Optimum Design of Hybrid Phase Locked Loops

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This article deals with the design procedure of phase locked loops in which the analog loop filter is replaced by a digital computer. Specific design curves are given for the step and ramp input changes in phase. It is shown that the designed digital filter depends explicitly on the product of the sampling time and the noise bandwidth of the phase locked loop. The technique of optimization developed in this article can be applied to the design of digital-analog loops for other applications.

I. Introduction

This article deals with the design of phase locked loops in which the analog loop filter is replaced by a digital computer. The optimum analog phase locked loop design, based on the Wiener filtering theory, has been analyzed by various researchers (Ref. 1). The effort described in this article is a procedure of digitizing the analog loop filters by properly designed digital filters. An applicable example is the tracking loop in the Multimegabit Telemetry System. Various techniques have been developed to replace the analog loop filters by their discrete counterparts (Ref. 2). However, Gupta (Ref. 3) pointed out that the optimum hybrid loop is not the discrete version of the analog loop. In this article, an optimum digital filter is designed to replace the conventional analog loop filter by putting an inequality constraint on the noise bandwidth of the phase locked loop. The Kuhn-Tucker theorem, together with the calculus of variation, is used to find the optimum structure of the digital filter. Both the interior optimum as well as the boundary optimum are evaluated. Based on the Kuhn-Tucker theorem, the interior optimum is computed as an unconstrained optimum while the boundary optimum is obtained via the Lagrange multiplier technique (Ref. 4).

II. The Mathematical Model

The mathematical model of the analog-digital phase locked loop is shown in Fig. 1. The digital filter, $D(z)$, together with the sample and hold circuit replaces the conventional analog loop filter. A VCO is modelled by a pure integrator, and the output of the VCO is fed back to produce an error signal. The input $\theta(t)$ is assumed to be deterministic and is corrupted by an additive white noise $n_i(t)$ which has a one-sided power spectral density of N_o w/Hz. Let $e(nT)$ be the sampled error between the actual input phase $\theta(t)$ and the output phase of the VCO $\psi(t)$ in the absence of noise. Then the sum squared sample error σ_e^2 is defined as

$$\sigma_e^2 = \sum_{n=0}^{\infty} e^2(nT) = \sum_{n=0}^{\infty} [\theta(nT) - \psi(nT)]^2 \qquad (1)$$

where T is the sampling period.

We want to minimize the σ_e^2 subject to the constraint that the average power of the output noise $n_o(t)$ is kept below a constant. Specifically, if B_N is the required noise bandwidth of the phase locked loop, we want to solve the following problem.

Problem: Minimize σ_e^2 subject to

$$\sigma_n^2 = E\left[n_o^2(t)\right] \leqslant N_o \text{ (w/Hz)} \cdot B_N \text{(Hz)} \tag{2}$$

In the following sections, we first evaluate the quantities σ_n^2 and σ_e^2 in terms of the digital filter $D(z)$. Then we apply the Lagrange multiplier technique and the calculus of variation to minimize σ_e^2 subject first to the equality constraint and then secondly to the strict inequality constraint. The resulting optimum digital filter $\hat{D}(z)$ is then evaluated explicitly for the cases where $\theta(t)$ are step and ramp inputs.

III. The Optimum Digital Filter

Let $G(s)$ be the transfer function of the cascaded system consisting of the zero order hold and the VCO. Then

$$G(s) = \frac{1 - e^{-sT}}{s} \frac{K}{s} \tag{3}$$

A. Evaluation of σ_e^2

Let $E(z)$ be the z transform of the sampled error sequence $e(nT)$. Then

$$\sigma_e^2 = \sum_{n=0}^{\infty} e^2(nT) = \sum_{n=0}^{\infty} e(nT)\left[\frac{1}{2\pi j}\oint_C E(z)z^{n-1}dz\right]$$

$$= \frac{1}{2\pi j}\oint_C E(z)\left[\sum_{n=0}^{\infty} e(nT)z^n\right]\frac{dz}{z}$$

$$= \frac{1}{2\pi j}\oint_C E(z)E(z^{-1})\frac{dz}{z} \tag{4}$$

where C is the counterclockwise closed contour in the region of convergence of $E(z)$ and encircling the origin of the z plane.

Let $G(z)$ be the z transform of the sampled impulse response of $G(s)$, and $\theta(z)$, $\psi(z)$ be the z transform of the sampled input sequence $\theta(nT)$ and the output sequence $\psi(nT)$. Then

$$E(z) = [1 - W(z)G(z)]\,\theta(z) \tag{5}$$

where

$$W(z) = \frac{D(z)}{1 + D(z)G(z)} \tag{6}$$

Substituting Eqs. (5) and (6) in (4), we obtain σ_e^2.

B. Evaluation of σ_n^2

The average power of the output noise σ_n^2 is given by

$$\sigma_n^2 = E\left\{n_o^2(t)\right\} = \frac{1}{2\pi j}\frac{1}{T}\int_0^1 dm \oint_{C'} \Phi_{n_o}(z,m)\frac{dz}{z} \tag{7}$$

where $\Phi_{n_o}(z,m)$ is the modified pulse spectral density (Ref. 6) of the output noise $n_o(t)$ and C' is the counterclockwise closed contour in the region of convergence of $\Phi_{n_o}(z,m)$, $0 < m < 1$, and encircling the origin of the z-plane. Then, from Appendix A (with extensions to the modified z transforms),

$$\sigma_n^2 = \frac{1}{2\pi j}\frac{1}{T}\int_0^1 dm \oint_{C'} H(z,m)H(z^{-1},m)\,\Phi_{n_i}(z)\frac{dz}{z} \tag{8}$$

where $H(z,m)$ is the modified z transform of the system transfer function between the input $n_i(t)$ and the output $n_o(t)$, and $\Phi_{n_i}(z)$ is the pulse spectral density of the input noise $n_i(t)$.

From Fig. 1,

$$H(z,m) = W(z)\,G(z,m) \tag{9}$$

where $W(z)$ is given by Eq. (6) and $G(z,m)$ is the modified z transform of the impulse response of $G(s)$.

For white noise input

$$\Phi_{n_i}(z) = N_o$$

Hence

$$\sigma_n^2 = \frac{1}{2\pi j}\oint_{C'} \frac{N_0}{T} A(z)\,W(z)\,W(z^{-1})\frac{dz}{z} \tag{10}$$

where

$$A(z) = \int_0^1 G(z, m)\, G(z^{-1}, m)\, dm$$

C. Constrained Optimum Digital Filter

Now we want to find an optimum $W(z)$, say $\hat{W}(z)$, that minimizes (2) with the equality constraint so that the optimum digital filter $\hat{D}(z)$ is given by

$$\hat{D}(z) = \frac{\hat{W}(z)}{1 - \hat{W}(z)G(z)} \tag{11}$$

Define

$$J[W(z)] = \frac{1}{2\pi j} \oint_\Gamma [1 - W(z)G(z)]\,[1 - W(z^{-1})G(z^{-1})]\,\theta(z)\,\theta(z^{-1})\frac{dz}{z}$$

$$+ \lambda\left[\frac{1}{2\pi j} \oint_\Gamma A(z)W(z)W(z^{-1}) \frac{dz}{z} - B_N T\right] \tag{12}$$

where λ is the Lagrange multiplier and Γ is the counterclockwise closed contour in the region of convergence of both $E(z)$ and $\Phi_{n_o}(z, m)$ and encircling the origin in the z plane. Since contour integration is independent of path within the region of convergence, the contour Γ is taken as the unit circle which is the same as C or C'.

Using the calculus of variation and evaluating

$$\left.\frac{\partial}{\partial \epsilon} J[\hat{W}(z) + \epsilon V(z)]\right|_{\epsilon=0} = 0$$

we obtain

$$\frac{1}{2\pi j} \oint_\Gamma \left[\frac{1}{z}P(z)\hat{W}(z^{-1}) - \frac{1}{z}G(z)\theta(z)\theta(z^{-1})\right] V(z)dz$$

$$+ \frac{1}{2\pi j} \oint_\Gamma \left[\frac{1}{z}P(z)W(z) - \frac{1}{z}G(z^{-1})\theta(z)\theta(z^{-1})\right] V(z^{-1})dz = 0$$

where

$$P(z) = G(z)G(z^{-1})\theta(z)\theta(z^{-1}) + \lambda A(z). \tag{13}$$

Defining

$$P(z) = P^+(z)\,P^-(z)$$

where $P^+(z)$ has all the poles and zeroes inside the unit circle of the z plane and $P^-(z)$ has all the poles and zeroes outside the unit circle of the z plane, we obtain the optimum $\hat{W}(z)$ as

$$\hat{W}(z) = \frac{z}{P^+(z)}\left[\frac{G(z^{-1})\theta(z)\theta(z^{-1})}{z\,P^-(z)}\right]_+ \tag{14}$$

where $[\cdot]_+$ represents that part of the partial fraction expansion of $[\cdot]$ whose poles are inside or on the unit circle.

In (14) the optimum $\hat{W}(z)$ is a function of the Lagrange multiplier λ. By putting $\hat{W}(z)$ in (10), and using

$$\sigma_n^2 = N_o B_N,$$

we can determine λ and hence the constrained optimum filter $\tilde{D}(z)$.

D. Unconstrained Optimum Digital Filter

Equation (14) defines the optimum filter $\tilde{W}(z)$ for the boundary minimum of (2). From the Kuhn-Tucker theorem, the interior minimum can be evaluated by forcing the Lagrange multiplier to zero in (12). In this case, the optimum digital filter given by (11) will satisfy the strict inequality of the constraint

$$\sigma_n^2 < N_o B_N$$

From (14) and (10), we obtain the unconstrained optimum digital filter, $\tilde{D}(z)$. The crossover point between the interior minimum and the boundary minimum is given by

$$(B_N T)_c = \frac{1}{2\pi j} \oint_\Gamma A(z)\tilde{W}(z)\tilde{W}(z^{-1}) \frac{dz}{z} \tag{15}$$

where $\tilde{W}(z)$ is the unconstrained optimum of (2).

IV. Examples

In this section, the optimum digital filter $\hat{D}(z)$ is evaluated for two types of inputs: the step and ramp change in phase inputs. Poles and zeroes of the optimum digital filter are described by the design curves for various time-bandwidth products.

Example 1. Step input

For step inputs,

$$\theta(t) = \begin{cases} 1 & t \geqslant 0 \\ 0 & t < 0 \end{cases}$$

Then

$$\theta(z) = \sum_{n=0}^{\infty} \theta(nT) z^{-n} = \frac{z}{z-1}$$

From the z transform table of Ref. 5, we obtain from (3)

$$G(z) = \frac{KT}{z-1}$$

and

$$G(z,m) = \frac{KT}{z}\left(m + \frac{1}{z-1}\right)$$

(i) Constrained optimum filter

From (13),

$$P(z) = \frac{\lambda K^2 T^2}{6} \frac{-(z^2 + z^{-2}) - 2(z + z^{-1}) + 6 + 6/\lambda}{(z-1)^2 (z^{-1}-1)^2}$$

$$= P^+(z) P^-(z)$$

where

$$P^+(z) = \sqrt{\frac{\lambda K^2 T^2}{6}} \frac{az^2 + bz + c}{(z-1)^2}$$

$$P^-(z) = \sqrt{\frac{\lambda K^2 T^2}{6}} \frac{az^{-2} + bz^{-1} + c}{(z^{-1}-1)^2}$$

and a, b, c satisfy the following equations

$$\begin{aligned} ac &= -1 \\ ab + bc &= -2 \\ a^2 + b^2 + c^2 &= 6 + 6/\lambda \end{aligned} \tag{16}$$

Then from (14)

$$\hat{W}(z) = \frac{a+b+c}{KT} \frac{z(z-1)}{az^2 + bz + c}$$

and from (12)

$$\hat{D}(z) = \frac{a+b+c}{KT} \frac{z}{az - c} = \frac{g_1}{KT} \frac{z}{z + P_1}$$

The constrained optimum digital filter for the step phase input is a single pole, single zero filter with pole located at c/a. Putting in the constraint that

$$B_N T = \frac{1}{2\pi j} \oint_{\substack{\text{unit}\\ \text{circle}}} A(z) \hat{W}(z) \hat{W}(z^{-1}) \frac{dz}{z}$$

we arrive

$$B_N T = \frac{(2a + 2c - b)(a + b + c)}{3(a-c)(a+c-b)} \tag{17}$$

Therefore given any $B_N T$, we can determine the variables a, b, c and λ from (16) and (17) and hence the optimum filter $\hat{D}(z)$. Since $B_N T \geqslant 0$, and $|c/a| < 1$ for a causal digital filter, only a range of values for a, b, and c can satisfy (17).

(ii) Unconstrained optimum filter

From (13)

$$P(z) = K^2 T^2 \frac{1}{(z-1)^2 (z^{-1}-1)^2} = P^+(z) P^-(z)$$

where

$$P^+(z) = KT \frac{z^2}{(z-1)^2}$$

$$P^-(z) = KT \frac{z^{-2}}{(z^{-1}-1)^2}$$

Then

$$\widetilde{W}(z) = \frac{1}{KT}\frac{z-1}{z}$$

$$\widetilde{D}(z) = \frac{1}{KT}$$

and from (15),

$$(B_N T)_c = \frac{1}{2\pi j}\oint_\Gamma A(z)\widetilde{W}(z)\widetilde{W}(z^{-1})\frac{dz}{z} = \frac{2}{3}$$

Figures 2 and 3 plot the gain coefficient g_1 and the pole location of the optimum digital filter $D(z)$ for the step phase input. Note that $\hat{D}(z)$ becomes $\widetilde{D}(z)$ for $B_N T \geqslant (B_N T)_c$.

Example 2. Ramp input

For ramp inputs,

$$\theta(t) = \begin{cases} t & t \geqslant 0 \\ 0 & t < 0 \end{cases}$$

Then

$$\theta(z) = \frac{Tz}{(z-1)^2}$$

(i) Constrained optimum filter

From (13),

$$P(z) = \frac{\lambda K^2 T^2}{b}\frac{(z^3+z^{-3}) - 9(z+z^{-1}) + 16 + 6T^2/\lambda}{(z-1)^3(z^{-1}-1)^3}$$

$$= P^+(z)P^-(z)$$

where

$$P^+(z) = \sqrt{\frac{\lambda K^2 T^2}{6}}\frac{az^3+bz^2+cz+d}{(z-1)^3}$$

$$P^-(z) = \sqrt{\frac{\lambda K^2 T^2}{6}}\frac{az^{-3}+bz^{-2}+cz^{-1}+d}{(z^{-1}-1)^3}$$

and a, b, c, d satisfy the following equations,

$$\begin{aligned} ad &= 1 \\ ac+bd &= 0 \\ ab+bc+cd &= -9 \\ a^2+b^2+c^2+d^2 &= 16+6T^2/\lambda \end{aligned} \tag{18}$$

Then

$$\hat{W}(z) = \frac{1}{KT}\frac{z(z-1)(ez+f)}{az^3+bz^2+cz+d}$$

and

$$\hat{D}(z) = \frac{1}{KT}\frac{z(ez+f)}{(z-1)(az+d)} = \frac{g_2}{KT}\frac{z(z-z_2)}{(z-1)(z+p_2)} \tag{19}$$

where

$$e = 2a+b-d$$

$$f = -a+c+2d$$

Unlike for the step change in phase, the optimum digital filter for the ramp change in phase is a double pole, double zero filter. Putting in the constraint that

$$B_N T = \frac{1}{2\pi j}\frac{2+\sqrt{3}}{6}\oint_\Gamma F(z)F(z^{-1})\frac{dz}{z}$$

where

$$F(z) = \frac{ez^2 + [(2-\sqrt{3})e+f]z + (2-\sqrt{3})f}{az^3+bz^2+cz+d}$$

Then

$$B_N T = \frac{1}{3}\frac{(2e^2+ef+2f^2)Q_0 - (e^2+4ef+f^2)Q_1 + efQ_2}{(a^2-d^2)Q_0 - (ab-cd)Q_1 + (ac-bd)Q_2}$$

where

$$Q_0 = a(a+c) - d(b+d)$$

$$Q_1 = ab - cd \tag{20}$$

$$Q_2 = b(b+d) - c(a+c)$$

As in example 1, for any given $B_N T$, the variables a, b, c, d and λ are determined from (18) and (20). The optimum digital filter $\hat{D}(z)$ is then given by (19).

(ii) Unconstrained optimum filter

From (13),

$$P(z) = \frac{K^2 T^4}{(z-1)^3 (z^{-1}-1)^3} = P^+(z) P^-(t)$$

where

$$P^+(z) = KT^2 \frac{z^3}{(z-1)^3}$$

$$P^-(z) = KT^2 \frac{z^{-3}}{(z^{-1}-1)^3}$$

Then

$$\widetilde{W}(z) = T \frac{2z-1}{(z-1)^2}$$

$$\widetilde{D}(z) = \frac{1}{KT} \frac{2z-1}{z-1}$$

and from (15),

$$(B_N T)_c = \frac{1}{2\pi j} \oint_\Gamma A(a) \widetilde{W}(z) \widetilde{W}(z^{-1}) \frac{dz}{z} = \frac{8}{3}$$

Figures 4, 5 and 6 plot the gain coefficient g_2, the pole location and the zero location of the optimum digital filter for the ramp phase input. Again we see that $\hat{D}(z)$ goes to $\widetilde{D}(z)$ when $B_N T \geqslant (B_N T)_c$.

V. Conclusion

An optimum design procedure to replace the conventional analog loop filter in a phase locked loop by a digital computer is given. Specific examples of step and ramp change in phase have been described. In both cases, the filter gain coefficients, pole locations, and zero locations are plotted for various time-bandwidth products.

References

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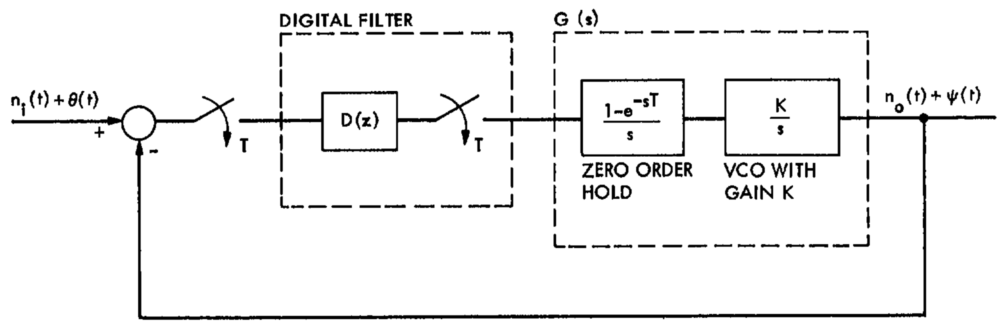

Fig. 1. Digital-analog phase locked loop

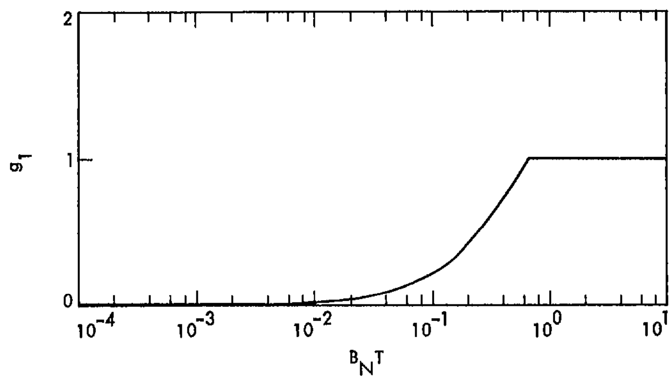

Fig. 2. Gain vs $B_N T$ curve for loop with step change in phase

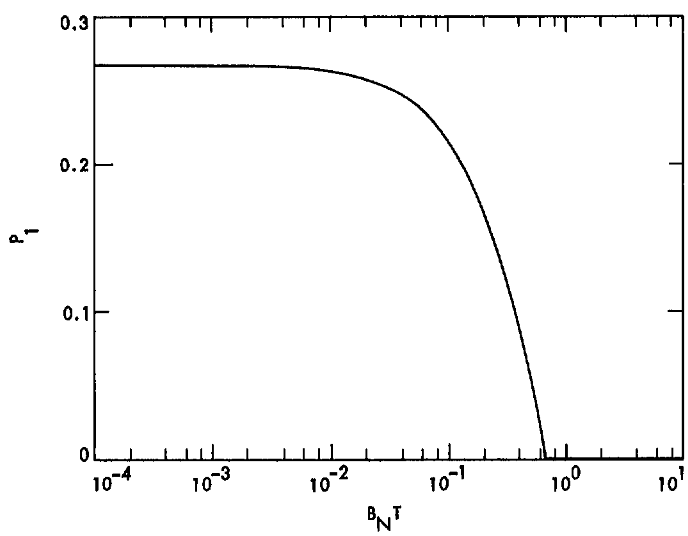

Fig. 3. Location of pole vs $B_N T$ curve for loop with step change in phase

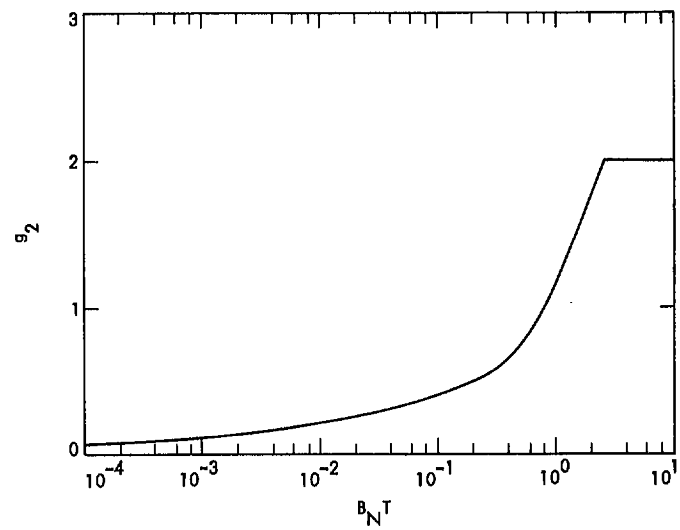

Fig. 4. Gain vs $B_N T$ curve for loop with ramp change in phase

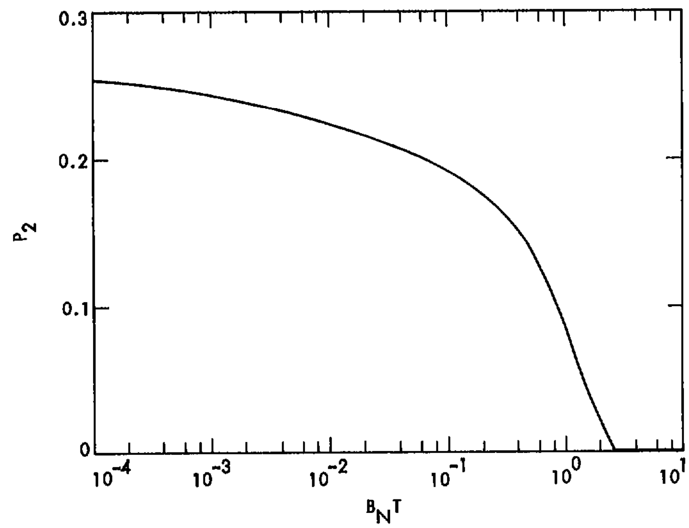

Fig. 5. Location of pole vs $B_N T$ curve for loop with ramp change in phase

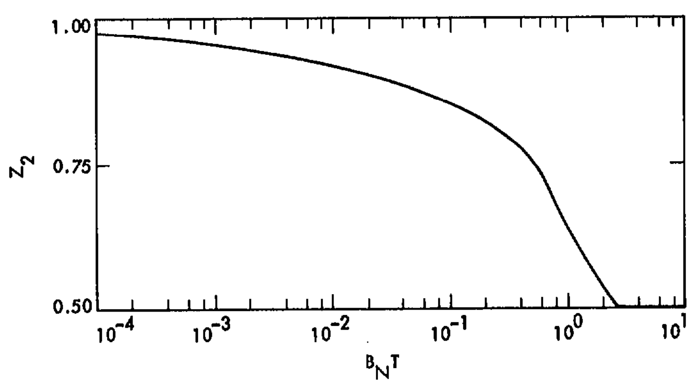

Fig. 6. Location of zero vs $B_N T$ curve for loop with ramp change in phase

Appendix A
Evaluation of the Sampled Power Spectral Density for the Output of Digital Filters

Let $[x_n]_{-\infty}^{\infty}$ be the stationary random sequence to the input of a digital filter having the impulse response $[h_n]_{-\infty}^{\infty}$. The output $[y_n]_{-\infty}^{\infty}$ can be written as

$$y_n = \sum_{k=-\infty}^{\infty} h_{n-k}\, x_k$$

Then

$$E[y_n y_m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h_{n-k}\, h_{m-l}\, E[x_k x_l]$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h_k h_l E[x_{n-k} x_{m-l}]$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h_k h_l r_{(m-n)+(k-l)}$$

where

$$r_k = E[x_n\, x_{n+k}]$$

Therefore $[y_n]_{-\infty}^{\infty}$ is also a stationary sequence with

$$E[y_n y_{n+m}] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h_k h_l r_{m+(k-l)}$$

Defining the sampled power spectral densities of sequences $[y_n]_{-\infty}^{\infty}$ and $[x_n]_{-\infty}^{\infty}$ as

$$\Phi_y(z) = T \sum_{m=-\infty}^{\infty} E[y_n y_{n+m}]\, z^{-m}$$

and

$$\Phi_x(z) = T \sum_{j=-\infty}^{\infty} E[x_n x_{n+j}]\, z^{-j} \quad \text{for all } n$$

we obtain

$$\Phi_y(z) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h_k h_l T \sum_{m=-\infty}^{\infty} r_{m+(k-l)} z^{-m}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h_k h_l z^{k-l}\, \Phi_x(z)$$

$$= H(z)\, H(z^{-1})\, \Phi_x(z)$$

where $H(z)$ is the z transform of the impulse response $[h_n]_{-\infty}^{\infty}$

Hence

$$E[y_n^2] = \frac{1}{T} \cdot \frac{1}{2\pi j} \oint_c \Phi_y(z)\, \frac{dz}{z}$$