# A Binary Coherent Optical Receiver for the Free-Space Channel

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*The free-space channel is an ideal medium for communicating by means of spatially and temporally coherent optical fields. Here we derive the structure of a maximum a-posteriori optical homodyne receiver for binary antipodal signals. The effects of background radiation and phase referencing errors on receiver performance are also examined. It is shown that this receiver structure is relatively insensitive to random phase errors, while moderate background radiation has virtually no effect on receiver performance.*

## I. Introduction

The structure of a binary optical receiver for processing spatially and temporally coherent optical fields is derived, based on the maximum a-posteriori (MAP) decoding criterion. The effects of background radiation and phase error on receiver performance are also examined. The receiver structure is shown in block diagram form in Fig. 1. The received field $f_r(t)$ is assumed to be in the form of a coherent plane wave. The optical antenna consists of a lens (or properly shaped mirror) of total collecting area $(A_r + A_e)$, which focuses the captured field towards the active surface of the photodetector. A fraction of the captured field is diverted to the phase-estimator subsystem by the beam splitter $B_1$, with power transmission factor $T_1 = A_r/(A_r + A_e)$, while only a negligible fraction of the received field is lost in propagating through the beam splitter $B_2$ (power transmission factor $T_2 \leqslant 1$), which is used to spatially combine the received and local fields. The total signal power reaching the detector therefore appears to have been collected by an effective collecting area $A_r$. The output of the phase estimator subsystem is a continuous estimate of the received phase, $\hat{\phi}_r(t)$, which is used to control the phase of the local laser. The local optical field combines with the received field on the surface of $B_2$ and the sum field, composed of the reflected portion of the local field and the transmitted part of the received field, propagates to the active surface of the photodetector. Even though only a small fraction of the local field is reflected by $B_2$, this is usually of little concern since generally ample local power is available. It is further assumed that the local field is processed so that the spatial distribution of the local and received fields over the detector surface are identical.

It is well known that if the amplitude of the local field is sufficiently great, then the effects of thermal noise generated within the receiver are suppressed, and shot-noise limited operation is achieved (Ref. 1). Assuming arbitrarily large detector bandwidth, the detector output can therefore be modelled as a point process with average rate determined entirely by the optical power of the detected local and received fields. An optical filter (with bandwidth $\Delta\nu$ Hz), a polarizer, and a spatial filter are also employed to reduce the intensity of the background radiation reaching the photodetector. To minimize the average bit-error probability, the output of the photodetector is processed by a MAP decoder. The decoder structure derived here is the MAP decoder only for the case of negligible phase error and background radiation. However, the performance of this receiver in the presence of random phase errors and background fields will also be examined.

## II. MAP Receiver Structure

The received optical field is assumed to be in the form of a linearly polarized plane wave, normally incident on the receiver aperture. Therefore it can be represented entirely in terms of its temporal characteristics as

$$f_r(t) = a_r\, p(t) \exp\left[j(\omega t + \phi_r(t))\right] \tag{1}$$

where $p(t) = (-1)^{i+1}$, $i = 0, 1$ over the interval $(0, \tau)$, $a_r$ is the real received field amplitude, $\omega$ is the radian frequency of the optical field, and $\phi_r(t)$ is a random phase process due to phase instabilities within the transmitter laser. The local field can be referred back to the receiver aperture and represented as

$$f_L(t) = a_L \exp\left[j(\omega t + \phi_L(t))\right] \tag{2}$$

where again $a_L$ is the (real) local field amplitude and $\phi_L(t)$ is a random phase process due to phase variations within the local laser.

The photodetector responds to the total optical power collected over its active surface. The count intensity at the output of the photodetector is due to the total power generated by the sum of the received and (equivalent) local fields over the receiver aperture (with effective collecting area $A_r$), and can be represented as

$$n(t) = \alpha A_r |f_r(t) + f_L(t)|^2 = \alpha A_r \left\{a_L^2 + a_r^2 + 2a_L a_r p(t) \cos \phi_e(t)\right\} \tag{3}$$

where $\alpha = \eta/h\nu$ ($\eta$ is the detector quantum efficiency, $h$ is Planck's constant, and $\nu = \omega/2\pi$ is the optical frequency) and $\phi_e(t) = \phi_r(t) - \phi_L(t)$ is defined as the phase-error process. If a sufficiently small phase-error can be maintained, then we can let $\cos \phi_e(t) \cong 1$. Since $p(t)$ takes on the values $\pm 1$ (depending on the binary hypothesis), the count-intensity is seen to be the sum of a constant count-rate due to the local and received fields, and a hypothesis-dependent rate that is either added to or subtracted from the constant level. If background radiation can be ignored, then the total count over each synchronous $\tau$-second signaling interval can be modeled as a Poisson random variable, with average count

$$K_i = \alpha A_r \int_0^\tau n(t|H_i)\, dt = \alpha A_r \tau \left\{a_L^2 + a_r^2 + 2(-1)^{i+1} a_L a_r\right\}; \quad i = 0, 1 \tag{4}$$

where $H_i$ denotes the $i$-th hypothesis.

It is well known that the bit-error probability is minimized if a maximum a-posteriori (or MAP) decoding strategy is employed. For equilikely hypotheses, the MAP decoder decides on the basis of the log-likelihood test

$$\ln[P(k|H_0)] \underset{H_1}{\overset{H_0}{\gtrless}} \ln[P(k|H_1)] \tag{5}$$

where $k$ is the observed count, with probability density

$$P(k|H_i) = \frac{K_i^k}{k!} e^{-K_i} \tag{6}$$

and the average counts $K_i$ are defined by Eq. (4). Substituting Eqs. (4) and (6) into Eq. (5) yields the test

$$k \,\ell n\, K_0 - K_0 \underset{H_1}{\overset{H_0}{\gtrless}} k \,\ell n\, K_1 - K_1 \tag{7}$$

or, equivalently

$$k \underset{H_0}{\overset{H_1}{\gtrless}} \frac{K_1 - K_0}{(\ell n\, K_1 - \ell n\, K_0)} \triangleq T_h \tag{8}$$

Equation (8) is seen to be a simple threshold test. The MAP decoder therefore observes the total count at the end of each $\tau$-second interval, and selects on the basis of a threshold comparison. Note that for $a_r << a_L$ (which is generally the case) we can write

$$\ell n\, K_1 - \ell n\, K_0 = \ell n \left(1 + \frac{2a_r}{a_L}\right) - \ell n \left(1 - \frac{2a_r}{a_L}\right) \approx \frac{4a_r}{a_L} \tag{9}$$

from which it follows that

$$T_h = \alpha A_r a_L^2 \tau \tag{10}$$

Therefore, the threshold is the average count due to the local field over the counting interval.

The performance of the binary homodyne MAP receiver can be obtained by summing the count random-variables above and below threshold. Assuming that $T_h$ is not integer-valued, and defining $\varepsilon$ as the greatest integer contained in $T_h$, the average bit-error probability becomes

$$P(E) = \frac{1}{2}\left[\sum_{k=\varepsilon+1}^{\infty} \frac{K_0^k}{k!} e^{-k_0} + \sum_{k=0}^{\varepsilon} \frac{K_1^k}{k!} e^{-K_1}\right] = \frac{1}{2}\left[\frac{\Gamma(1+\varepsilon, K_1)}{\varepsilon!} + \frac{\gamma(1+\varepsilon, K_0)}{\varepsilon!}\right] \tag{11}$$

where

$$\Gamma(a,x) = \int_x^{\infty} e^{-t} t^{(a-1)}\, dt$$

and

$$\gamma(a,x) = \int_{o}^{x} e^{-t} t^{(a-1)} dt$$

This expression does not yield to further simplification by analytic means. However, we can invoke the Central Limit Theorem, and argue that since the average counts $K_i$ are large, each observed count can be considered the sum of a large number of independent Poisson random variables, and therefore we can approximate the discrete sums in Eq. (11) by integrals of the normal density (with mean and variance $K_i$). Defining the random variables $y_i$ as

$$y_i = \frac{x_i - K_i}{\sqrt{K_i}}; \quad i = 0, 1 \tag{12}$$

we obtain

$$P(E) = \frac{1}{2} \left[ \sum_{i=0}^{1} \int_{T_{hi}}^{\infty} \frac{e^{-y_i^2/2}}{\sqrt{2\pi}} dy_i \right] \tag{13}$$

$$T_{hi} = (-1)^i \frac{T_h - K_i}{\sqrt{K_i}} \simeq \sqrt{4K_S} \tag{14}$$

where we again assumed that $a_r/a_L \ll 1$ and $K_S = \alpha A_r a_r^2 \tau$ is the average count due to the signal field. This expression can be written in terms of the function $Q(u)$, where

$$Q(u) = \frac{1}{\sqrt{2\pi}} \int_{u}^{\infty} e^{-\beta^2/2} d\beta \tag{15}$$

Substituting Eq. (15) into Eq. (13) yields

$$P(E) \simeq Q\left(\sqrt{4K_S}\right) \tag{16}$$

It is clear, therefore, that when background radiation and phase estimation errors can be neglected, the performance of the homodyne receiver depends exponentially on the sum of the count-energies in the binary signals, $2K_s$. This can be seen from the asymptotic upper bound

$$Q(u) \leqslant (2\pi u^2)^{-1/2} e^{-u^2/2} \tag{17a}$$

or

$$Q\left(\sqrt{4K_s}\right) \leqslant (8\pi K_s)^{-1/2} e^{-2K_s} \tag{17b}$$

Receiver performance for the ideal case discussed here is shown in Fig. 2. Next we examine the effects of background fields on receiver performance.

## III. The Effects of Background Fields on Receiver Performance

To assess the effects of background radiation on receiver performance, we shall represent the background radiation in complex form as (Ref. 2):

$$b(t) = [B_c(t) - jB_s(t)]\, e^{j\omega t} \tag{18}$$

where $B_c(t)$ and $B_s(t)$ are uncorrelated, zero mean, narrowband Gaussian noise envelopes with two-sided spectral level $N_0$. If the optical detector is preceded by a narrowband optical filter of bandwidth $\Delta\nu$ Hz, then the first and second moments of the noise field become

$$E[b(t)] = 0 \tag{19a}$$

$$E[|b(t)|^2] = E[B_c^2(t)] + E[B_s^2(t)] = 2N_0 \Delta\nu \tag{19b}$$

For homodyne detection, the detector responds to the total optical power generated by the sum of the local, received, and background fields. Assuming that phase-errors can be neglected, and that the amplitude of the local field is much greater than either the amplitude of the received field or the standard deviation of the noise field (collected in a single spatial mode) the count-intensity at the detector's output can be represented as

$$n(t) = \alpha A_r |a_L + a_r p(t) + B_c(t) - jB_s(t)|^2 \simeq \alpha A_r \,[a_L^2 + 2a_L\,(a_r p(t) + B_c(t))] \tag{20}$$

The use of a narrowband optical filter generally ensures the validity of this approximation. It is apparent therefore that with negligible phase-error, the detector responds to the local field, the signal field, and the in-phase component of the noise envelope. The average count at the end of each $\tau$-second signaling interval is the time-integral of the count-intensity process

$$K_i = \alpha A_r a_L^2 \tau + (-1)^{i+1}\, 2a_L a_r \tau \alpha A_r + y \tag{21}$$

where we define the (Gaussian) random variable $y$ as the integral of the background-induced noise process

$$y = 2\alpha\, a_L A_r \int_0^{\tau} B_c(t)\, dt \tag{22}$$

Clearly, $E[y] = 0$. The variance can be evaluated by writing $y^2$ as a product of integrals, and taking the expectation over the integrand:

$$E[y^2] = 4\alpha^2 a_L^2 A_r^2 \int_0^{\tau} d\zeta_1 \int_0^{\tau} d\zeta_2 E\,[B_c(\zeta_1)\, B_c(\zeta_2)] = 4\alpha^2 a_L^2 A_r^2 \int_0^{\tau} d\zeta_1 \int_0^{\tau} d\zeta_2 R_c(\zeta_1 - \zeta_2) \tag{23}$$

where we define the autocorrelation function $R_c(\zeta)$ as

$$R_c(\zeta) = N_0 \int_{-\Delta\nu/2}^{\Delta\nu/2} e^{j2\pi f\zeta}\, df = N_0 \Delta\nu \left[\frac{\sin(\pi\Delta\nu\zeta)}{\pi\Delta\nu\zeta}\right] \tag{24}$$

Letting $\zeta = \zeta_1 - \zeta_2$, Eq. (23) can be rewritten as

$$E[y^2] = 4\alpha^2 a_L^2 A_r^2 \tau \int_{-\tau}^{\tau} \left(1 - \frac{|\zeta|}{\tau}\right) R_c(\zeta)\, d\zeta \simeq 4\alpha^2 a_L^2 A_r^2 \tau \int_{-\infty}^{\infty} R_c(\zeta)\, d\zeta = 4\alpha^2 a_L^2 A_r^2 \tau N_0 \triangleq \sigma_y^2 \tag{25}$$

In evaluating the first integral of Eq. (25), we made use of the fact that the coherence-time of the noise envelope is much smaller than the signaling interval, $\tau$.

The performance of the homodyne receiver can be evaluated by again invoking the Central Limit Theorem, approximating the sum of the Poisson density (conditioned on $y$) by a properly defined integral of the normal density, and averaging over the density of $y$. Since $y$ is Gaussian with zero mean and variance $\sigma_y^2$, the average bit error-probability becomes

$$P(E) = \frac{1}{2} \int_{-\infty}^{\infty} dy \frac{e^{-y^2/2\sigma_y^2}}{\sqrt{2\pi\sigma_y^2}} \left\{ \int_{T_{h1}}^{\infty} \frac{e^{-(x_1 - (T_h - m + y))^2/2(T_h - m + y)}}{\sqrt{2\pi(T_h - m + y)}}\, dx_1 \right.$$

$$\left. + \int_{-\infty}^{T_{h2}} \frac{e^{-(x_2 - (T_h + m + y))^2/2(T_h + m + y)}}{\sqrt{2\pi(T_h + m + y)}}\, dx_2 \right\} \tag{26}$$

where

$$m = 2\alpha A_r \tau a_L a_r$$

$$T_{h1} = (m - y)/\sqrt{T_h - m + y}$$

$$T_{h2} = (-m - y)/\sqrt{T_h + m + y}$$

If we make the change of variables

$$z_1 = \frac{x_1 - (T_h - m + y)}{\sqrt{T_h - m + y}} \tag{27a}$$

$$z_2 = \frac{x_2 - (T_h + m + y)}{\sqrt{T_h + m + y}} \tag{27b}$$

then Eq. (26) can be written as

$$P(E) = \frac{1}{2} \int_{-\infty}^{\infty} dy \, \frac{e^{-y^2/2\sigma_y^2}}{\sqrt{2\pi\sigma_y^2}} \left\{ Q\left( \frac{m-y}{\sqrt{T_h}\left(1+\frac{m+y}{T_h}\right)^{1/2}} \right) + Q\left( \frac{m+y}{\sqrt{T_h}\left(1-\frac{m-y}{T_h}\right)^{1/2}} \right) \right\} \tag{28}$$

where $Q(\cdot)$ is defined in Eq. (15).

The $Q(\cdot)$ functions can be expanded as

$$Q_1 \triangleq Q\left( \frac{m-y}{\sqrt{T_h}\left(1-\frac{m-y}{T_h}\right)^{1/2}} \right) = Q\left( \frac{m-y}{\sqrt{T_h}} \left[ 1 + \frac{m-y}{2T_h} + \cdots \right] \right) \tag{29a}$$

$$Q_2 \triangleq Q\left( \frac{m+y}{\sqrt{T_h}\left(1+\frac{m+y}{T_h}\right)^{1/2}} \right) = Q\left( \frac{m+y}{\sqrt{T_h}} \left[ 1 - \frac{m+y}{2T_h} + - \cdots \right] \right) \tag{29b}$$

Note that $(m \pm y)/\sqrt{T_h}$ is a function of $a_r$, but not of $(a_r/a_L)$, while $(m \pm y)/T_h$ is a function of $(a_r/a_L)$. If we hold $a_r$ fixed, then as $(a_r/a_L) \to 0$, $(m \pm y)/T_h \to 0$ as well, whereas $(m \pm y)/\sqrt{T_h}$ remains constant. Therefore, as $(a_r/a_L) \to 0$, the bracketed term on the right-hand side approaches one, i.e.,

$$\left[ 1 \pm \frac{m \mp y}{2T_h} + \cdots \right] \to 1 \tag{30}$$

and the approximations

$$Q_1 \simeq Q\left( \frac{m-y}{\sqrt{T_h}} \right) \tag{31a}$$

$$Q_2 \simeq Q\left( \frac{m+y}{\sqrt{T_h}} \right) \tag{31b}$$

become valid. Note that these are exactly the results we would have obtained if the inner integrals in Eq. (26) were integrals of Gaussian random variables with mean $(T_h \pm m + y)$, and variance $T_h$. Substituting Eq. (31) into (28), averaging over $y$, and making use of Eqs. (10) and (25) yields the limiting form

$$P(E) \simeq Q\left( \sqrt{\frac{m^2}{T_h\left(1+\frac{\sigma_y^2}{T_h}\right)}} \right) = Q\left( \sqrt{\frac{4K_s}{1+4\alpha A_r N_0}} \right) \tag{32}$$

in the limit as $(a_r/a_L) \to 0$.

It is perhaps more meaningful to relate receiver performance to the noise-field parameter $K_b = 2\alpha A_r N_0 \Delta\nu\tau$, which is defined as the average number of counts generated by a multimode noise field at the output of a direct-detection receiver with effective collecting area $A_r$. When expressed in terms of $K_b$, the error probability of the homodyne receiver becomes

$$P(E) \simeq Q\left(\sqrt{\frac{4K_s}{1+\delta}}\right)$$

$$\delta = \frac{2K_b}{\Delta\nu\tau} = 4\alpha A_r N_0 \tag{33}$$

Note that the use of very narrowband optical filters for background suppression is not required, since $\delta$ is not a function of $\Delta\nu$. The optical filter bandwidth must be narrow enough only to guarantee that Eq. (20) remains valid.

The ability of the homodyne receiver to suppress background radiation is evident from Eq. (33), since in typical applications the product of the optical bandwidth $\Delta\nu$ and the bit duration $\tau$ is a very large number ($\Delta\nu\tau >> 1$). This implies that the performance of the homodyne receiver remains unaffected by background radiation, even when operating in relatively high-background environments corresponding to $K_b >> 1$. In contrast, the performance of receiver structures that direct detect the noise field deteriorate significantly when background fields of comparable intensity are present (Ref. 1).

The effect of background radiation on receiver performance is shown in Fig. 3, as a function of the noise parameter $\delta$. Note that over the range of error-probabilities considered, $P(E)$ remains essentially unaffected by background radiation until the noise intensity becomes so great that $\delta$ exceeds ~0.1. In typical applications, $\Delta\nu \gtrsim 3 \times 10^{11}$ Hz (corresponding to an optical bandwidth of $\Delta\lambda \gtrsim 10$ Å at a wavelength of $\lambda = 1\mu$), while the bit-duration is generally greater than $10^{-9}$ seconds; this implies that the background field must be intense enough to generate a large number of equivalent direct-detected counts ($K_b \gtrsim 15$) over the bit interval. In well designed receivers, background fields of this intensity are encountered only if the Sun, or other bright interference, is directly included in the receiver's field of view (Ref. 3).

## IV. The Effects of Phase-Error on Receiver Performance

When the received field is corrupted by background radiation, and the effects of phase error also have to be taken into account, the count-intensity becomes

$$n(t) = \alpha A_r \left[a_L^2 + 2a_L a_r p(t) \cos\phi_e(t) + 2a_L (B_c(t) \cos\phi_e(t) + B_s(t) \sin\phi_e(t)\right] \tag{34}$$

The count intensity is therefore influenced both by random phase-error, and by random background noise. If we assume that the phase error is narrowband compared to both the modulation and the noise envelope, then the phase error can be considered constant over the signaling interval, and treated as a random variable. The noise term now contains in-phase and quadrature components weighted by sinusoidal functions of the phase error. Conditioned on a given phase error, the count intensity becomes

$$K_i(\phi_e) = \alpha A_r a_L^2 \tau + 2(-1)^{i+1} a_L a_r \tau\alpha A_r \cos\phi_e + y$$

$$y = 2\alpha A_r a_L \left(\cos\phi_e \int_0^\tau B_c(t)\,dt + \sin\phi_e \int_0^\tau B_s(t)\,dt\right) \tag{35}$$

Clearly, $y$ is a zero-mean Gaussian random variable, with variance $\sigma_y^2 = 4\alpha^2 a_L^2 A_r^2 N_0 \tau$, exactly as before. The conditional error probability, $P(E|\phi_e)$, is therefore given by an expression similar to Eq. (33), but with $\sqrt{4K_s}$ replaced by $\sqrt{4K_s} \cos\phi_e$:

$$P(E \mid \phi_e) = Q\left(\sqrt{\frac{4K_s}{1+\delta}} \cos\phi_e\right) \tag{36}$$

The bit-error probability is the average of the conditional error-probability over the density of the random phase error:

$$P(E) = \overline{Q\left(\sqrt{\frac{4K_s}{1+\delta}} \cos\phi_e\right)}^{\phi_e} \tag{37}$$

The probability density of the phase-error can often be modelled in terms of the parameterized Tikhonov density (Ref. 4):

$$p(\phi_e) = \frac{e^{\sigma_e^{-2} \cos\phi_e}}{2\pi I_0 (\sigma_e^{-2})} \; ; \quad |\phi_e| \leqslant \pi \tag{38}$$

where $I_0(\sigma_e^{-2})$ is the modified Bessel function of zero order, argument $\sigma_e^{-2}$, and where $\sigma_e^2$ is roughly the variance of the phase error for $\sigma_e^2 \ll 1$. Numerical integration was employed to evaluate Eq. (37), using the phase error density of Eq. (38). Receiver performance in the presence of phase error, but with negligible background interference, is shown in Fig. 4a. It is apparent that phase error effects do not become significant until the rms phase error exceeds 0.14 radians (corresponding to $\sigma_e^2 \gtrsim 0.02$) in the range of error probabilities considered: therefore, we can conclude that the performance of the homodyne receiver is not very sensitive to phase error effects.

The effects of background radiation and random phase errors are shown in Fig. 4b, corresponding to $\delta = 0.1$, or $K_b = 0.2\, \Delta\nu\tau$. We emphasize that since $\Delta\nu\tau$ is generally a large number, $\delta = 0.1$ implies very intense background radiation. However, since the background radiation enters the error-probability expression only as $\sqrt{1+\delta}$ in Eq. (33), even relatively high-intensity radiation has little effect on the average bit-error probability, as can be seen by comparing the performance curves of Figs. 4a and 4b.

## V. Summary and Conclusions

The structure of a coherent binary optical MAP receiver has been derived, and the effects of background radiation and phase estimation errors on receiver performance were evaluated. It was found that random phase errors do not affect the performance of the coherent receiver significantly, as long as the standard deviation of the phase error remains less than roughly 0.14 radians (or 8 degrees). It was also shown that background radiation is effectively suppressed by the optical homodyne receiver: Significant deterioration in receiver performance is observed only in the presence of extremely intense background fields. The most notable characteristic of the binary optical homodyne receiver therefore appears to be its ability to achieve quantum-limited performance in high-background environments, provided that sufficiently accurate estimates of the received phase can be obtained.

# References

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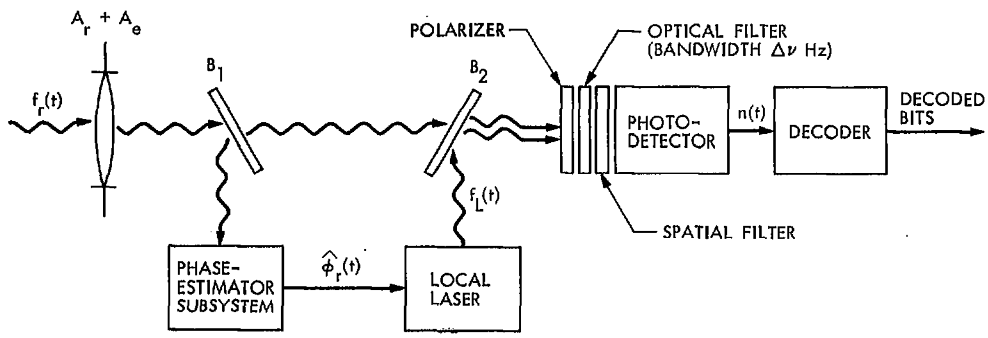


**Fig. 1. Coherent optical receiver block diagram**

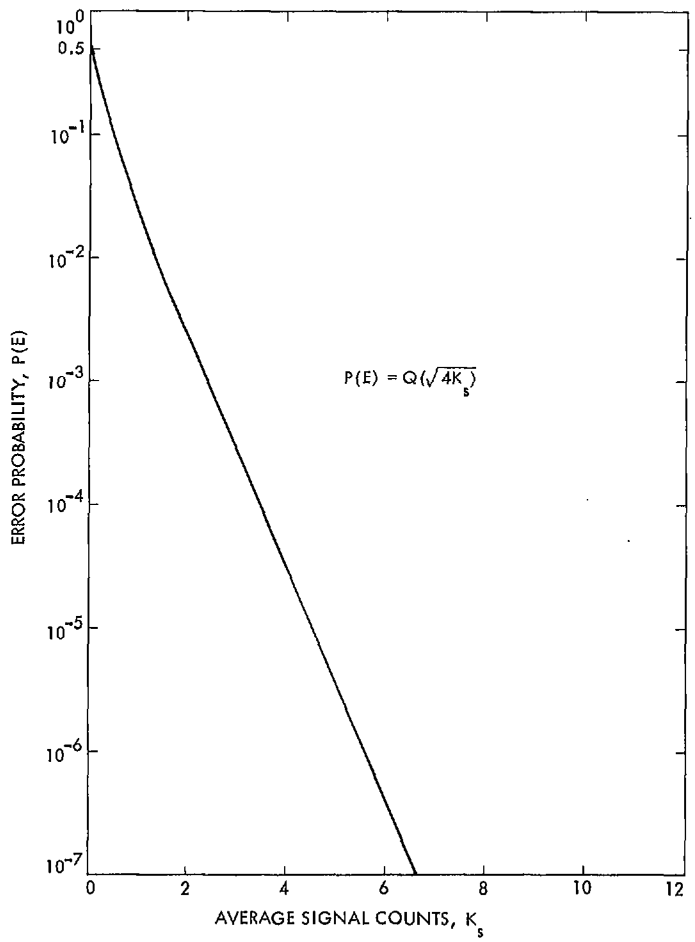


**Fig. 2. Receiver performance under ideal conditions: negligible background radiation and phase error**

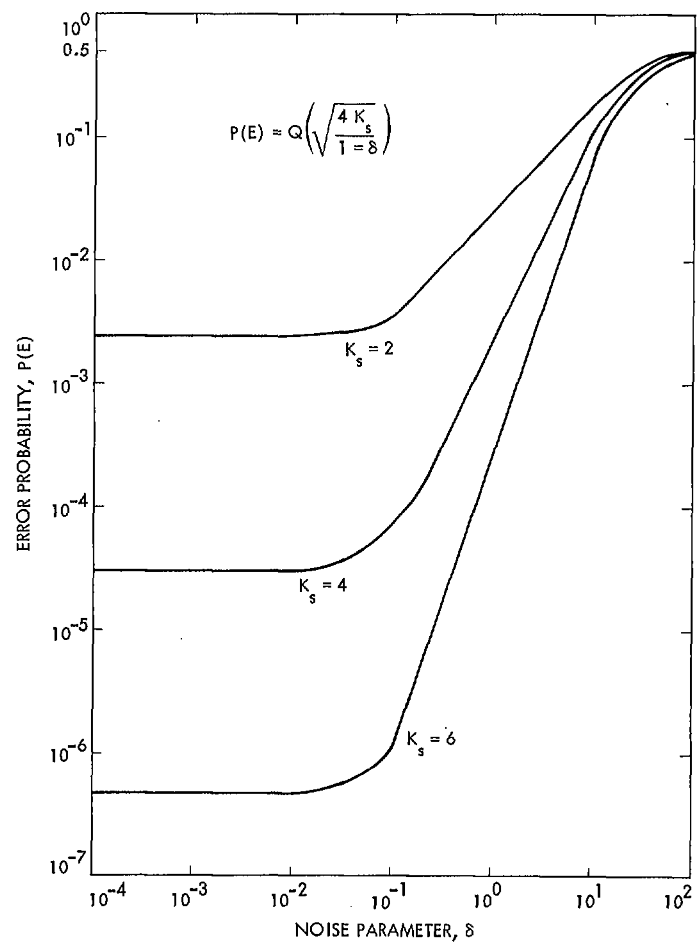


**Fig. 3. Receiver performance in the presence of background radiation; negligible phase error**

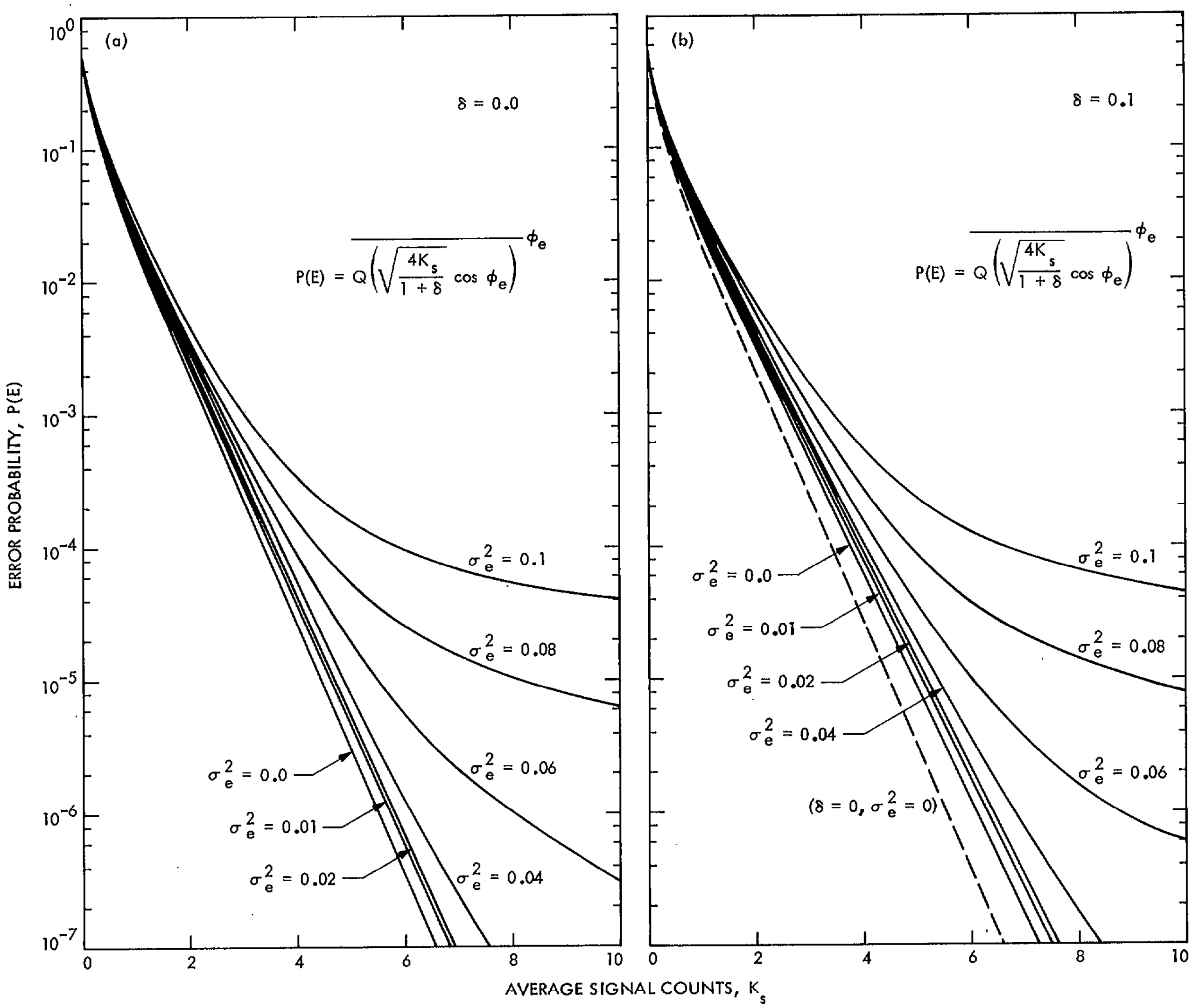


**Fig. 4. Receiver performance in the presence of phase error: (a) negligible background; (b) high-intensity background**