Maximizing Throughput Over an Average-Power-Limited and Band-Limited Optical Pulse Position Modulation Channel

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Given an optical pulse position modulation (PPM) channel, with an average power constraint and a bandwidth constraint, this article determines the word length needed to maximize the information throughput achievable by the channel. It is shown that, to achieve the maximal capacity, the channel must be operated with a high erasure probability. This implies that coding schemes capable of compensating for a high percentage of erasures are needed for the PPM channel.

I. Introduction

Many investigators have shown that pulse position modulation (PPM) is an effective method to use over the photon channel. PPM has been shown to maximize various channel parameters when the bandwidth goes to infinity (Refs. 1 and 2).

Because of the optimality of the PPM technique for infinite bandwidth, it has been thought of as a technique to use for a band-limited (realistic) channel. If it is decided to use PPM over a band-limited, average-power-limited photon channel, the PPM word size is still an arbitrary parameter. By selecting the PPM word size Q, we control the channel throughput (measured in nats/slot, which is equivalent to nats/sec since the bandwidth is fixed).

The question naturally arises as to the optimal choice of Q (denoted by Q^* for a specified system). The capacity of a channel and the R_{comp} (computation cutoff rate of the channel, also known as R_0) of a channel are both valid measures of a channel's throughput.

Section II of this article determines the word size to maximize capacity; Section III determines the word size to maximize R_0. Section IV is a discussion.

II. Maximizing Capacity

Let us suppose that the PPM channel allows an average power expenditure of P watts and permits us to send N slots per second. Given P, N we would like to find the word size Q (slots/word) that maximizes the capacity C (nats/slot) of the channel. We can define the average energy (in photons) per slot $\bar{s}$ by $\bar{s} = P/Nh\nu$, where $h = 6.62 \times 10^{-34}$ j-sec is Planck's constant and ν is the frequency of the optical radiation in Hz. Since P and N are fixed system parameters, $\bar{s}$ is a known constant.

Let us define the following quantities:

$$W = N/Q = \text{word rate (words/sec)}$$

$$\bar{P} = \bar{s}N/W = \text{energy/word (photons/word)}$$

$$\epsilon = e^{-\bar{P}} = \text{erasure probability/word}$$

For a derivation of this formula for ϵ, see Ref. 1. Recall that the Q-ary PPM channel is equivalent to a Q-ary erasure channel. A simple calculation gives for the Q-ary erasure channel:

$$C = \text{capacity} = (1 - \epsilon) \log Q \text{ nats/word}$$

Since there are W words/sec:

$$C = W(1 - \epsilon) \log Q \text{ nats/sec}$$

Since there are N slots/sec:

$$C = \frac{W}{N}(1 - \epsilon) \log Q \text{ nats/slot}$$

Substituting for W and ϵ we find:

$$C = \left(\frac{1 - e^{-\overline{s}Q}}{Q}\right) \log Q \text{ nats/slot} \tag{1}$$

This is the form we must maximize by varying Q. Recall that $\overline{s}$ is derivable from the given channel parameters.

It should be noted at this point that in Katz et al. (Ref. 3), an equation very similar to our Eq. (1) was developed to maximize the nats/photon of the channel, when the nats/sec of the channel was given. This study was conducted to minimize the power requirements of the system. The Katz paper can be viewed as a technique for the communications engineer to use when apportioning power on a spacecraft. The present paper is capable of determining the coding scheme *after* the power devoted to the communication system has been decided upon.

Figure 1 shows contours of constant C for varying Q and $\overline{s}$ (utilizing Eq. 1). Note that for each $\overline{s}$, there is a best value of Q, Q^*, such that Eq. (1) is maximized. This value can be found from Fig. 1 by drawing the line $\overline{s}$ = constant, finding the C = constant curve that is tangent to this line, and, at the point of tangency, looking at the ordinate to read the value of Q^*. This has been done for various values of $\overline{s}$, and the result is Fig. 2 (which plots $\overline{s}$ vs Q^*). It is surprising how linear Fig. 2 appears. In the range $Q^* \epsilon [10, 1000]$ this relationship is well approximated by

$$Q^* \overline{s}^{0.817} = 1.36$$

Now recall that ϵ, the erasure probability, is given by $\epsilon = \exp(-\overline{P}) = \exp(\overline{s}N/W) = \exp(-Q\overline{s})$. Therefore, given an $\overline{s}$ we can find Q^* and then we find $\epsilon^* = \exp(-\overline{s}Q^*)$. This value of ϵ, ϵ^*, is the erasure probability that the channel must operate at to achieve the capacity given by $\overline{s}$ and Q^*.

Figure 3 is a graph of Q^* vs ϵ^* (note that Q^* implies $\overline{s}$ by Fig. 2). Figure 3 shows that a value of ϵ^* implies a single value of Q^*, and so also a value of $\overline{s}$. Therefore a value of ϵ^* supplies all the necessary information to compute (1). Figure 4 shows this relationship between C and ϵ^*.

III. Maximizing R_{comp}

Using the same notation as in Section II for $\overline{s}$, N, $\overline{P}$ and ϵ we can obtain very analogous results for the computational cutoff rate R_{comp}. We easily find:

$$R_0 = R_{comp} = -\log\left[\frac{1-\epsilon}{Q} + \epsilon\right] \text{ nats/word}$$

Since there are W words/sec:

$$R_0 = -W \log\left[\frac{1-\epsilon}{Q} + \epsilon\right] \text{ nats/sec}$$

Since there are N slots/sec:

$$R_0 = -\frac{W}{N} \log\left[\frac{1-\epsilon}{Q} + \epsilon\right] \text{ nats/slot}$$

Substituting for W and ϵ we find:

$$R_0 = -\frac{\log}{Q}\left[\frac{1 - e^{-\overline{s}Q}}{Q} + e^{-\overline{s}Q}\right] \text{ nats/slot} \tag{2}$$

Now everything that was done in Section II for capacity can be done for R_{comp}. Specifically, Fig. 5 shows contours of constant R_0 and Fig. 6 gives the value of Q^* that maximizes (2) for a given value of $\overline{s}$. Figure 7 is a graph of Q^* vs ϵ^*, and Fig. 8 is a graph of R_0 vs ϵ^*.

It is worth noting how linear Fig. 6 appears. A good approximation in the range $Q^* \epsilon [30,800]$ is given by

$$Q^* \overline{s}^{0.984} = 1.75$$

(An easier, still reasonable, approximation is $Q^* \overline{s} = 1.56$.)

IV. Discussion

What Sections II and III have shown is that there is a triplet $(\bar{s}, \epsilon^*, Q^*)$, any one of which determines the other two. Since $\bar{s}$ is determined by the channel, there is an optimal ϵ and Q for each channel. This allows comparison of channels.

Suppose it had been decided to use a 256-ary PPM channel with an erasure probability of 0.01. This is within the realm of current thinking, and it is a *bad choice of parameters.* With $Q = 256$, $\epsilon = 0.01$ we have $\bar{P} = -\log \epsilon = 4.61$ and $\bar{s} = \bar{P}/Q = 0.018$. From these values of Q and $\bar{s}$ we obtain $C = 0.021$ nats/slot (using formula 1).

We can do a lot better, however, with a different value of Q and the *same value of* $\bar{s}$. With $\bar{s} = 0.018$, Fig. 2 gives $Q^* = 35$. Using $Q^* = 35$ in Fig. 3 we find $\epsilon^* = 0.53$. Using $\epsilon^* = 0.53$ in Fig. 4 we find $C = 0.047$ nats/slot. Hence we see that by changing Q from 256 to $Q^* = 35$, we increase the throughput from 0.021 nats/slot to 0.047 nats/slot, an increase of 124%! The peak power requirement also goes down from $-\log(\epsilon) = 4.61$ to $-\log(\epsilon^*) = 0.63$, a decrease of 86%!

The only drawback is that this new set of parameters dictates the erasure probability to be $\epsilon = 0.53$. This means that coding schemes considered for this channel must have the capability of compensating for an unusually large number of erasures. Of course, the calculation above could also have been done for R_0, using Figs. 6, 7, and 8.

The conclusion is that, to effectively utilize the PPM channel, the erasure probability must be rather large. For example, associated with $Q^* = 256$ is $\epsilon^* = 0.68$ (utilizing Fig. 3). This high erasure probability forces the use of lower rate (and hence more complex) codes if one desires to use the channel optimally.

References

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2. Tan, H. H., "Capacity of a Multi-mode Direct Detection Optical Communication Channel," (to be published in TDA Progress Report).
3. Katz, J., Butman, S. A., and Lesh, J. R., "Practical Limitations on Noiseless Optical Channel Capacity," in *The Deep Space Network Progress Report 42-55*, Jet Propulsion Laboratory, Pasadena, Calif., Feb. 15, 1980.

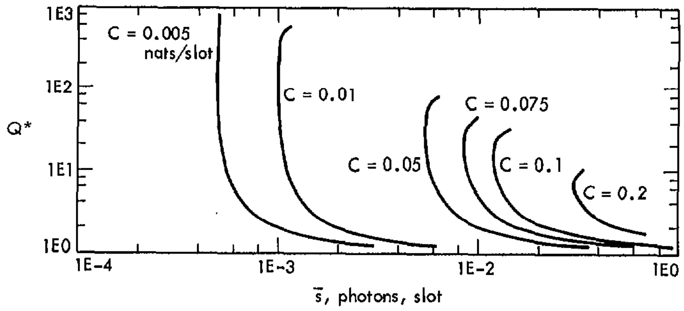

Fig. 1. Contours of constant *C*

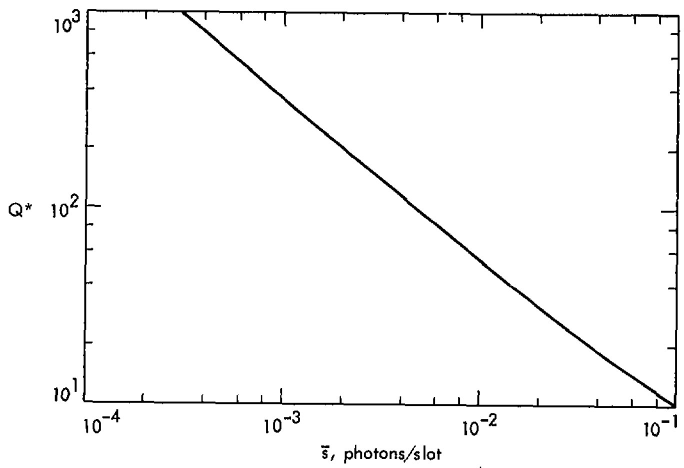

Fig. 2. Optimal Q^* such that *C* is maximal, given $\overline{s}$

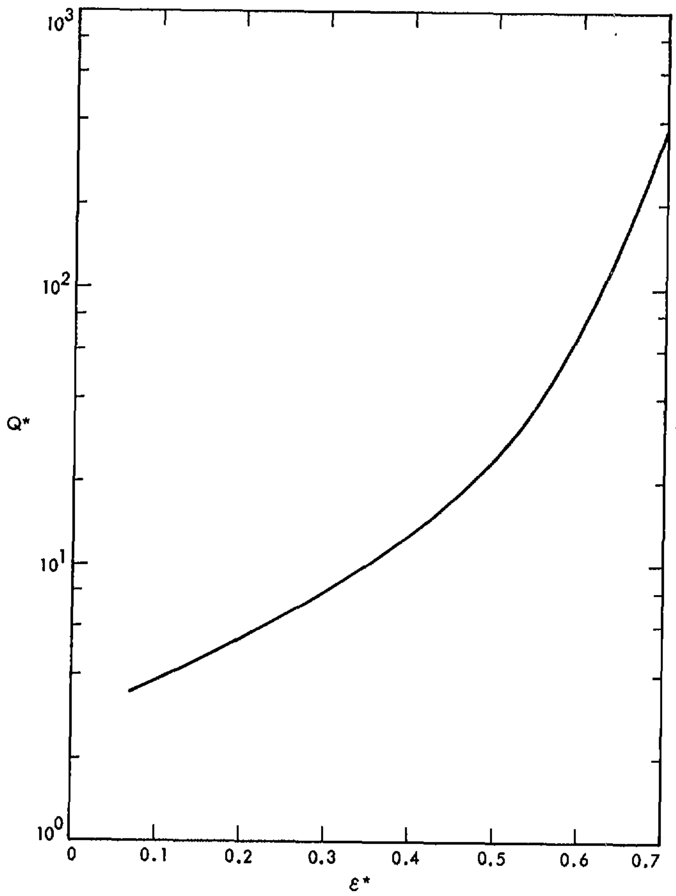

Fig. 3. Q^* vs ϵ^* utilizing capacity as functional to be maximized

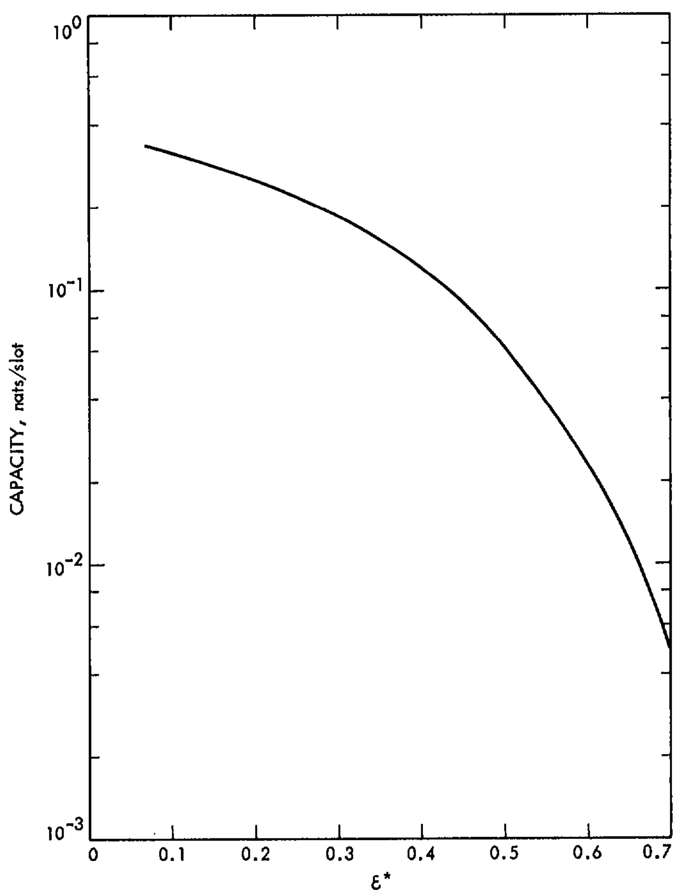

Fig. 4. Capacity given ϵ^*

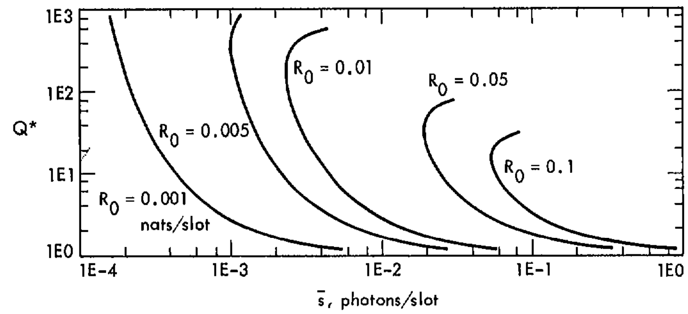

Fig. 5. Contours of constant R_{comp}

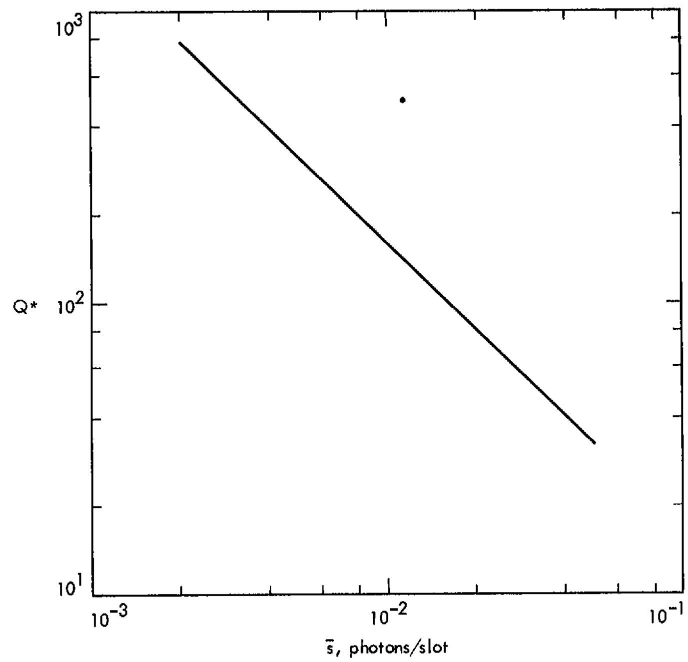

Fig. 6. Optimal Q^* such that R_0 is maximal, given $\bar{s}$

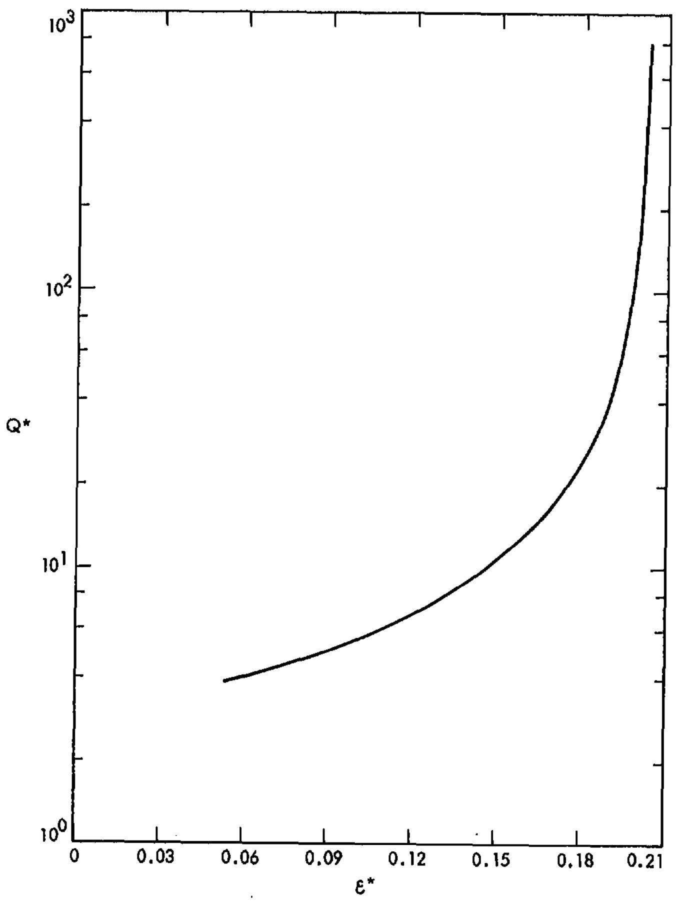

Fig. 7. Q^* vs ϵ^* utilizing R_{comp} as functional to be maximized

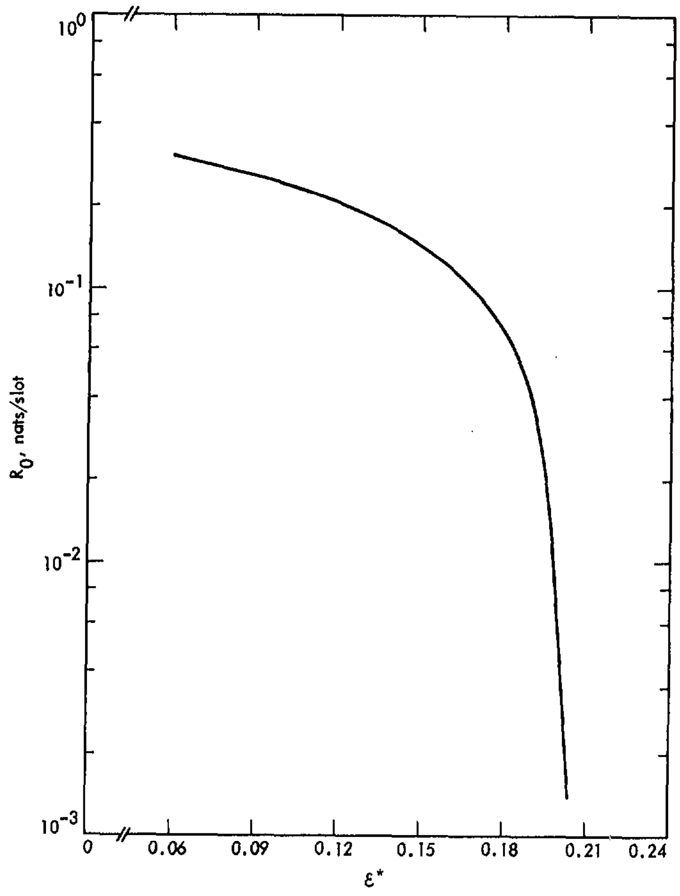

Fig. 8. R_{comp} given ϵ^*