Posterior Error Probability in the Mu-II Sequential Ranging System

C. W. Coyle
Communications Systems Research Section

An expression is derived for the posterior error probability in the Mu-II Sequential Ranging System. An algorithm is developed which closely bounds the exact answer and can be implemented in the machine software. A computer simulation is provided to illustrate the improved level of confidence in a ranging acquisition using this figure of merit as compared to that using only the prior probabilities. In a simulation of 20,000 acquisitions, with an experimentally determined threshold setting, the algorithm detected 90 percent of the actual errors and made false indications of errors on 0.2 percent of the acquisitions.

I. Introduction

In the interest of economizing on the acquisition time in the Mu-II Ranging System, it is desirable to find an upper bound on the error probability in a range determination, given suitable estimates of the signal and noise levels and the entire set of receiver correlator outputs. A complete description of this ranging machine can be found in Ref. 1.

The Mu-II operates by receiving a sequence of square waves, each member of the sequence having twice the period of the previous one (Fig. 1). The highest frequency component provides the required precision, while each subsequent component is used to resolve the remaining ambiguity, which is equal to the previous component's period. Upon correlation of each component with a locally generated reference, the receiver shifts the local reference in an attempt to force the next component onto a peak of its triangular correlation curve. A quadrature output is also produced by correlation of the signal with a 90-deg shifted version of the reference.

An initial assumption made here is that the 2π modulus phase delay of the first (highest resolution) component was measured exactly (τ in Fig. 1). For the second component, and for all subsequent components, a binary decision must be made as to whether the in-phase correlator output is at a positive or negative peak, as discussed in Ref. 2. It is the probability of correctness of this set of decisions that will be examined here.

II. A General Expression for the Posterior Probability

If $\overline{R}$ is a vector of all in-phase correlator outputs I_n and all quadrature correlator outputs Q_n, the posterior probability that all decisions were made correctly is, by Bayes's rule,

$P(\text{correct}|\overline{R})$

$$= \frac{P(\overline{R}|\text{correct})\,P(\text{correct})}{P(R)}$$

$$= \frac{P(\overline{R}|\text{correct})\,P(\text{correct})}{P(\overline{R}|\text{correct})\,P(\text{correct}) + P(\overline{R}|\text{incorrect})\,P(\text{incorrect})}$$

$$= \left[1 + \frac{P(\text{incorrect})\,P(\overline{R}|\text{incorrect})}{P(\text{correct})\,P(\overline{R}|\text{correct})}\right]^{-1} \tag{1}$$

As nothing is known about the transmitted vector, $P(\text{correct}) = 2^{-N}$ and $P(\text{incorrect}) = 1-2^{-N}$, giving

$$P(\text{correct}|\overline{R}) = \left[1 + (2^N - 1)\frac{P(\overline{R}|\text{incorrect}}{P(\overline{R}|\text{correct})}\right]^{-1} \tag{2}$$

$P(\overline{R}|\text{correct})$ is simple to compute because, given that components were received correctly, the means of all the Q_n are zero. The probability that component n will yield output values Q_n, I_n is

$$P_n(I_n, Q_n|S_0) = P_n(\overline{R}_n|S_0) \tag{3}$$

where S_0 is the mean of I_n. By the receiver decision rule, S_0 will have a sign equal to the sign of I_n.

These probabilities are mutually independent when conditioned on a sequence of correct decisions, so

$$P(\overline{R}|\text{correct}) = \prod_{n=1}^{N} P_n(\overline{R}_n|S_0) \tag{4}$$

$P(\overline{R}|\text{incorrect})$ is much more difficult to determine. This is because the location of the mean for a given component depends on the sequence of prior errors. The mean will, in general, be located off the I axis. This probability can be expanded as

$$P(\overline{R}|\text{incorrect}) = \sum_{\text{all } \overline{S}} P(\overline{R}|\overline{S})\,P(\overline{S})$$

$$= \sum_{\text{all } \overline{S}} \prod_{n=1}^{N} P(\overline{R}_n|\overline{S})\,P(\overline{S}), \tag{5}$$

where $\overline{S}$ is an N bit binary sequence other than the sequence chosen by the receiver. As there are $2^N - 1$ equally likely sequences $\overline{S}$, $P(\overline{S}) = (2^N - 1)^{-1}$. The desired expression is then

$$P(\text{correct}|\overline{R}) = \left[1 + \sum_{\text{all } \overline{S}} \prod_{n=1}^{N} \frac{P_n(\overline{R}_n|\overline{S})}{P_n(\overline{R}_n|S_0)}\right]^{-1} \tag{6}$$

To evaluate each $P_n(\overline{R}_n|\overline{S})$, the position of the mean for component n is determined by assuming $\overline{S}$ to be the correct sequence and comparing this with the sequence produced by the receiver, thus making known the nature and location of all errors in the receiver decision under this hypothesis.

III. Development of a Useful Bound

Evaluating the sum in Eq. (6), with its $2^N - 1$ terms, each containing N factors, is clearly impractical for the usual values of N (often > 10) encountered. In order to be useful as an indicator of performance in the Mu-II, a lower bound must be found for (6) which is simple to compute but sufficiently tight to be meaningful. At this point, we observe that all of the $\overline{R}_n$ are Gaussian random variables of equal variance with probability densities of the form

$$p_n(\overline{R}_n|\overline{\mu}_n) = (2\pi\sigma^2)^{-1} \exp\left\{-\frac{1}{2\sigma^2}\left[(Q_n - \mu_{nq})^2 + (I_n - \mu_{ni})^2\right]\right\} \tag{7}$$

where $\overline{\mu}_n$ is the mean vector resulting from the specified condition on $P_n(\overline{R}_n)$. This substitution makes (6) become

$$P(\text{correct}|\overline{R}) = \left[1 + \sum_{\overline{S}} \prod_{n} F_n(\overline{S})\right]^{-1} \tag{8}$$

where

$$F_n(\overline{S}) = \exp\left\{\frac{1}{2\sigma^2}\left[m^2 - (\mu_{ni}^2 + \mu_{nq}^2) + 2(I_n\mu_{ni} + Q_n\mu_{nq} - |I_n|m)\right]\right\} \tag{9}$$

Here, (μ_{ni}, μ_{nq}) is the mean vector derived for $P_n(\overline{R}_n|\overline{S})$ and m is the magnitude of S_0. The (μ_{ni}, μ_{nq}) are the component means seen by the receiver if its decision was incorrect and the $\overline{S}$ under consideration is correct.

The $2^N - 1$ possible sequences $\overline{S}$ can be classified into types. Each type is denoted by the number of agreements with receiver decisions before a disagreement occurs. Thus, if there are N components, there will be N types of decision sequences, numbered from 0 to $N - 1$. (A sequence with N agreements corresponds to the receiver decision and the set $\overline{S}$ specifically excludes this case.) It is easy to see that there will be 2^{N-p-1} members of a type p sequence.

Now, if there are p agreements before a disagreement, the first p values of (μ_{ni}, μ_{nq}) will be simply $(S_0, 0)$ and the first p values of $F_n(\overline{S})$ will, using (9), be unity. Component $p + 1$ will still have its mean on the I axis, specifically at $(-S_0, 0)$. For $n = p + 2$, the vector (μ_{ni}, μ_{nq}) becomes either $(0, m)$ or $(0, -m)$, because of error at component $p + 1$ shifts the mean onto the Q axis. The value of $F_{p+2}(\overline{S})$ can be upper bounded by always choosing the sign of m to minimize the distance between the mean and $\overline{R}_n$. Using (9),

$$F_{p+1}(\overline{S}) = \exp\left\{\frac{-2m}{\sigma^2} |I_{p+1}|\right\} = \overline{F}_{p+1}, \tag{10}$$

and

$$F_{p+2}(\overline{S}) \leqslant \exp\left\{\frac{-m}{\sigma^2}\left[|I_{p+2}| - |Q_{p+2}|\right]\right\} = \overline{\overline{F}}_{p+2}. \tag{11}$$

Equation (8) can now be lower bounded by

$$P(\text{correct}|\overline{R}) \geqslant \left[1 + \sum_{p=0}^{N-1} G_p\right]^{-1} \tag{12}$$

where G_p is an upper bound on the sum of products over a particular $\overline{S}$ type:

$$G_p \leqslant 2^{N-p-1}\, \overline{F}_{p+1}\, \overline{\overline{F}}_{p+2} \prod_{n>p+2} F_n(\overline{S}), \tag{13}$$

where all factors with subscripts greater than N are taken equal to unity.

A simplifying approximation can be used to determine the $F_n(\overline{S})$ when $n > p + 2$. Upper bounds on these can be computed by choosing, for each component, a mean vector (μ_{ni}, μ_{nq}) that will result in the largest possible value of F_n. Such worst case values are independent of $\overline{S}$ and need only be computed once for each component.

Such a bound can be obtained by realizing that the possible means associated with a particular sequence $\overline{S}$ are confined to a locus of points as shown in Fig. 2. (Ref. 2). Attention has been restricted to the first quadrant as would be done in practice by using the absolute values of I_n and Q_n. For $\overline{R}_n$ in the shaded region bordering the I axis, the nearest possible mean (the one that will maximize F_n) is the point $(m, 0)$. If $\overline{R}_n$ is in the shaded region bordering the Q axis, the nearest mean is the point $(0, m)$. For all other $\overline{R}_n$, the nearest mean is the intersection of the locus of means with a perpendicular passing through $\overline{R}_n$, yielding a vector (μ_{ni}, μ_{nq}) where

$$\begin{aligned} \mu_{ni} &= \frac{1}{2}[m - (|Q_n| - |I_n|)] \\ \mu_{nq} &= \frac{1}{2}[m + (|Q_n| - |I_n|)]. \end{aligned} \tag{14}$$

Using Eq. (9)

a) $F_n = 1$ for $|I| > m$ and $|Q| < |I| - m$,

b) $F_n = \exp\left[\frac{m}{\sigma^2}(|Q_n| - |I_n|)\right]$

for $|Q| > m$ and $|I| < |Q| - m$

c) $F_n = \exp\left\{\frac{1}{4\sigma^2}\left[(m - |I_n|) + |Q_n|\right]^2\right\}$ otherwise.

(15)

It is apparent from (13) that the F_n need not be computed for $n < 3$.

An improvement in the tightness of the bound can be realized by taking into account the discrete nature of the possible locations for the means. It is clear that the first component can only have a mean at $(m, 0)$. Each subsequent component has $2^{n-2} + 1$ possible means equally spaced along the line joining $(m, 0)$ and $(0, m)$. In general, the location of these will be given by

$$\mu'_{ni} = mi/2^{n-2} \tag{16}$$

and

$$\mu'_{nq} = m - \mu'_{ni} = m(1 - i/2^{n-2}),$$

$$n > 1 \qquad i = 0, 1, 2 \ldots, 2^{n-2}.$$

An upper bound on F_n is found by again dropping a perpendicular from $\overline{R}_n$ to the diagonal locus of means. In general, this intersection will not satisfy (16). It is easily seen that the nearest mean satisfying (15) is found by choosing a value for i that will minimize $|\mu_{ni} - \mu'_{ni}|$. Substituting values from (14) and (16),

$$\left.\begin{aligned} i_n &= \frac{\mu'_{ni}}{m} 2^{n-2} \\ &= \left(1 + \frac{|I_n| - |Q_n|}{m}\right) 2^{n-3} \end{aligned}\right\} \begin{array}{l}\text{rounded to} \\ \text{nearest integer} \\ n > 1\end{array} \qquad (17)$$

The resulting μ'_{ni} and μ'_{nq} can be inserted into (9) and the result

$$F_n = \exp\left(\frac{m}{\sigma^2}\left\{(1 - i_n 2^{2-n})\left[(i_n m 2^{2-n} - (|I_n| - |Q_n|)\right]\right\}\right) \qquad (18)$$

for $n > 1$ can be used in place of (15c).

This expression will provide noticeable reductions in the F_n compared with (15c), especially for low values of n where the nearest μ'_{ni} is often far from μ_{ni}.

To reiterate the procedure of finding a bound for $P(\text{correct}|\overline{R})$,

(a) Compute a set of F_n $(n = 3, 4, \ldots, N)$

from

$$F_n = 1 \qquad \text{for } |I| > m \text{ and } |Q| < |I| - m$$

$$F_n = \exp\left[\frac{m}{\sigma^2}(|Q_n| - |I_n|)\right] \qquad \text{for } |Q| > m \text{ and } |I| < |Q| - m$$

$$F_n = \exp\left(\frac{m}{\sigma^2}\left\{(1 - i_n 2^{2-n})\left[i_n m 2^{2-n} - (|I_n| - |Q_n|)\right]\right\}\right)$$

Otherwise, (19)

where

$$i_n = \left(1 + \frac{|I_n| - |Q_n|}{m}\right) 2^{n-3}$$

rounded to the nearest integer.

(b) Compute a set of $\overline{F}_n$ $(n = 1, 2, \ldots, N)$

from

$$\overline{F}_n = \exp\left\{\frac{-2m}{\sigma^2}|I_n|\right\}. \qquad (20)$$

(c) Compute a set of $\overline{\overline{F}}_n$ $(n = 2, 3, \ldots, N)$

from

$$\overline{\overline{F}}_n = \exp\left\{\frac{-m}{\sigma^2}(|I_n| - |Q_n|)\right\} \qquad (21)$$

(d) Compute a set of $N - 1$ products

$$G_p = 2^{N-p-1}\, \overline{F}_{p+1}\, \overline{\overline{F}}_{p+2} \prod_{n>p+2} F_n \qquad p = 0, \ldots, N-1. \quad (22)$$

The result is then given as

$$P(\text{correct}|\overline{R}) \geqslant \left[1 + \sum_{p=1}^{N-1} G_p\right]^{-1} \qquad (23)$$

IV. Simulation

A computer program was developed to simulate a ranging acquisition by generating a random binary vector, suitably corrupted with Gaussian noise. The vector was sequentially detected, care being taken to properly shift the means of subsequent components when a detection error was made. A modified version of the algorithm found in Ref. 3 was used.

The simulation consisted of 20,000 trial acquisitions, each with 10 lower frequency components. The value of m/σ was 3.5, for a postcorrelation SNR of 7.9 dB. Using Ref. 2, the prior probability of a correct acquisition was determined to be approximately 0.998. The expected number of errors over the sample is then about 40. The number of errors that actually occurred was 39. Various thresholds were considered and an error was assumed to have occurred when the algorithm returned a number below threshold. Table 1 summarizes the results.

It is important to note that all the undetected errors (with the exception of one additional at 0.01 threshold) occurred on the last component of the acquisition. Such an error is particularly difficult to detect because there is no following component from which to observe the shift of the mean onto the Q axis. If the rest of the components were received reliably, the error probability will be essentially that of the last component, as if it were the only one transmitted. This probability will never be less than 0.5. The problem could be circumvented by transmitting one more code component than is necessary to resolve the range ambiguity. The received component could then be used to check the reliability of the previous decision but would not be used in the range calculation.

The reason why it is possible to detect virtually all errors even with a rather low threshold is found in the behavior of the probability bound for "marginal" acquisitions. In such cases the number returned by the algorithm is often considerably less than the actual probability of being correct. Thus it becomes necessary to experimentally determine a threshold setting for the signal-to-noise ratios of interest. When an error was actually made in the simulation on any but the last component, the algorithm reacted very strongly by returning values almost always below 0.01 and typically less than 10^{-5}. Overall, the average figure of merit for the 20,000 trials was 0.991. It is assumed that an average over the true posterior probabilities would converge to the prior correctness probability, 0.998.

V. Conclusion

It should be possible to reduce the integration time for receiving the lower frequency components if this algorithm is used to determine a figure of merit for each acquisition. Such a reduction may be of importance for very long distance ranging when signal power is low and integration times are correspondingly long.

Also, during superior solar conjunction, a ranging acquisition should be performed in as short a time as possible to reduce the change of bursts of noise interfering strongly with the signal. Ranging via several short acquisitions has been seen to be effective in such highly dynamic noise situations (Ref. 4). It is expected that the posterior error probability bound would be a very useful figure of merit given the resultingly low signal-to-noise ratios.

References

1. Martin, W. L., and Zygielbaum, A. I., *Mu-II Ranging*, Technical Memorandum, 33-768, Jet Propulsion Laboratory, Pasadena, Calif., May 15, 1977.
2. Goldstein, R. M., "Ranging With Sequential Components," in *Space Programs Summary*, 37-52, Vol. II, pp. 46-49, Jet Propulsion Laboratory, Pasadena, Calif., July 31, 1968.
3. Timor, U., *Sequential Ranging with the Viterbi Algorithm*, JPL Technical Report 32-1526, Vol. II, pp. 75-79, Jet Propulsion Laboratory, Pasadena, Calif., April 15, 1971.
4. Zygielbaum, A. I., "Near Sun Ranging," in *The Deep Space Network Progress Report* 42-41, pp. 43-50, Jet Propulsion Laboratory, Pasadena, Calif., October 15, 1977.

Table 1. Performance of the figure of merit algorithm for various threshold settings

Threshold	Errors detected	Errors not detected	False error indications
0.99	39	0	579
0.9	38	1	236
0.8	38	1	183
0.7	37	2	140
0.6	36	3	113
0.5	35	4	91
0.4	35	4	79
0.3	35	4	68
0.2	35	4	53
0.1	35	4	37
0.01	34	5	16

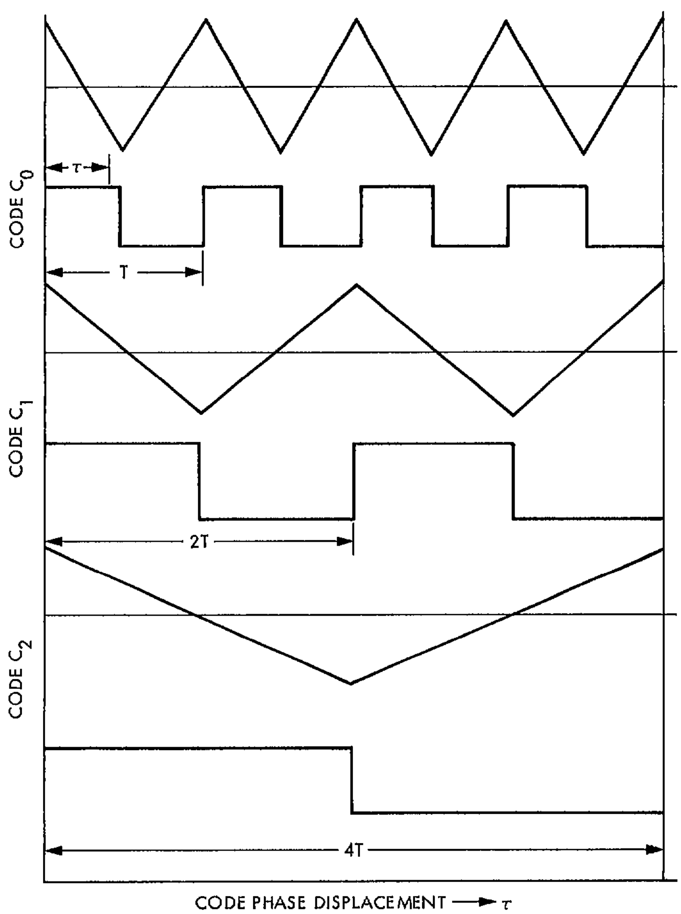

Fig. 1. Transmitted codes with respective correlation functions. The measured phase of the highest frequency component is τ

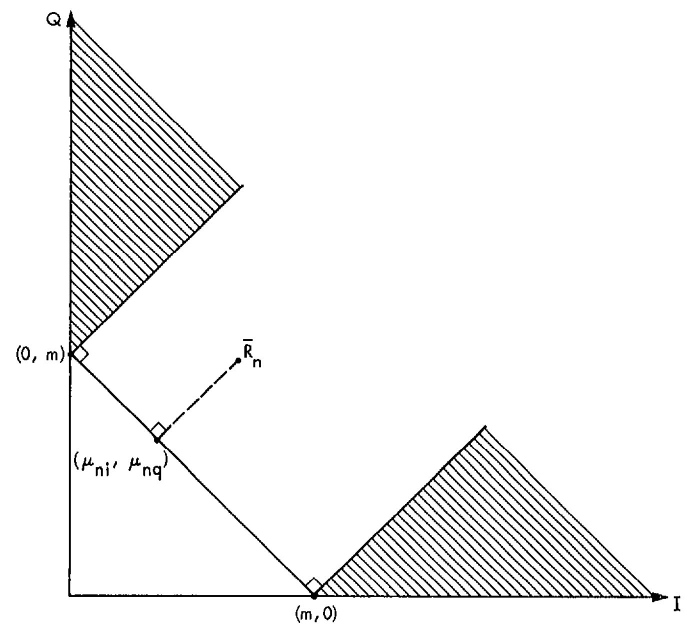

Fig. 2. Determining the nearest possible mean to the received vector $\overline{R}_n$