

# The Effects of Pointing Errors on the Performance of Optical Communications Systems

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*Optical communications systems operating over interplanetary distances require the use of extremely narrow optical beams for maximum power concentration near the receiver. Consequently, pointing errors must be kept to a small fraction of a beamwidth to avoid severe deterioration in receiver performance, due to the decrease in received power associated with pointing errors. In this article, the mathematical models required for studying the effects of random pointing errors are developed and applied to the problem of quantifying the effects of pointing errors on the performance of coherent and incoherent optical receivers.*

## I. Introduction

Long-distance optical communications systems generally operate with narrow optical beams, in order to maximize the signal power-density in the vicinity of the receiver. While the minimum attainable beamwidth is limited by diffraction effects, optical antennas can generate beams with divergence angles on the order of microradians. Therefore, accurate beam pointing becomes a formidable task, and even minute pointing errors can lead to severe deterioration in system performance. Here we consider an idealized long-distance optical communications system model consisting of a diffraction-limited optical transmitter and an optical receiver located in the far-field of the transmitted beam. In order to simplify the analysis, it is assumed that there is no relative motion between the receiver and the transmitter. We concentrate on modelling the optical field at the receiver in the presence of random pointing errors and on developing a useful model for the probability density of the pointing error. The model is applied to the optical communications problem in order to determine the effects of constant pointing offsets and random pointing errors on the performance of both coherent and direct-detection (or incoherent) optical communications systems.

## II. Mathematical Models for the Received Field

Consider the field-propagation model shown in Fig. 1. A circular transmitter aperture  $\mathcal{A}_t$  is assumed to be centered in the transmitter plane (coordinates  $x_1, y_1$ ). The transmitter aperture is illuminated by a temporally modulated, normally incident plane-wave  $U_t(t; x_1, y_1)$ , which we model as

$$U_t(t; x_1, y_1) = \begin{cases} U_1(t_1); & x_1^2 + y_1^2 \leq (d_t/2)^2 \\ 0 & ; \text{ elsewhere} \end{cases} \quad (1)$$

where we define  $t_1 = t + (z/c)$  in order to account for propagation delay,  $d_t$  is the diameter of the transmitter aperture and  $c$  is the speed of light. The temporal component is defined as

$$U_1(t) = \left( \frac{U_1}{\sqrt{A_t}} \right) m(t) \exp [j(\omega t + \phi_1(t))] \quad (2)$$

where  $A_t$  is the area of the transmitter aperture,  $U_1/\sqrt{A_t}$  is the normalized field amplitude,  $m(t)$  is the modulating waveform ( $|m(t)| \leq 1$ ),  $\omega$  is the radian frequency of the optical carrier, and  $\phi_1(t)$  is a random phase process associated with the optical source. The normalized field amplitude  $U_1/\sqrt{A_t}$  can be interpreted as an equivalent field amplitude that generates an average photon rate of  $n_t = U_1^2/h\nu$  photons/second, independent of the area of the transmitting aperture (here  $h$  is Planck's constant, and  $\nu = \omega/2\pi$  is the optical carrier frequency). The beam axis is defined to be a line normal to the transmitter plane, passing through the origin. The receiver aperture  $\mathcal{A}_r$  (with collecting area  $A_r$ ) is assumed to be located a distance  $z$  from the origin of the transmitter plane, perpendicular to the line between the centers of the transmitter and receiver apertures. The pointing error  $\theta_e$  is defined as the angle between this line and the beam axis, as shown in Fig. 1.

The receiver aperture is assumed to be in the "far-field," or Fraunhofer region, of the transmitted beam. If the dimensions of the receiver aperture are much smaller than the beam dimensions, then amplitude and phase variations over the aperture can be neglected, and the complex field at the receiver can be represented as

$$U(t; z, \theta_e) = A_r U_1(t) f(z) G(\theta_e) \quad (3)$$

where

$$f(z) = \frac{e^{j2\pi z/\lambda}}{j\lambda z} \quad (4)$$

$$G(\theta_e) \approx 2 \frac{J_1(\pi d_t \theta_e / \lambda)}{(\pi d_t \theta_e / \lambda)} \quad (5)$$

and  $J_1(\cdot)$  is the Bessel function of order one (Ref. 1). The amplitude gain function  $G(\theta_e)$  is the normalized diffraction pattern of the transmitter aperture. The amplitude gain function  $G(\theta_e)$  and the intensity gain function  $G^2(\theta_e)$  are shown in Fig. 2. Note that the first zero occurs at  $\zeta \triangleq (\pi d_t / \lambda) \theta_e = 3.82$ , clearly defining the dimensions of the main lobe.

When the standard deviation of the pointing error is much less than one radian ( $\theta_e \ll 1$  radian), the pointing error can be conveniently decomposed into orthogonal components  $\theta_x$  and  $\theta_y$ , where

$$\theta_x = \theta_e \cos \psi \quad (6a)$$

$$\theta_y = \theta_e \sin \psi \quad (6b)$$

and

$$\psi = \tan^{-1} \left( \frac{\theta_y}{\theta_x} \right) \quad (6c)$$

We assume that  $\theta_x$  and  $\theta_y$  are independent random variables with mean values  $\eta_x$  and  $\eta_y$ , and variance  $\sigma_x^2$  and  $\sigma_y^2$  respectively. The total pointing error process can then be expressed in terms of  $\theta_x$  and  $\theta_y$  as

$$\theta_e(t) = \left[ \theta_x^2(t) + \theta_y^2(t) \right]^{1/2} \quad (7)$$

It is convenient to define the parameter  $\eta$  as

$$\eta = \left[ \eta_x^2 + \eta_y^2 \right]^{1/2} \quad (8)$$

which can be interpreted as the pointing error induced by constant pointing offsets in the  $\theta_x$  and  $\theta_y$  directions.

The analysis becomes somewhat tractable if we assume that  $\theta_x(t)$  and  $\theta_y(t)$  are Gaussian random processes. Suppressing the time dependence for notational simplicity, and letting  $\sigma_x^2 = \sigma_y^2 = \sigma^2$  the probability density for the independent Gaussian random variables  $\theta_x$  and  $\theta_y$  is given by the expression

$$p_{xy}(\theta_x, \theta_y) = \frac{1}{2\pi\sigma^2} \exp \left[ -\frac{(\theta_x - \eta_x)^2 + (\theta_y - \eta_y)^2}{2\sigma^2} \right] \quad (9)$$

The density of  $\theta_e$  can be determined by a straightforward transformation (Ref. 2); using Eq. (6), the joint density of the random variables  $\theta_e$  and  $\psi$  can be expressed as

$$p(\theta_e, \psi) = \frac{\theta_e}{2\pi\sigma^2} \exp \left[ -\frac{(\theta_e \cos \psi - \eta_x)^2 + (\theta_e \sin \psi - \eta_y)^2}{2\sigma^2} \right] \quad (10)$$

It follows therefore that

$$p(\theta_e) = \int_0^{2\pi} p(\theta_e, \psi) d\psi = \frac{\theta_e}{\sigma^2} \exp \left[ -\frac{(\theta_e^2 + \eta^2)}{2\sigma^2} \right] \left( \frac{1}{2\pi} \int_0^{2\pi} \exp \left[ \frac{\theta_e}{\sigma^2} (\eta_x \cos \psi + \eta_y \sin \psi) \right] d\psi \right) \quad (11)$$

Defining the angle  $\psi_1$  as

$$\psi_1 = \tan^{-1} \left( \frac{\eta_y}{\eta_x} \right) \quad (12)$$

we can rewrite the exponent inside the integral as

$$\eta_x \cos \psi + \eta_y \sin \psi = \eta \cos(\psi_1 - \psi) \quad (13)$$

The integral is now recognized as a representation of the modified Bessel function of order zero:

$$\frac{1}{2\pi} \int_0^{2\pi} \exp \left[ \frac{\theta_e}{\sigma^2} \eta \cos(\psi_1 - \psi) \right] d\psi = I_0 \left( \frac{\theta_e \eta}{\sigma^2} \right) \quad (14)$$

Substituting Eq. (14) into Eq. (11) yields

$$p(\theta_e) = \frac{\theta_e}{\sigma^2} \exp \left[ -\frac{1}{2\sigma^2} (\theta_e^2 + \eta^2) \right] I_0 \left( \frac{\theta_e \eta}{\sigma^2} \right) \quad (15)$$

which is seen to be the well-known Rice density. Note that since  $I_0(0) = 1$ , Eq. (15) reduces to the familiar Rayleigh density in the limit as the pointing error  $\eta \rightarrow 0$ . In the following sections, we shall apply the above results to examine the effects of pointing error on the performance of direct detection and coherent optical receivers.

### III. Performance of Optical Receivers in the Presence of Pointing Errors

In this section, the effects of pointing errors on receiver performance are examined. First, we consider the effects of pointing errors on direct-detection receivers, assuming that  $M$ -ary PPM signal sets are observed. Such signals can be generated by letting  $m(t) = 1$  over one of  $M$  time slots, and zero over the remaining  $(M - 1)$ . The performance of  $M$ -ary PPM receivers in the presence of background radiation has been studied elsewhere (Ref. 3). Here we shall assume that the effects of background radiation are negligible, and concentrate on the effects of random pointing errors.

The symbol-error probability can be expressed in terms of the symbol erasure probability  $\epsilon$  as

$$P_d(E) = \frac{M-1}{M} \epsilon \quad (16a)$$

where

$$\epsilon = \exp [-K_s] \quad (16b)$$

and  $K_s$  is the average count per symbol, in the absence of any pointing errors. For pulses that are much narrower than the correlation time of the pointing-error process, the average pulse count can be related to the received field, conditioned on a given pointing error,  $\theta_e$ , as

$$K_s(\theta_e) = \frac{\rho}{h\nu} \iint_{\mathcal{A}_r} \left[ \int_0^\tau |U(t; \theta_e)|^2 dt \right] dx dy = \left( \frac{\rho}{h\nu} \right) A_r \left( \frac{U_1^2}{\Omega_r z^2} \right) G^2(\theta_e) = K_s G^2(\theta_e) \quad (17)$$

where  $\tau$  is the pulse duration,  $\rho$  is the quantum efficiency of the photodetector,  $h$  is Planck's constant,  $\nu$  is the optical frequency, and  $\Omega_r = \lambda^2/A_r$  is the divergence of the transmitted beam, measured in steradians. The unconditional erasure probability is obtained by averaging the conditional erasure probability over the density of the pointing-error:

$$\epsilon = \int_0^\infty \exp [-K_s(\theta_e)] p(\theta_e) d\theta_e \quad (18)$$

This expression is accurate whenever the Gaussian approximation for  $\theta_x$  and  $\theta_y$  can be invoked. For the pointing-error density of Eq. (15), the erasure probability becomes

$$\epsilon = \frac{1}{\sigma^2} \int_0^\infty \theta_e \exp \left[ -K_s G^2(\theta_e) - \frac{(\theta_e^2 + \eta^2)}{2\sigma^2} \right] I_0 \left( \frac{\theta_e \eta}{\sigma^2} \right) d\theta_e \quad (19)$$

In the limit as  $\sigma^2 \rightarrow 0$ ,  $p(\theta_e) \rightarrow \delta(\theta_e - \eta)$ , and the erasure probability reduces to

$$\epsilon = \exp [-K_s G^2(\eta)] \quad (20)$$

This erasure probability is shown in Fig. 3 as a function of the normalized pointing error  $\eta_e$  for several values of  $K_s$ , where  $\eta_e = (\pi d_r / \lambda) \eta$ . In terms of these units, a normalized mean value of  $\eta_e = 3.82$  corresponds to the planar half-angle of the main lobe (the actual main lobe half-angle is, of course,  $(\lambda / \pi d_r)$  times as great). The points where  $\epsilon = 1$  correspond to the zeros of the antenna pattern.

The effects of random pointing errors on the erasure probability are shown in Fig. 4(a) through 4(c). (Numerical integration of Eq. (19) was employed to obtain these graphs.) For a given  $K_s$ , the erasure probability is shown as a function of the normalized mean pointing error  $\eta_e = (\pi d_t / \lambda) \eta$  for various values of the normalized variance  $\sigma_e^2 = (\pi d_t / \lambda)^2 \sigma^2$ . Recall that in typical applications,  $(\lambda / \pi d_t) \approx 10^{-6}$ , which means that typical beam half angles are on the order of microradians. Note that for high values of  $K_s$ , ( $K_s \gtrsim 10$ ) the erasure probability increases dramatically with increasing (normalized) variance, emphasizing the importance of reducing the variance of the pointing error to a small fraction of the main-lobe divergence when operating at low error probabilities. At low values of  $K_s$ , ( $K_s \lesssim 5$ ), the effects of mean pointing offsets and random pointing errors become much less pronounced, suggesting that under these conditions the requirements on pointing accuracy can be relaxed.

The performance of coherent receivers can be analyzed in a similar manner. Coherent homodyne reception requires the addition of a local field prior to photodetection. We model the local field as an equivalent plane-wave with temporal variation

$$U_L(t) = \left( \frac{U_L}{j} \right) \exp [j(\omega t + \phi_L(t))] \quad (21)$$

where  $U_L$  is the field amplitude, and  $\phi_L(t)$  is a random phase process due to phase instabilities within the local laser. Assuming that  $U_L \gg U_1 / \sqrt{\Omega_t z}$  (which is generally true for long-range communications systems) the average count generated by a binary antipodal signal ( $m(t) = \pm 1$ ), given that hypothesis  $H_i$  is true ( $i = 0, 1$ ) can be expressed as

$$K_i(\theta_e) = \frac{\rho}{h\nu} \iint_{\mathcal{A}_r} \left[ \int_0^\tau |U(t; \theta_e) + U_L(t)|^2 dt \right] dx dy = \frac{\rho}{h\nu} A_r \left[ U_L^2 + 2(-1)^{(i+1)} \frac{U_L U_1}{\sqrt{\Omega_t z}} G(\theta_e) \cos \phi_e \right] \quad (22)$$

where  $\phi_e = [\phi_1 - \phi_L + (2\pi z / \lambda)]$  is the phase process of the detected field. If the phase-tracking error  $\phi_e(t)$  is assumed to be negligible, then we can let  $\cos \phi_e \approx 1$ . If the pointing errors were also negligible ( $\theta_e = 0$ ), then the error probability of the above coherent receiver could be expressed as (Ref. 4):

$$P_c(E) = Q(\sqrt{4K_s})$$

$$Q(u) = \frac{1}{2\pi} \int_u^\infty e^{-z^2/2} dz \quad (23)$$

where  $K_s$  is again the average number of signal counts over a given bit interval. We have observed before that the effect of the pointing error  $\theta_e$  is to decrease the average number of observed counts in proportion to the normalized antenna pattern of the transmitter aperture. Therefore, the conditional error probability of the coherent binary (MAP) receiver can be expressed as

$$P_c(E|\theta_e) = Q(\sqrt{4K_s} G(\theta_e)) \quad (24)$$

(Note that the argument of Eq. (24) may take on negative values due to the phase-sensitive detection technique we have employed, which responds to negative values of  $G(\theta_e)$ ). The unconditional error probability is again the average of the conditional error-probability over the pointing-error statistics:

$$P_c(E) = \frac{\left[ \exp - \left( \frac{\eta^2}{2\sigma^2} \right) \right]}{\sigma^2} \int_0^\infty \theta_e Q(\sqrt{4K_s} G(\theta_e)) \exp \left[ - \frac{\theta_e^2}{2\sigma^2} \right] I_0 \left( \frac{\theta_e \eta}{\sigma^2} \right) d\theta_e \quad (25)$$

When only pointing offsets are present, the error probability again becomes a function of  $\eta$ :

$$P_c(E) = Q(\sqrt{4K_s} G(\eta)) \quad (26)$$

Figure 5 shows the error probability of the coherent receiver as a function of  $\eta_e$ , in the limit as  $\sigma_e^2 \rightarrow 0$ , which again corresponds to the idealized pointing error density  $p(\theta_e) = \delta(\theta_e - \eta)$ . Note that the error probability may exceed a half because the antenna pattern can assume negative values, in which case the receiver almost certainly commits an error. As the variance of the pointing error increases ( $\sigma_e^2 > 0$ ) receiver performance deteriorates, as can be seen in Fig. 6(a) through 6(c). (Again, numerical integration was employed to evaluate Eq. (25)). As before, we observe that the performance deterioration due to pointing error is most severe when the receiver is operating at low error probabilities, and tends to become less serious as the average on-axis signal bit count  $K_s$  decreases.

#### IV. Summary and Conclusions

We have developed a general model for evaluating the effects of random pointing errors on the received field in long-range optical communications systems. The probability density of the pointing-error random process has been derived for the case of independent, equal-variance Gaussian pointing error components. This model was then applied to the problem of determining the effects of pointing errors on the performance of direct-detection and coherent optical receivers. The results indicate that pointing errors tend to cause a severe deterioration in receiver performance only when the optical receivers are operating at very low error probabilities. Therefore, the ultimate performance of long distance optical communications systems may well be limited by the ability of the transmitter to point the downlink beam toward the intended receiver.

#### References

1. Born, M., and Wolf, E., *Principles of Optics*, Pergamon Press, New York, 1975.
2. Papoulis, A., *Probability, Random Variables and Stochastic Processes*, McGraw-Hill, New York, 1965.
3. Gagliardi, M., and Karp, S., *Optical Communications*, J. Wiley, New York, 1976.
4. Vilmrotter, V. A., "A Binary Coherent Optical Receiver for the Free-Space Channel," in *The TDA Progress Report 42-61*, pp. 27-38, Jet Propulsion Laboratory, Pasadena, Calif., Feb. 15, 1981.

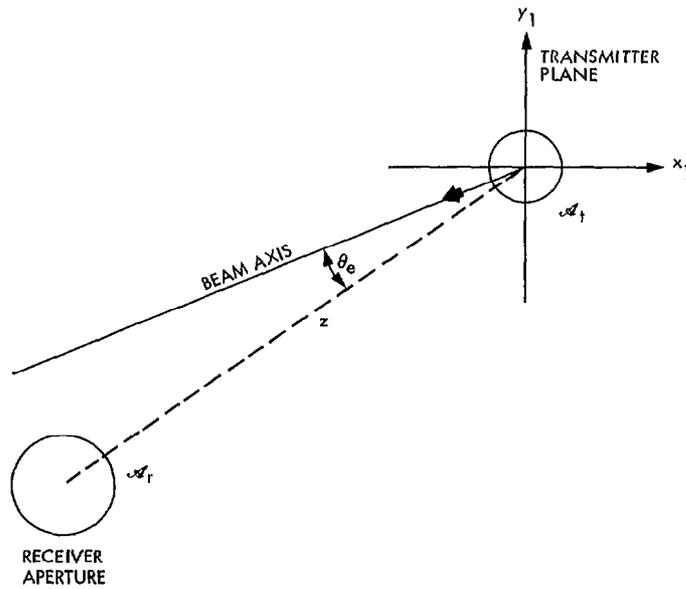


Fig. 1. Optical field propagation geometry

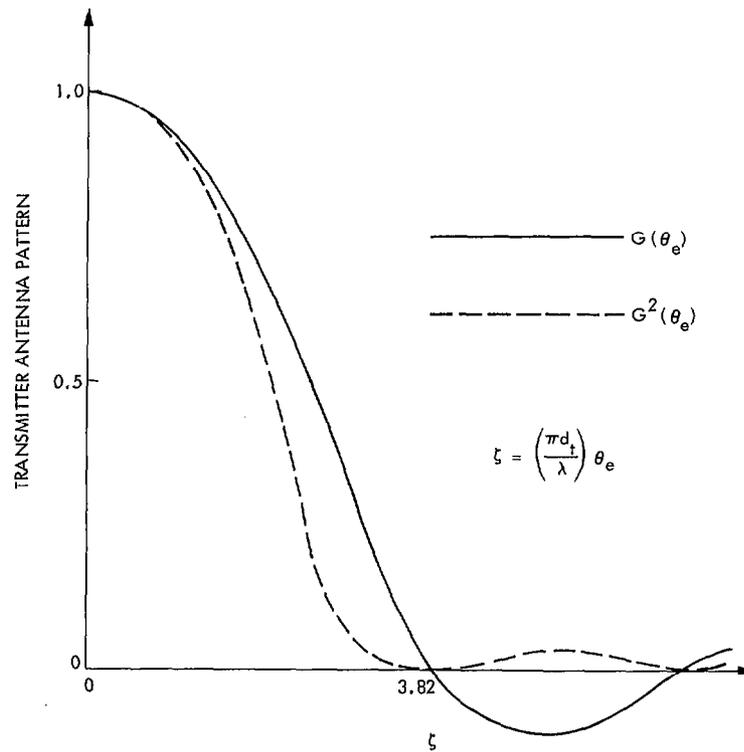
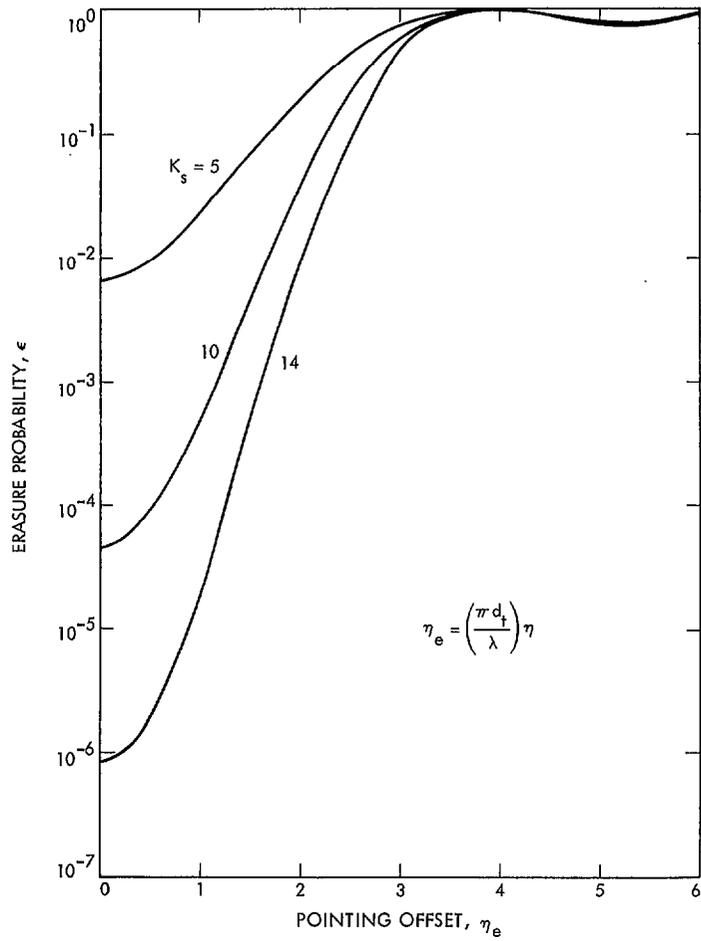


Fig. 2. Amplitude and intensity patterns generated by a circular transmitter aperture



**Fig. 3. Erasure probability in the presence of pointing offset**

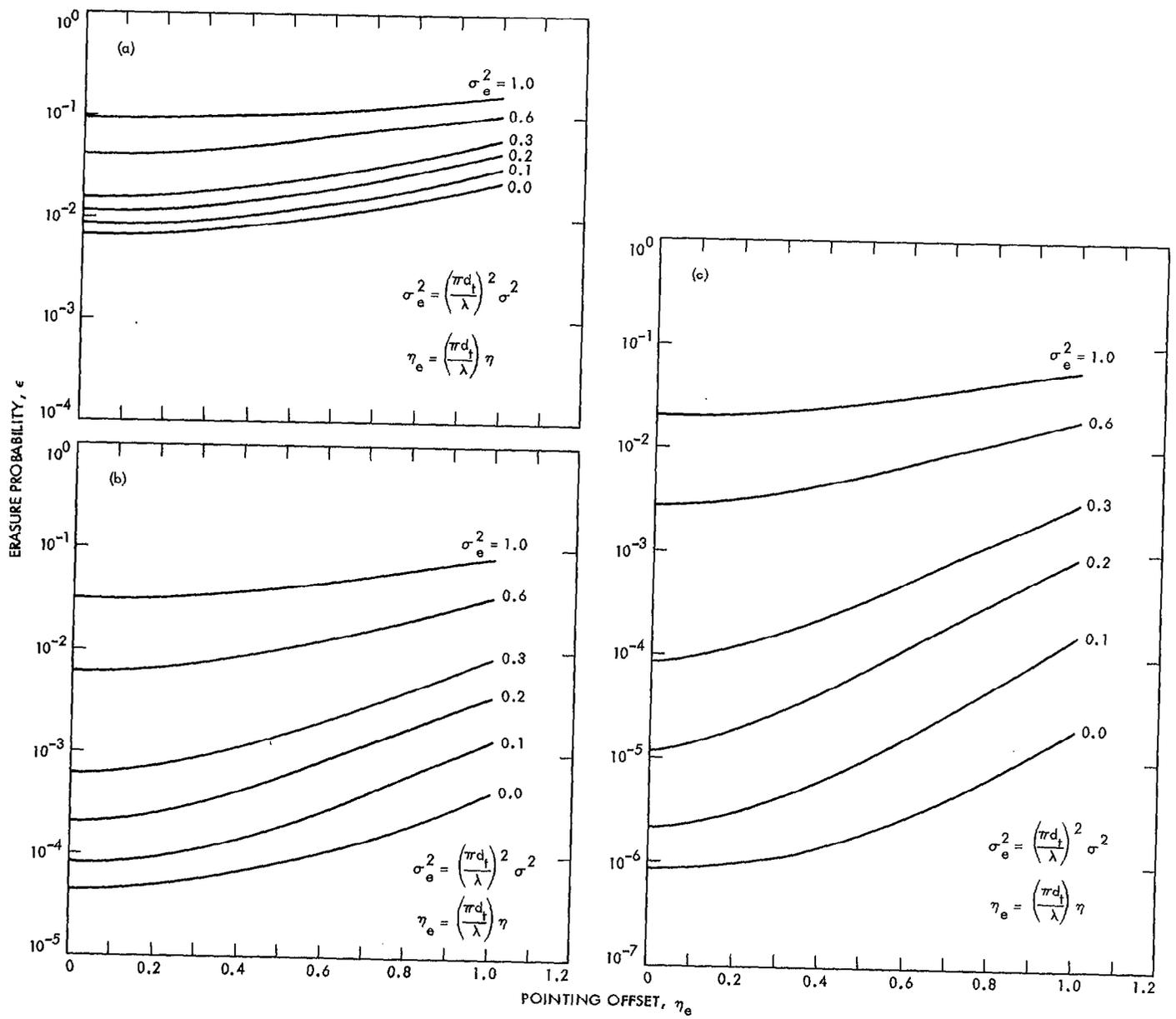
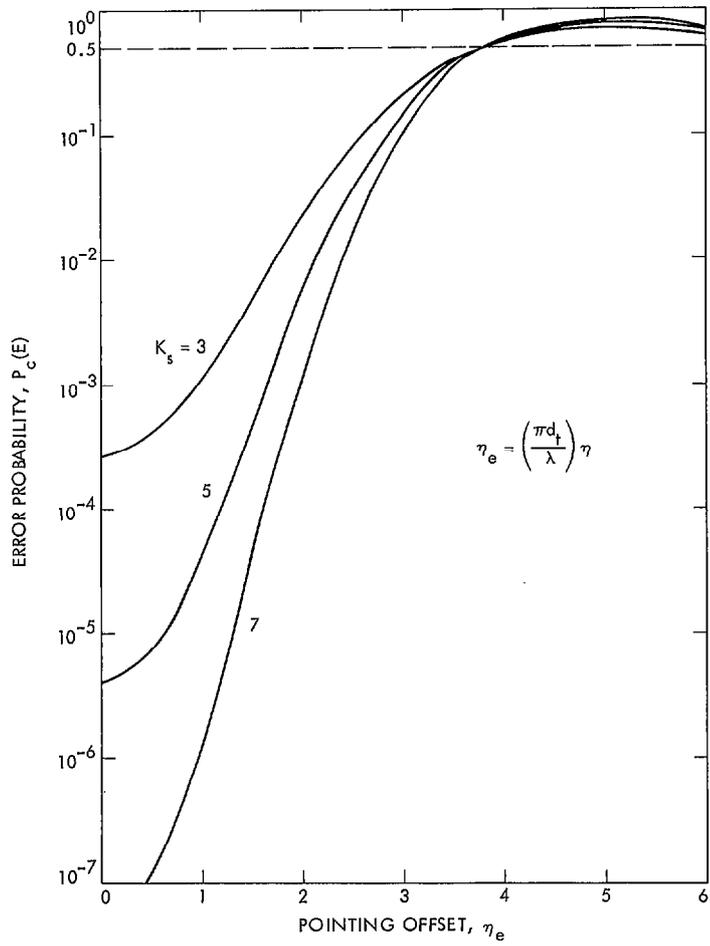


Fig. 4. Erasure probability in the presence of pointing error: (a)  $K_s = 5$ ; (b)  $K_s = 10$ ; (c)  $K_s = 14$



**Fig. 5. Performance of binary coherent receiver in the presence of pointing offset**

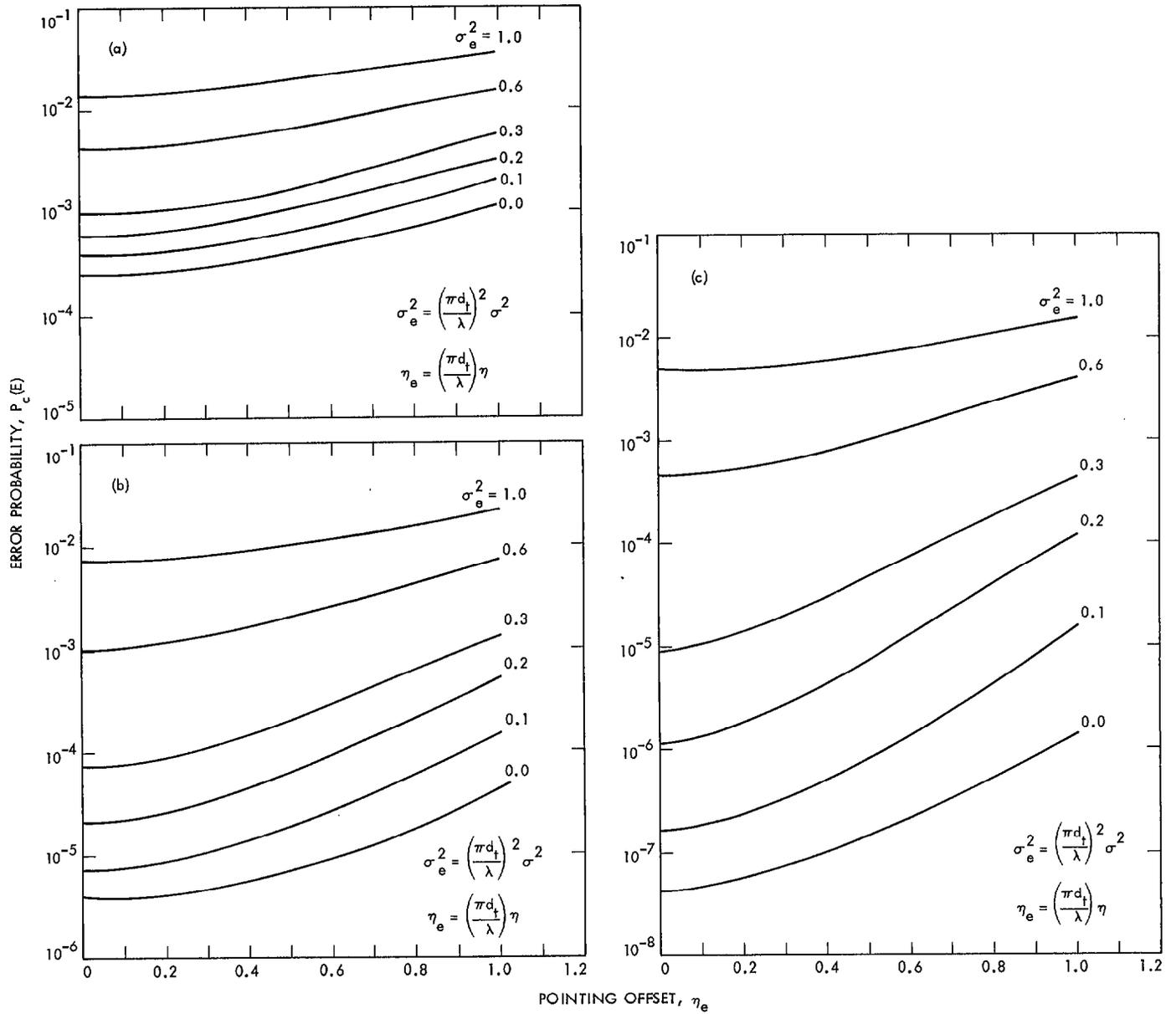


Fig. 6. Performance of binary coherent receiver in the presence of pointing error: (a)  $K_s = 3$ ; (b)  $K_s = 5$ ; (c)  $K_s = 7$