c^2 \\
& + vR_1 \cos(\gamma_0 + \alpha - \epsilon_0 - \omega t)/c^2] \\
& + \Big\{ [1 - (v/c)^2]^{-2} \, [t + (vd_0 \cos\alpha)/c^2 \\
& - vR_1 \cos(\gamma_0 + \alpha - \epsilon_0 - \omega t)/c^2]^2 \\
& + [1 - (v/c)^2]^{-1} \, [(d_0^2 + R_1^2 - 2R_1 d_0 \cos(\gamma_0 \\
& - \epsilon_0 - \omega t))/c^2 - t^2] \Big\}^{1/2}
\end{aligned} \tag{3}$$

$$\begin{aligned}
\phi_2(t,\gamma_0) = {} & (1 - (v/c)^2)^{-1} \, [-(v/c)^2 t - vd_0 \cos\alpha)/c^2 \\
& + VR_2 \cos(\gamma_0 + \alpha \quad \epsilon_0 - \eta - \omega t)/c^2] \\
& + \Big\{ (1 - (v/c)^2)^{-2} \, [t + (vd_0 \cos\alpha)/c^2 \\
& - vR_2 \cos(\gamma_0 + \alpha - \epsilon_0 - \eta - \omega t)/c^2]^2 \\
& + (1 - (v/c)^2)^{-1} \, [(d_0^2 + R_2^2 - 2R_2 d_0 \cos \\
& - (\gamma_0 - \epsilon_0 - \eta - \omega t))/c^2 - t^2] \Big\}^{1/2}
\end{aligned} \tag{4}$$

The receiver's function is to estimate γ_0 based on observations $(y_1(t), y_2(t))$, $-T_1 \leqslant t \leqslant T_2$, with a goal of minimizing the mean square estimation error. The minimum mean square error (MMSE) estimator is the conditional mean estimator when the prior distribution of γ_0 is known. In this case, the conditional mean estimator is nonlinear. Moreover, it appears that the problem of determining explicit estimator equations is not tractable. An approach to overcome this problem is to derive suboptimum receivers that can be implemented instead. In the next section of this paper we consider one method of obtaining a suboptimal receiver by using an extended Kalman filter estimation approach. This approach results in a relatively simple receiver structure.

It is also of interest to determine the optimum mean square estimation error so that the performance of suboptimal receivers can be evaluated. Of course, the minimum attainable mean square estimation error is of interest by itself. Unfortunately, it appears in this case that the problem of determining this optimum performance value is also not tractable. However, it is possible to obtain lower bounds (Refs. 1-3) on the minimum mean square estimation error. The Cramer-Rao lower bound (Ref. 1, p. 275) appears to be the most tractable to use. In the case when γ_0 is an unknown but nonrandom parameter, the Cramer-Rao lower bound on the mean square estimation error of any unbiased estimator $\hat{\gamma}_0$ is given by (Ref. 1, p. 275):

$$E[(\hat{\gamma}_0 - \gamma_0)^2] \geqslant \left[\sum_{i=1}^{2} (2/N_i) \int_{-T_1}^{T_2} \dot{s}^2(t - \phi_i(t,\gamma_0)) \left(\frac{\partial \phi_i(t,\gamma_0)}{\partial \gamma_0} \right)^2 dt \right]^{-1} \tag{5}$$

where $\dot{s}(t) = ds(t)/dt$. In the case when γ_0 is a random parameter with known density $p(\gamma_0)$, the Cramer-Rao lower bound on the mean square estimation error of any estimator $\hat{\gamma}_0$ is given by (Ref. 1, p. 275):

$$E[(\hat{\gamma}_0 - \gamma_0)^2] \geqslant \left\{ E\left[\left(\sum_{i=1}^{2} (2/N_i) \int_{-T_1}^{T_2} \dot{s}^2(t - \phi_i(t,\gamma_0)) \left(\frac{\partial \phi_i(t,\gamma_0)}{\partial \gamma_0} \right)^2 dt \right) - \frac{\partial^2 \ln p(\gamma_0)}{\partial \gamma_0^2} \right] \right\}^{-1} \tag{6}$$

where the expectation in the right hand side of Eq. (6) is with respect to the prior distribution of γ_0. For a normal γ_0 with variance σ_γ^2, Eq. (6) reduces to:

$$E[(\hat{\gamma}_0 - \gamma_0)^2] \geqslant \left\{ E\left[\sum_{i=1}^{2} (2/N_i) \int_{-T_1}^{T_2} \dot{s}^2(t - \phi_i(t,\gamma_0)) \left(\frac{\partial \phi_i(t,\gamma_0)}{\partial \gamma_0} \right)^2 dt + 1/\sigma_\gamma^2 \right] \right\}^{-1} \tag{7}$$

These lower bounds will be used in the remainder of this article to estimate the performance of the suboptimal estimator as well as the optimum theoretically attainable performance.

III. The Estimator

Consider the problem of estimating γ_0 in the following equivalent state variable formulation. Let $\gamma(t)$ be a variable state satisfying

$$\dot{\gamma}(t) = 0, \qquad \gamma(-T_1) = \gamma_0 \tag{8}$$

Then $\gamma(t) = \gamma_0$, the parameter to be estimated, for all t. Rewrite Eqs. (1) and (2) as

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} s(t - \phi_1(t,\gamma(t))) + n_1(t) \\ s(t - \phi_2(t,\gamma(t))) + n_2(t) \end{bmatrix} \tag{9}$$

So the equivalent problem is to estimate $\gamma(T_2)$ based on observations of $y(t)$ in the interval $[-T_1, T_2]$.

As we noted previously, the problem of determining the MMSE estimator is not tractable. An alternative is to derive a suboptimal estimator that approximates the MMSE estimator. Another alternative is to abandon the MMSE criterion and to seek estimators based on the maximum likelihood (ML) or maximum a posteriori (MAP) criterion. However, it can be shown (Refs. 1, 4) that the optimum ML or MAP estimators are also not practically implementable. Hence, developing estimators using the ML or MAP criterion will also require consideration of suboptimal estimators. Since the ultimate performance measure of interest is still mean square error, it appears more appropriate to seek approximations of the MMSE estimator.

There are numerous ways (Refs. 4, 5) of determining such suboptimal estimators. Our approach will be to adopt one version (Ref. 4, p. 267) of the extended Kalman filter algorithm. This version is the Kalman filter operating on a linearization of the observation equations (9) about the state estimate. The reason for adopting this approach over others is its relative simplicity. In the nonlinear estimation folklore, the extended Kalman filter is regarded as being capable of performing as well as other suboptimal schemes in most problems. So there is a priori no reason to believe that constraining our approach to the extended Kalman filter is overly restrictive.

Let $\hat{\gamma}(t)$ denote the extended Kalman filter estimate of $\gamma(t)$. Then a straightforward application of the equations of Ref. 4, (p. 267) shows that $\hat{\gamma}(t)$ satisfies

$$\frac{d\hat{\gamma}(t)}{dt} = -P(t) \sum_{i=1}^{2} (1/N_i) \left[y_i(t) - s(t - \phi_i(t,\hat{\gamma}(t)))\right] \cdot \left[\dot{s}(t - \phi_i(t,\hat{\gamma}(t))) \frac{\partial \phi_i(t,\hat{\gamma}(t))}{\partial \hat{\gamma}} \right] \tag{10}$$

$$\frac{dP(t)}{dt} = -P^2(t) \sum_{i=1}^{2} (1/N_i) \left[\dot{s}(t - \phi_i(t,\hat{\gamma}(t))) \frac{\partial \phi_i(t,\hat{\gamma}(t))}{\partial \hat{\gamma}} \right]^2 \tag{11}$$

with initial conditions

$$\hat{\gamma}(-T_1) = \overline{\gamma}_0 \tag{12}$$

$$P(-T_1) = \sigma_{\gamma_0}^2 \tag{13}$$

where $\overline{\gamma}_0$ and $\sigma_{\gamma_0}^2$ are the prior mean and variance respectively of γ_0. (We shall denote

$$\left. \frac{\partial \phi_i(t,\gamma_0)}{\partial \gamma_0} \right|_{\gamma_0 = \hat{\gamma}(t)} \quad \text{by} \quad \frac{\partial \phi_i[t,\hat{\gamma}(t)]}{\partial \hat{\gamma}}$$

for simplicity). Also, in Eqs. (10), and (11), $P(t)$ represents an approximation of the conditional variance of $\hat{\gamma}(t)$. The solution of Eq. (11) can easily be shown to be

$$P(t) = \left\{ 1/\sigma^2_{\gamma_0} + \int_{-T_1}^{t} \sum_{i=1}^{2} (1/N_i) \left[\dot{s}(\tau - \phi_i(\tau,\hat{\gamma}(\tau))) \frac{\partial \phi_i(\tau,\hat{\gamma}(\tau))}{\partial \hat{\gamma}} \right]^2 d\tau \right\}^{-1} \tag{14}$$

Rewriting Eq. (10) as an integral equation gives

$$\hat{\gamma}(t) = \bar{\gamma}_0 - \int_{-T_1}^{t} P(\tau) \sum_{i=1}^{2} (1/N_i) \left[y_i(\tau) - s(\tau - \phi_i(\tau,\hat{\gamma}(\tau))) \right] \cdot \left[\dot{s}(\tau - \phi_i(\tau,\hat{\gamma}(\tau))) \frac{\partial \phi_i(\tau,\hat{\gamma}(\tau))}{\partial \hat{\gamma}} \right] d\tau \tag{15}$$

Thus, Eqs. (14) and (15) give the estimator structure with $\hat{\gamma}(T_2)$, the desired estimate of γ_0. The only prior statistical knowledge of γ_0 required is its mean and variance. A block diagram of the implementation of the estimator is given in Fig. 3. The expression for $\partial\phi_i[t, \hat{\gamma}(t)]/\partial\hat{\gamma}$ is given in Eqs. (B-2) and (B-3) of Appendix B. These waveforms are implemented in the receiver of Fig. 3 by adjusting the $\hat{\gamma}(t)$ phase contributions in the sinusoidal terms given in Eqs. (B-2) and (B-3). The structure of the estimator is somewhat similar to the MAP estimator with normal prior distribution for γ_0 (Ref. 1, p. 453). One substantial difference of the estimator here with the MAP estimator is that the gain term $P(\tau)$ in the integral in Eq. (15) is replaced by $2\sigma^2_{\gamma_0}$ in the MAP estimator. This is because the gain is updated to account for the change in the a posteriori variance of γ_0 based on the observations. This is not performed in the MAP estimator.

In Eqs. (14) and (15), $\phi_i(t, \gamma)$ is given by Eq. (3) and $\partial\phi_i(t, \gamma)/\partial\gamma$ by Eqs. (B-2) and (B-3) in Appendix B. Further simplification of Eqs. (14) and (15) can result from using the simpler approximations in Eqs. (24) to (27) given in section IV for ϕ_i and $\partial\phi_i/\partial\gamma$. Simplification of the basic estimator structure depicted in Fig. 3 apparently cannot be done without specific assumptions on the signal structure.

The performance of this algorithm unfortunately cannot be determined analytically. In evaluating extended Kalman filters, $P(t)$ is often regarded as a measure of the mean square error. However, care must be taken to adopt this conclusion since $P(t)$ is only an approximation to the conditional variance of $\hat{\gamma}(t)$ (Refs. 4, 5). Moreover, $P(t)$ depends on the observations and so cannot be determined other than from simulation runs of the filter. In spite of these pitfalls, let us examine Eq. (14) to obtain a heuristic estimate of the best possible performance of the estimator.

Assume that the estimator is performing well. Thus, $\hat{\gamma}(t)$ will be close to $\gamma(t) = \gamma_0$. Assume also that $P(T_2)$ is a good approximation of the mean square estimation error. Then, from Eq. (14) we have

$$P(T_2) = \frac{\left[\sum_{i=1}^{2} (1/N_i) \int_{-T_1}^{T_2} \dot{s}^2 (t - \phi_i(t,\hat{\gamma}(t))) \left(\frac{\partial \phi_i(t,\hat{\gamma}(t))}{\partial \hat{\gamma}} \right)^2 dt \right]^{-1}}{1 + \left[\sum_{i=1}^{2} (1/N_i) \int_{-T_1}^{T_2} \dot{s}^2 (t - \phi_i(t,\hat{\gamma}(t))) \left(\frac{\partial \phi_i(t,\hat{\gamma}(t))}{\partial \hat{\gamma}} \right)^2 dt \right]^{-1} \Big/ \sigma^2_{\gamma_0}}$$

$$\leqslant \left[\sum_{i=1}^{2} (1/N_i) \int_{-T_1}^{T_2} \dot{s}^2 (t - \phi_i(t,\hat{\gamma}(t))) \left(\frac{\partial \phi_i(t,\hat{\gamma}(t))}{\partial \hat{\gamma}} \right)^2 dt \right]^{-1} \tag{16}$$

Let γ_0 be the true value of the unknown angle. So, if $\hat{\gamma}(t) \cong \gamma_0$, replacing $\hat{\gamma}(t)$ by γ_0 in Eq. (16) shows that the upper bound on $P(T_2)$ is roughly twice the Cramer-Rao lower bound (Eq. (5)) on the optimum mean square error. Thus from the above heuristic point of view, the best possible performance of the estimator is roughly within a factor of 2 from the Cramer-Rao lower bound of Eq. (5).

IV. Optimum Theoretically Attainable Estimation Performance

As we noted previously the Cramer-Rao lower bound gives a lower bound on the optimum attainable angle mean square estimation error. In this section we shall examine the Cramer-Rao lower bound in a special case. In particular, we shall assume the following set of parameters:

$$\begin{aligned} d_0 &= 8 \times 10^8 \text{ km} \\ R_1 = R_2 &= 6.5 \times 10^3 \text{ km} \\ \nu &= 10 \text{ km/sec} \\ T_1 = T_2 &= 30 \text{ min} \end{aligned} \tag{17}$$

This set of parameters is consistent with the distances encountered in a Jupiter mission. We assume in addition that $N_1 = N_2$ for simplicity. We shall first analyze the effects of the relative angular positions and the rotation of the Earth on the Cramer-Rao lower bound (Eq. (5)). This, then, gives the dependence of the optimum attainable performance on these effects.

We first consider the effect of the angular positions γ_0, ϵ_0 and η given in Fig. 1. Since the problem of estimating γ_0 is nonlinear, the minimum attainable estimation error would generally depend on γ_0. Consider first the case when $\omega = \nu = 0$ for insight into this dependence. Using the parameters in Eq. (17) we have from Eqs. (3) and (4) that

$$\phi_1(t,\gamma_0) = \left[d_0^2 + R_1^2 - 2d_0R_1 \cos(\gamma_0 - \epsilon_0)\right]^{1/2} \Big/ c$$
$$\cong \left[(d_0^2 + R_1^2)^{1/2}/c\right] \left[1 - (R_1 d_0/(d_0^2 + R_1^2)) \cos(\gamma_0 - \epsilon_0)\right] \tag{18}$$

$$\phi_2(t,\gamma_0) = \left[d_0^2 + R_2^2 - 2d_0R_2 \cos(\gamma_0 - \epsilon_0 - \eta)\right]^{1/2} \Big/ c$$
$$\cong \left[(d_0^2 + R_2^2)^{1/2}/c\right] \left[1 - (R_2 d_0/(d_0^2 + R_2^2)) \cos(\gamma_0 - \epsilon_0 - \eta)\right] \tag{19}$$

so

$$\frac{\partial \phi_i(t,\gamma_0)}{\partial \gamma_0} \cong \left[R_1 d_0/c\,(d_0^2 + R_1^2)^{1/2}\right] \sin(\gamma_0 - \epsilon_0) \tag{20}$$

$$\frac{\partial \phi_2(t,\gamma_0)}{\partial \gamma_0} \cong \left[R_2 d_0/c\,(d_0^2 + R_2^2)^{1/2}\right] \sin(\gamma_0 - \epsilon_0 - \eta) \tag{21}$$

Since $[R_i d_0/c(d_0^2 + R_i^2)^{1/2}] = 2.17 \times 10^{-2}$, $\phi_i(t,\gamma_0)$ is relatively independent of γ_0. Thus an approximation of the Cramer-Rao lower bound (Eq. (5)) in the case when $\omega = \nu = 0$ is:

$$E\left[(\hat{\gamma}_0 - \gamma_0)^2\right] \geqslant \left[\frac{2R_1^2 d_0^2}{N_1 c^2 (d_0^2 + R_1^2)} \int_{-T_1}^{T_2} \dot{s}^2 \left(t - \frac{(d_0^2 + R_1^2)^{1/2}}{c}\right) \sin^2(\gamma_0 - \epsilon_0)\,dt + \frac{2R_2^2 d_0^2}{N_2 c^2 (d_0^2 + R_2^2)} \int_{-T_1}^{T_2} \dot{s}^2 \left(t - \frac{(d_0^2 + R_2^2)^{1/2}}{c}\right) \sin^2(\gamma_0 - \epsilon_0 - \eta)\,dt\right]^{-1} \tag{22}$$

Hence under the assumption that $R_1 = R_2$ and $N_1 = N_2$, Eq. (23) depends inversely on

$$f(\delta) = \sin^2 \delta + \sin^2(\delta - \eta) = 1 - \cos\eta \cos(2\delta - \eta) \tag{23}$$

where $\delta = \gamma_0 - \epsilon_0$. Note that $f(\delta)$ is symmetric about $\delta = \eta/2$, which corresponds to when the vehicle is halfway between the two stations (see Fig 1). When $0 < \eta < 90°$, $f(\delta)$ increases as δ deviates from $\eta/2$, or when the vehicle moves toward either station from the midpoint. So, when $0 < \eta < 90°$, the worst performance is when the vehicle is exactly halfway between the two stations. This is shown in Fig. 4. When $\eta > 90°$, the converse is true and the best performance is when the vehicle is exactly halfway between the two stations. Since $f(\delta)$ is independent of δ when $\eta = 90°$, this is the best value of η from the viewpoint of uniformity of performance over a range of γ_0. An examination of Eq. (23) shows that for $80° \leqslant \eta \leqslant 100°$, the variation of performance is less than 20% for δ from 0 to η.

The above considerations are when $\omega = \nu = 0$. Let us now consider when $\omega \neq 0$ and $\nu \neq 0$. In Appendix B, it is shown that approximate expressions for ϕ_1, ϕ_2, $\partial\phi_1/\partial\gamma_0$ and $\partial\phi_2/\partial\gamma_0$ are:

$$\phi_1(t,\gamma_0) \cong -\left(1 - \left(\frac{\nu}{c}\right)^2\right)^{-1} \left[\left(\frac{\nu}{c}\right)^2 + \frac{\nu d_0 \cos\alpha}{[(d_0^2 + R_1^2)(c^2 - \nu^2)]^{1/2}}\right] t + \left(\frac{d_0^2 + R_1^2}{c^2 - \nu^2}\right)^{1/2} - \frac{\nu d_0 \cos\alpha}{c^2 - \nu^2}$$

$$- \frac{R_1 d_0}{[(d_0^2 + R_1^2)(c^2 - \nu^2)]^{1/2}} \cos(\gamma_0 - \epsilon_0 - \omega t) \tag{24}$$

$$\phi_2(t,\gamma_0)$$

$$\cong - \left(1 - \left(\frac{\nu}{c}\right)^2\right)^{-1} \left[\left(\frac{\nu}{c}\right)^2 + \frac{\nu d_0 \cos\alpha}{[(d_0^2 + R_2^2)(c^2 - \nu^2)]^{1/2}}\right] t$$

$$+ \left(\frac{d_0^2 + R_2^2}{c^2 - \nu^2}\right)^{1/2} - \frac{\nu d_0 \cos\alpha}{c^2 - \nu^2}$$

$$- \frac{R_2 d_0}{[(d_0^2 + R_2^2)(c^2 - \nu^2)]^{1/2}} \cos(\gamma_0 - \epsilon_0 - \eta - \omega t) \tag{25}$$

$$\frac{\partial \phi_i(t,\gamma_0)}{\partial \gamma_0}$$

$$\cong \frac{R_1 d_0}{[(d_0^2 + R_1^2)(c^2 - \nu^2)]^{1/2}} \sin(\gamma_0 - \epsilon_0 - \omega t)$$

$$- \frac{\nu R_1}{c^2 - \nu^2} \left[1 - \left(\frac{d_0^2 + R_1^2}{c^2 - \nu^2}\right)^{-1/2} \left(1 - \left(\frac{\nu}{c}\right)^2\right)^{-1} t\right]$$

$$\cdot \sin(\gamma_0 - \epsilon_0 + \alpha - \omega t) \tag{26}$$

$$\frac{\partial \phi_2(t,\gamma_0)}{\partial \gamma_0}$$

$$\cong \frac{R_2 d_0}{[(d_0^2 + R_2^2)(c^2 - \nu^2)]^{1/2}} \sin(\gamma_0 - \epsilon_0 - \eta - \omega t)$$

$$- \frac{\nu R_2}{c^2 - \nu^2} \left[1 - \left(\frac{d_0^2 + R_2^2}{c^2 - \nu^2}\right)^{-1/2} \left(1 - \left(\frac{\nu}{c}\right)^2\right)^{-1} t\right]$$

$$\cdot \sin(\gamma_0 - \epsilon_0 + \alpha - \eta - \omega t) \tag{27}$$

In Eq. (26), the factor in front of $\sin(\gamma_0 - \epsilon_0 + \alpha - \omega t)$ is of the order 10^{-7} while the factor in front of $\sin(\gamma_0 - \epsilon_0 - \omega t)$ is of the order 10^{-2}. Hence, the second term in Eq. (26) can be neglected except when $\sin(\gamma_0 - \epsilon_0 + \alpha - \omega t)$ is sufficiently larger than $\sin(\gamma_0 - \epsilon_0 - \omega t)$. In an extreme case when $\gamma_0 - \epsilon_0 = 0°$ and $\alpha = 90°$, the first term in Eq. (26) is zero at $t = 0$. However, as t deviates sufficiently from 0, the first term will again dominate the second term. For example, if $|t| = 10$ sec, the first term is 10 times the second in Eq. (26). So, in instances when the observation time interval $T_1 + T_2$ is much larger than 10 sec, the contribution of the second term in Eq. (26) to the Cramer-Rao lower bound will be negligibly small. The same conclusion can be drawn for the second term in Eq. (29). Hence, neglecting these terms results in the following approximation to the Cramer-Rao lower bound (Eq. (5)):

$$E[(\hat{\gamma}_0 - \gamma_0)^2] \geqslant \left[\frac{2R_1^2 d_0^2}{N_1(c^2 - \nu^2)(d_0^2 + R_1^2)} \int_{-T_1}^{T_2} \dot{s}^2(t - \phi_1(t,\gamma_0)) \sin^2(\gamma_0 - \epsilon_0 - \omega t)\, dt + \frac{2R_2^2 d_0^2}{N_2(c^2 - \nu^2)(d_0^2 + R_2^2)} \int_{-T_1}^{T_2} \dot{s}^2(t - \phi_2(t,\gamma_0)) \sin^2(\gamma_0 - \epsilon_0 - \eta - \omega t)\, dt\right]^{-1} \tag{28}$$

where ϕ_1 and ϕ_2 are given by Eqs. (24) and (25) respectively.

Let us now compare Eq. (28) with Eq. (22) when $\omega = \nu = 0$. From Eqs. (24) and (25) it can be seen that the dependence of ϕ_1 and ϕ_2 on γ_0 is small. We may assume that ϕ_1 and ϕ_2 are both essentially independent of γ_0 in Eq. (28). So from the viewpoint of dependence on γ_0, the essential difference in the structure of Eq. (28) to the structure of Eq. (22) is the $\sin^2(\gamma_0 - \epsilon_0 - \omega t)$ and $\sin^2(\gamma_0 - \epsilon_0 - \eta - \omega t)$ factors in the integrands in Eq. (28) versus the corresponding $\sin^2(\gamma_0 - \epsilon_0)$ and $\sin^2(\gamma_0 - \epsilon_0 - \eta)$ factors in Eq. (22). Although the earth rotational angular velocity $\omega = 7.27 \times 10^{-5}$ rad/sec, for $t = 30$ minutes $\omega t = 7.5°$. Hence this difference is certainly not negligible. This points out a significant contribution to the estimation performance due to the rotation of the stations.

To assess the dependence of Eq. (28) on the angular position γ_0 we assume that ϕ_1 and ϕ_2 are essentially independent of γ_0 in Eq. (28). Under the assumption that $N_1 = N_2$ and $R_1 = R_2$, the integrand in Eq. (28) is directly proportional to

$$\sin^2(\gamma_0 - \epsilon_0 - \omega t) + \sin^2(\gamma_0 - \epsilon_0 - \eta - \omega t)$$

$$= 1 - \cos\eta \cos[2(\gamma_0 - \epsilon_0 - \omega t) - \eta] \ . \tag{29}$$

Comparing Eq. (29) to Eq. (23), we see that to a first order approximation, the conclusions regarding the dependence of performance on angular position $\gamma_0 - \epsilon_0$ in the case $\omega = \nu = 0$ still hold here. In particular, it is clear from Eq. (29) that from a viewpoint of uniformity of performance over a range of γ_0, angular positions near $\eta = 90^\circ$ are desirable.

Let us consider next the effect of varying the parameters R_1 and R_2 on the optimum attainable estimation performance. Recall that R_1 and R_2 are the distances from the stations to geocenter. In Eq. (17), the values of R_1 and R_2 are for ground-based stations. The other case of interest is when the two stations are in geostationary orbit with $R_1 = R_2$. We shall consider an orbital radius of up to 10^5 km. Hence we need to examine the dependence of Eq. (5) on $R_1 = R_2$ for a range of these parameters from 6.5×10^3 km to 10^5 km. We still take d_0, the distance of the vehicle to geocenter, to be as in Eq. (17). Hence d_0 is still much larger than R_1 and R_2. In this case, an examination of the derivation in Appendix B shows that we can still use the approximation in Eq. (28) to Eq. (5) with the expressions in Eqs. (24) and (25) for ϕ_1 and ϕ_2, respectively. Since d_0 is much larger than $R_1 = R_2$, an examination of Eqs. (24), (25) and (28) shows that the Cramer-Rao lower bound is proportional to $1/R_1^2 = 1/R_2^2$. In other words, the root mean square estimation error is directly proportional to $1/R_1 = 1/R_2$. Increasing, $R_1 = R_2$ from 6.5×10^3 km to 10^5 km will decrease the optimum root mean square estimation error by two orders of magnitude.

Finally, let us consider the effect of varying the observation duration $T_1 + T_2$ on the optimum attainable estimation performance. We assume that $T_1 + T_2$ is large compared to 10 seconds and that the other parameters are given as in Eq. (17). Then Eq. (28) is again a valid approximation of Eq. (5) with ϕ_1 and ϕ_2 approximated by Eqs. (24) and (25), respectively. We also assume that the frequency of the ranging signal $s(t)$ is much higher than $1/(T_1 + T_2)$ and also much higher than $\omega/2\pi$ (ω = rotational angular velocity of the stations). It is still difficult to assess the dependence of Eq. (28) on T_1 and T_2 in general because of the $\sin^2(\gamma_0 - \epsilon_0 - \omega t)$ and $\sin^2(\gamma_0 - \epsilon_0 - \eta - \omega t)$ terms in the integrals in Eq. (28). These terms change the value of the integrands as T_1 and T_2 are varied. To a first order approximation it appears that the right-hand side of Eq. (28) is inversely proportional to

$$(T_1 + T_2)\,[\sin^2(\gamma_0 - \epsilon_0 - \omega T_2) - \sin^2(\gamma_0 - \epsilon_0 + \omega T_1)$$

$$+ \sin^2(\gamma_0 - \epsilon_0 - \eta - \omega T_2) - \sin^2(\gamma_0 - \epsilon_0 - \eta + \omega T_1)]$$

$$= (T_1 + T_2)\,\{2 - \cos\eta \cos[2(\gamma_0 - \epsilon_0 - \omega T_2) - \eta)]$$

$$- \cos\eta \cos[2(\gamma_0 - \epsilon_0 + \omega T_1) - \eta]\}$$

In the case when $\eta = 90^\circ$, Eq. (30) reduces to $2(T_1 + T_2)$. Thus, when $\eta \cong 90^\circ$, the optimum attainable root mean square error performance is approximately inversely proportional to $\sqrt{(T_1 + T_2)}$.

Finally, we consider a specific ranging signal $s(t)$ and perform numerical computations of the Cramer-Rao lower bound.

Example

Consider a sinusoidal ranging signal of frequency f_c Hz. That is,

$$s(t) = \sqrt{2S} \cos(2\pi f_c t)$$

We assume that for $i = 1, 2$, the demodulated ranging signal power to noise spectral density ratio is

$$\frac{S}{N_i} = 10 \text{ dB.}$$

This signal-to-noise ratio is consistent with X-band carrier, 20-dB vehicle antennae gain, 53-dB station antenna gains, receiver noise temperatures of 50°K, 20 W vehicle transmitted power and a 3-dB modulation loss. We also assume that

$$d_0 = 8 \times 10^8 \text{ km}$$

$$\nu = 10 \text{ km/sec}$$

$$R_1 = R_2 = 6.5 \times 10^3 \text{ km}$$

These parameters are consistent with that encountered in a Jupiter mission with ground-based stations. We also assume that $T_1 = T_2$. Numerical Monte Carlo integration was used to compute the value of the Cramer-Rao lower bound for various values of signal frequency f_c and observation time duration $T_1 + T_2$. The numerical computations are within a 1% accuracy. These numerical results are summarized in Tables 1 and 2 below. The listed angle estimation accuracies in these tables are the square root of the Cramer-Rao lower bound.

Table 1 shows that the optimum angle estimation accuracy is inversely proportional to the frequency of the sinusoidal ranging signal. Although this particular relation between estimation accuracy and signal frequency does not hold in general, it can be easily seen from Eq. (5) that signals of higher frequency give a smaller Cramer-Rao lower bound. Also note that Table 2 shows that the estimation accuracy is approximately inversely proportional to $T_1 + T_2$, as we would expect, since $\eta = 90^\circ$.

We note that the above angle estimation accuracy was obtained using the Cramer-Rao lower bound, Eq. (5), which is valid when γ_0 is an unknown but nonrandom parameter. Suppose instead that γ_0 is a random parameter and can a priori be assumed to be normally distributed. Then the relevant lower bound on mean square estimation error is Eq. (7). We claim that if the a priori variance of γ_0 is much larger than the lower bound Eq. (5), then the above estimation accuracy calculation is still valid. This follows because Eq. (5) is essentially independent of γ_0, in this case since $\eta = 90^\circ$. Hence, the expectation term in Eq. (7) is 1/(lower bound Eq. (5)).

V. Conclusion

This work has considered the problem of estimating the angular position of a moving vehicle using two rotating stations. The optimum attainable angle mean square estimation error was derived along with an implementable suboptimal estimation algorithm. A situation comparable to that encountered in a Jupiter mission was further analyzed. In this situation it was shown that the optimum angle between the two stations from a viewpoint of uniformity of estimation performance is 90°. It was also shown that the optimum attainable estimation accuracy varies inversely with the distance of the stations from geocenter and approximately inversely with the square root of the observation time duration. The optimum attainable angular estimation accuracy was numerically computed for a sinusoidal ranging signal. These computations show that the optimum attainable estimation accuracy is 0.02 μrad for a 2 MHz signal and an observation time of one hour.

This work has only considered the problem of estimating one angle with the range, velocity and other angles of the vehicle known. Further work should be done to include some or all of these parameters as parameters to be estimated along with the angle considered in this work.

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Table 1. Estimation accuracy vs signal frequency

Signal frequency f_c	Optimum angle estimation accuracy
2 MHz	0.018 μrad
5 MHz	0.0072 μrad
10 MHz	0.0036 μrad
20 MHz	0.0018 μrad

Table 2. Estimation accuracy vs observation time

Observation time duration $(T_1 + T_2)$	Optimum angle estimation accuracy
10 min	0.046 μrad
30 min	0.026 μrad
60 min	0.018 μrad
90 min	0.015 μrad

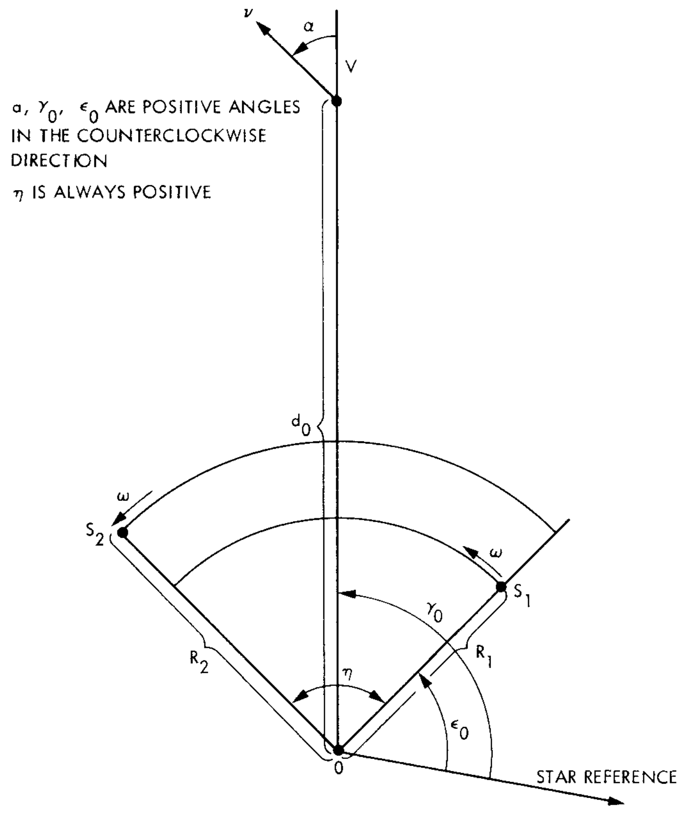

Fig. 1. Vehicle and station locations

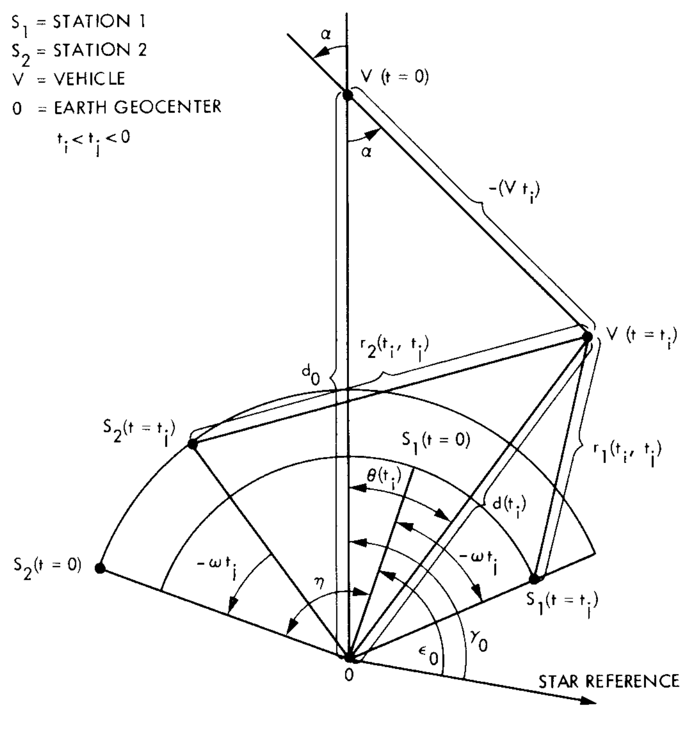

Fig. 2. Relative locations for $t = 0, t_i, t_j$

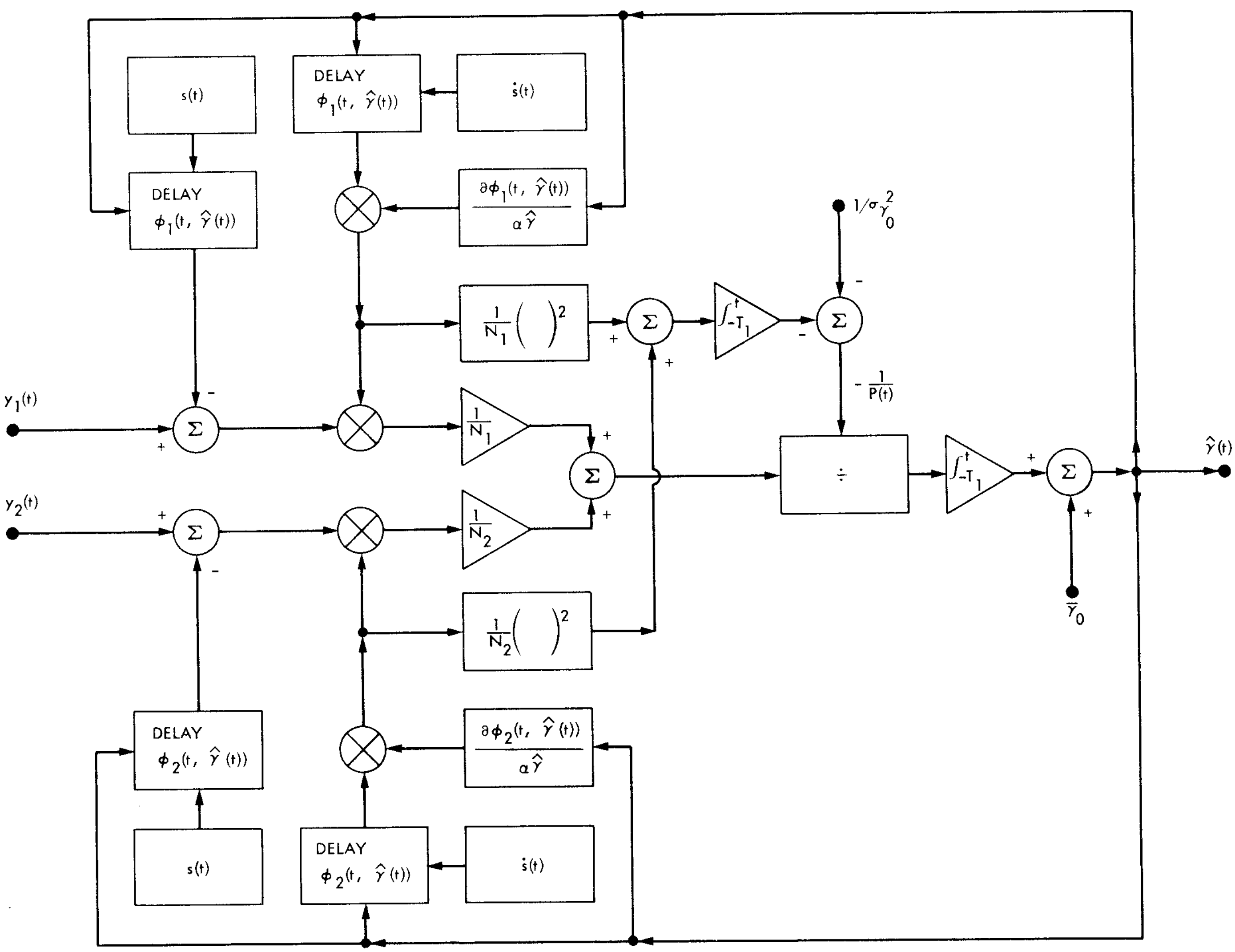

Fig. 3. Estimator block diagram

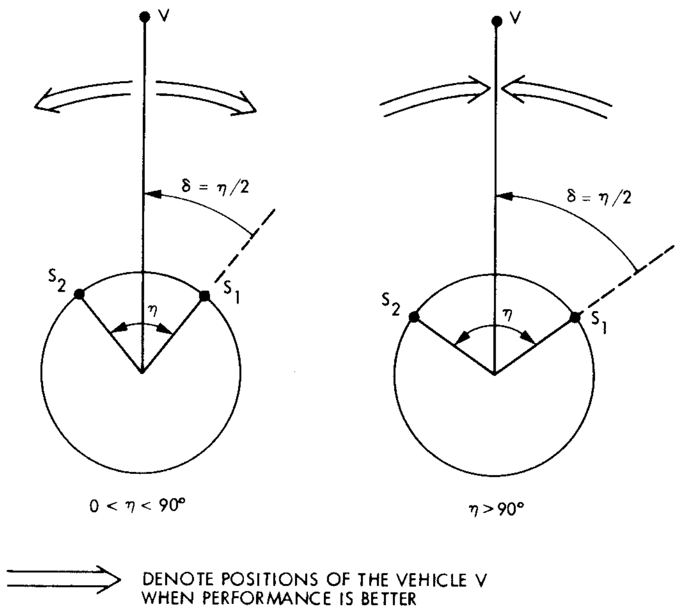

Fig. 4. Dependence of estimation performance on the angle between the stations

Appendix A
Derivation of Expressions for ϕ_1 and ϕ_2

It follows from Eqs. (1) and (2) that the signal value received at station S_i at time t is the signal value transmitted by the vehicle at time $t - \phi_i(t,\gamma_0)$ for $i = 1,2$. Define

$$\tau_i(t) = t - \phi_i(t,\gamma_0), \quad i = 1,2 \qquad \text{(A-1)}$$

For time instants t_i and t_j let

$r_1(t_i,t_j)$ = distance between the vehicle at time t_i and station S_1 at time t_j

$r_2(t_i,t_j)$ = distance between the vehicle at time t_i and station S_2 at time t_j.

It is then clear from Eqs. (1), (2) and (A-1) that

$$\phi_i(t,\gamma_0) = \frac{r_i\,[\tau_i(t),t]}{c} \qquad \text{(A-2)}$$

or equivalently,

$$\tau_i(t) = t - \frac{r_i\,[\tau_i(t),t]}{c} \qquad \text{(A-3)}$$

for $i = 1,2$. Hence if Eq. (A-3) can be solved for $\tau_i(t)$, then $\phi_i(t,\gamma_0)$ may be obtained using Eq. (A-1). In order to accomplish this we must first obtain expressions for $r_1(t_i,t_j)$ and $r_2(t_i,t_j)$. Since $\tau_i(t) < t$ in (A-3), we are interested in the cases $0 < t_i < t_j$, $t_i < 0 < t_j$ and $t_i < t_j < 0$ only in determining these expressions. Consider first the case when $t_i < t_j < 0$ as given in Fig. 2. Also we shall assume that $0 \leqslant \alpha \leqslant 180°$ in Fig. 2. Let

$d(t_i)$ = distance between the vehicle and Earth geocenter at time t_i.

$\theta(t_i)$ = relative angle at geocenter between the vehicle positions at $t = 0$ and $t = t_i$.

From the geometry of Fig. 2 we can write the following relations

$$d^2(t_i) = v^2 t_i^2 + d_0^2 + 2 d_0 v t_i \cos\alpha \qquad \text{(A-4)}$$

$$\frac{d(t_i)}{\sin\alpha} = \frac{-(vt_i)}{\sin\theta(t_i)} \qquad \text{(A-5)}$$

$$r_1^2(t_i,t_j) = d_1^2(t_i) + R_1^2 - 2R_1 d_1(t_i) \cos\,[\gamma_0 - \epsilon_0 - \omega t_j - \theta(t_i)] \qquad \text{(A-6)}$$

$$r_2^2(t_i,t_j) = d_1^2(t_i) + R_2^2 - 2R_2 d_1(t_i) \cos\,[\eta + \epsilon_0 - \gamma_0 + \theta(t_i) + \omega t_j] \qquad \text{(A-7)}$$

Next, using Eqs. (A-4) and (A-5) we get

$$\cos\theta\,(t_i) = [1 - \sin^2\theta(t_i)]^{1/2} = \frac{d_0 + vt_i\cos\alpha}{d(t_i)} \qquad \text{(A-8)}$$

Now, substitution of Eqs. (A-5) and A-8) into Eqs. (A-6) and (A-7) yield the following expressions for $r_1^2(t_i,t_j)$ and $r_2^2(t_i,t_j)$:

$$r_1^2\,(t_i,t_j) = d^2(t_i) + R_1^2 - 2R_1 d_0 \cos(\gamma_0 - \epsilon_0 - \omega t_j) - 2R_1 v t_i \cos(\alpha + \gamma_0 - e_0 - \omega t_j) \qquad \text{(A-9)}$$

$$r_2^2(t_i,t_j) = d^2(t_i) + R_2^2 - 2R_2 d_0 \cos(\eta + \epsilon_0 - \gamma_0 + \omega t_j) - 2R_2 v t_i \cos(\alpha - \eta - \epsilon_0 + \gamma_0 - \omega t_j) \qquad \text{(A-9)}$$

So, Eqs. (A-4), (A-8) and (A-9) define $r_1^2(t_i,t_j)$ and $r_2^2(t_i,t_j)$ for the case $t_i < t_j < 0$ and $0 \leqslant \alpha \leqslant 180°$. Arguments similar to the above can be used to establish Eqs. (A-4), (A-8) and (A-9) when $0 < t_i < t_j$ or $t_i < 0 < t_j$ and also when $180° \leqslant \alpha \leqslant 360°$. Hence, we may conclude that these equations hold for all cases of interest in solving Eq. (A-3) for $\tau_i(t)$. For simplicity let us denote $\tau_i(t)$ by τ_i for the remainder of the appendix. Squaring Eq. (A-3) yields

$$c^2(\tau_i - t)^2 = r_i^2(\tau_i,t), \quad i = 1,2 \qquad \text{(A-10)}$$

Next, substitution of Eqs. (A-4), (A-8) and (A-9) into Eq. (A-10) yields the following quadratic equations satisfied by τ_i and τ_2:

$$(c^2 - v^2)\tau_1^2 - 2[c^2 t + vd_0 \cos\alpha - vR_1 \cos(\alpha + \gamma_0 - \epsilon_0 - \omega t)]\tau_1$$

$$- [d_0^2 + R_1^2 - 2R_1 d_0 \cos(\gamma_0 - \epsilon_0 - \omega t) - c^2 t^2] = 0 \tag{A-11}$$

$$(c^2 - v^2)\tau_2^2 - 2[c^2 t + vd_0 \cos\alpha - vR_2 \cos(\alpha - \eta - \epsilon_0 + \gamma_0 - \omega t)]\tau_2$$

$$- [d_0^2 + R_2^2 - 2R_2 d_0 \cos(\eta + \epsilon_0 - \gamma_0 + \omega t) - c^2 t^2] = 0 \tag{A-12}$$

The constraint that $\tau_i < t$ identifies the roots of Eqs. (A-11) and (A-12) that give τ_1 and τ_2 respectively. These are:

$$\tau_1 = (1 - (v/c)^2)^{-1} [t + (vd_0 \cos\alpha)/c^2 - vR_1 \cos(\gamma_0 + \alpha - \epsilon_0 - \omega t)/c^2] - \Big\{(1 - (v/c)^2)^{-2} [t + (vd_0 \cos\alpha)/c^2 - vR_1 \cos(\gamma_0 + \alpha - \epsilon_0 - \omega t)/c^2]^2 + (1 - (v/c)^2)^{-1} [(d_0^2 + R_1^2 - 2R_1 d_0 \cos(\gamma_0 - \epsilon_0 - \omega t))/c^2 - t^2]\Big\}^{1/2} \tag{A-13}$$

$$\tau_2 = (1 - (v/c)^2)^{-1} [t + (vd_0 \cos\alpha)/c^2 - vR_2 \cos(\gamma_0 + \alpha - \epsilon_0 - \eta - \omega t)/c^2] + \Big\{(1 - (v/c)^2)^{-2} [t + (vd_0 \cos\alpha)/c^2 - vR_2 \cos(\gamma_0 + \alpha - \epsilon_0 - \eta - \omega t)/c^2]^2 + (1 - (v/c)^2)^{-1} [(d_0^2 + R_2^2 - 2R_2 d_0 \cos(\gamma_0 - \epsilon_0 - \eta - \omega t))/c^2 - t^2]\Big\}^{1/2} \tag{A-14}$$

Finally, Eqs. (A-1), (A-13) and (A-14) give the expressions in Eqs. (3) and (4) for $\phi_1(t,\gamma_0)$ and $\phi_2(t,\gamma_0)$, respectively.

Appendix B

Approximations to ϕ_1, ϕ_2, $\frac{\partial \phi_1}{\partial \gamma_0}$ and $\frac{\partial \phi_1}{\partial \gamma_0}$

Consider first ϕ_1 given by Eq. (3). Ignoring terms smaller than 10^{-9}, we have

$$\Big\{(1-(v/c)^2)^{-2}\,[t+(vd_0\cos\alpha)/c - vR_1\cos(\gamma_0+\alpha-\epsilon_0 - \omega t)/c^2]^2 + (1-(v/c)^2)^{-1}\,[(d_0^2+R_1^2-2R_1d_0 \cos(\gamma_0-\epsilon_0-\omega t))/c^2 - t^2]\Big\}^{1/2}$$

$$\cong \left(\frac{d_0^2+R_1^2}{c^2-v^2}\right)^{1/2}\left[1+\frac{2vd_0\cos\alpha}{(d_0^2+R_1^2)(1-(v/c)^2)}\,t - \frac{2R_1d_0}{d_0^2+R_1^2}\cos(\gamma_0-\epsilon_0-\omega t)\right]^{1/2}$$

$$\cong \left(\frac{d_0^2+R_1^2}{c^2-v^2}\right)^{1/2}\left[1+\frac{2vd_0\cos\alpha}{(d_0^2+R_1^2)(1-(v/c)^2)}\,t - \frac{R_1d_0}{d_0^2+R_1^2}\cos(\gamma_0-\epsilon_0-\omega t)\right] \tag{B-1}$$

where the second line in Eq. (B-1) is obtained by using the approximation $(1+x)^{1/2} \cong 1+(1/2)x$ with accuracy of the order 10^{-10}. Substitution of (B-1) into Eq. (3) and ignoring terms of order 10^{-7} and smaller gets Eq. (26). A similar argument using Eq. (4) gets Eq. (27). Consider next $\partial\phi_1/\partial\gamma_0$ and $\partial\phi_2/\partial\gamma_0$. Differentiating Eq. (3) and Eq. (4) yields

$$\frac{\partial\phi_1(t,\gamma_0)}{\partial\gamma_0} = -\frac{vR_1}{c^2-v^2}\sin(\gamma_0-\epsilon_0+\alpha-\omega t) + \Big\{(1-(v/c)^2)^{-2}\,[t+(vd_0\cos\alpha)/c^2 - vR_1\cos(\gamma_0-\epsilon_0+\alpha-\omega t)/c^2]^2 + (1-(v/c)^2)^{-1}\,[(d_0^2+R_1^2-2R_1d_0\cos(\gamma_0-\epsilon_0 - \omega t)/c^2 - t^2]\Big\}^{-1/2}$$
$$\cdot\Big\{(1-(v/c)^2)^{-2}\,[t+(vd_0\cos\alpha)/c^2 - vR_1\cos(\gamma_0-\epsilon_0+\alpha-\omega t)/c^2] \cdot (vR_1/c^2)\sin(\gamma_0-\epsilon_0+\alpha-\omega t) + \frac{R_1d_0}{c^2-v^2}\sin(\gamma_0-\epsilon_0-\omega t)\Big\} \tag{B-2}$$

$$\frac{\partial\phi_2(t,\gamma_0)}{\partial\gamma_0} = -\frac{vR_2}{c^2-v^2}\sin(\gamma_0-\epsilon_0+\alpha-\eta-\omega t) + \Big\{(1-(v/c)^2)^{-2}\,[t+vd_0\cos\alpha/c^2 - vR_2\cos(\gamma_0-\epsilon_0+\alpha-\eta-\omega t)/c^2]^2 + (1-(v/c)^2)^{-1}\,[(d_0^2+R_2^2-2R_2d_0\cos(\gamma_0-\epsilon_0 - \eta-\omega t)/c^2 - t^2]\Big\}^{-1/2}$$
$$\cdot\Big\{(1-(v/c)^2)^{-1}\,[t+(vd_0\cos\alpha)/c^2 - vR_2\cos(\gamma_0-\epsilon_0+\alpha-\eta-\omega t)/c^2] \cdot (vR_2/c^2)\sin(\gamma_0-\epsilon_0+\alpha-\eta-\omega t) + \frac{R_2d_0}{c^2-v^2}\sin(\gamma_0-\epsilon_0-\eta-\omega t)\Big\} \tag{B-3}$$

An approximation similar to that used in (B-1) can then be used with (B-2) and (B-3) to establish Eqs. (28) and (29).