On the Correction for Quantization Effects in Signal-To-Noise Ratio Estimation

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In sampled data digital telemetry systems the signal-to-noise ratio (SNR) is typically derived as a function of the moments of the digitized input stream (e.g., the receiver output). This analog-to-digital conversion process is itself an additional noise source known as the "quantization" noise. Thus a digitally measured SNR will only approximately represent the SNR of the analog input signal. This report presents a procedure (Sheppard's corrections) for correcting moments of any order for this quantization effect.

I. Introduction

Experience gained over the last two years using the existing subsystems for arraying (Ref. 1) at Jupiter (Voyager 2) and Saturn (Pioneer 11, Voyagers 1 and 2) shows that one of the weak points of the current system is the ability to accurately monitor telemetry signal-to-noise ratio (SNR) at levels acceptable for real-time array performance validation. As part of a development task to improve performance in this area a review and analysis of the problems expected from digital signal processing and SNR estimation has been undertaken. The first of a series of reports on this work is a treatment of quantization effects.

II. Quantization Effects

Quantization error is the term applied to the errors introduced by representation of analog values by a finite (usually few) number of bits. Other sources of numerical error may be introduced during internal calculations; but they are seldom of the significance of the initial quantization errors from the analog-to-digital conversion process, and they can usually be avoided by careful hardware design.

The most extreme case of quantization error occurs when only the sign of the analog signal (+ or −) survives the digital conversion process. This process is often referred to as hard-limiting the signal, and the output is called the sign bit. Single bit estimators are important in that they minimize hardware complexity, often with an acceptable degradation in performance.

There should be a separate treatment of n bit quantization for each $n = 1, 2, 3 \ldots$ ad nauseum. Fortunately an n bit quantizer has 2^n states or quantization levels, each state occupying 2^{-n} of full-scale input range (nonlinear or companding converters will not be treated here, although companding offers interesting possibilities). For $n > 4$ the quantization levels become close enough together that we may assume the sampled distribution is constant over the width of a level. This allows us to treat all cases of sufficiently large n as if the conversion was infinitely accurate ($n \to \infty$) but also introduces an additive noise component (dependent on n) called the quantization noise.

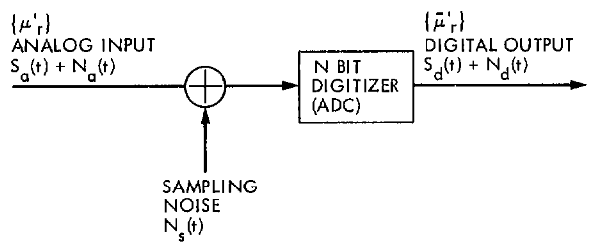
For an n bit quantizer the quantization levels are spaced 2^{-n} of full range apart; call this value ϵ. The quantized output can exactly represent the input only if the input is exactly at one of these quantization levels. Imagine now that for any arbitrary analog input (within range), an additional "noise" source was added so that the resulting analog signal was moved to the nearest quantization level.

This is the "quantization noise." From what has already been said, its distribution will be uniform from $-\epsilon/2$ to $+\epsilon/2$ about zero (and thus $dF = dx/\epsilon$) and the associated noise power is thus

$$P_{QNoise} = \int_{-\epsilon/2}^{\epsilon/2} x^2 \left(\frac{1}{\epsilon}\right) dx = \frac{\epsilon^2}{12} \tag{1}$$

The effect of quantization noise is to increase the effective noise component of the signal-to-noise ratio during the act of digitization.

A block diagram of the process is:



where $\{\mu'_r\}$ = moments of the left-hand side of the distribution, and $\{\bar{\mu}'_r\}$ = moments of the right-hand side of the distribution. Since the analog input power is usually held constant (to keep within the ADC range), the digitally estimated SNR asymptotically approaches a constant value as the analog SNR increases.

Consider the general effect on moments over the distribution. Analog moments on the left-hand side (l.h.s.) of the block diagram are given by

$$\mu'_r = \int_{-\infty}^{\infty} x^r \, dF(x) = \sum_{K=0}^{2^N} \int_{V_K}^{V_{K+1}} x^r dF(x) \tag{2}$$

where

$F(x)$ = input distribution

μ'_r = r^{th} central moment (l.h.s.)

V_K = K^{th} quantization level, $K = 1, \ldots, 2^N$

$(V_0 \to -\infty, V_{2+1} \to +\infty)$

Digital moments on the right hand side (r.h.s.) of the block diagram are given by

$$\mu'_r = \sum_{K=1}^{2^N} V_K^r \int_{V_K}^{V_{K+1}} dF(x) = \sum_{K=1}^{2^N} V_K^r \left[F(V_K) - F(V_{K-1})\right] \tag{3}$$

The problem of approximating integrals like Eq. (2) by finite sums like Eq. (3) is the subject of the Euler-Maclaurin sum formula (see Ref. 2). The estimation of continuous moments from quantized or group data is fortunately a familiar problem in statistics. The results are known as Sheppard's corrections, and the detailed development may be found in statistics texts such as Ref. 3. The formulas for Sheppard's corrections are summarized in the next section.

III. Quantization Effect Corrections

The results are as follows:

$$\bar{\mu}'_r = \sum_{j=0}^{\left[\frac{r}{2}\right]} \binom{r}{2j} \left(\frac{1}{2}\epsilon\right)^{2j} \frac{1}{2j+1} \mu'_{r-2j} \tag{4}$$

where

$\left[\frac{r}{2}\right]$ is the integral part of $\frac{r}{2}$

which approximates the r.h.s. moments in terms of l.h.s. moments, and

$$\mu'_r = \sum_{j=0}^{r} \binom{r}{j} (2^{1-j} - 1)\, B_j \epsilon^j \bar{\mu}'_{r-j} \tag{5}$$

which approximates l.h.s. moments in terms of r.h.s. moments.

The B_j are the Bernoulli numbers, and obey the generating equation

$$\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} \frac{B_j t^j}{j!} \tag{6}$$

The first 14 are: $B_0 = 1$; $B_1 = -1/2$; odd B's = 0 except B_1; $B_2 = 1/6$; $B_4 = -1/30$; $B_6 = 1/42$; $B_8 = -1/30$; $B_{10} = 5/66$; $B_{12} = -691/2730$; $B_{14} = 7/6$.

For the lower moments

$$\bar{\mu}_1' = \mu_1'$$

$$\bar{\mu}_2' = \mu_2' + \frac{\epsilon^2}{12}$$

$$\bar{\mu}_3' = \mu_3' + \frac{1}{4}\mu_1' \epsilon^2$$

$$\bar{\mu}_4' = \mu_4' + \frac{1}{2}\mu_2' \epsilon^2 + \frac{1}{80}\epsilon^4$$

$$\bar{\mu}_5' = \mu_5' + \frac{5}{6}\mu_3' \epsilon^2 \frac{1}{16}\mu_1' \epsilon^4$$

$$\bar{\mu}_6' = \mu_6' + \frac{1}{4}\mu_4' \epsilon^2 \frac{3}{80}\mu_2' \epsilon^4 + \frac{1}{448}\epsilon^6 \tag{7}$$

and

$$\mu_1' = \bar{\mu}_1'$$

$$\mu_2' = \bar{\mu}_2' - \frac{\epsilon^2}{12}$$

$$\mu_3' = \bar{\mu}_3' - \frac{1}{4}\bar{\mu}_1' \epsilon^2$$

$$\mu_4' = \bar{\mu}_4' - \frac{1}{2}\bar{\mu}_2' \epsilon^2 + \frac{7\epsilon^4}{240}$$

$$\mu_5' = \bar{\mu}_5' - \frac{5}{6}\bar{\mu}_3' \epsilon^2 + \frac{7}{48}\bar{\mu}_1' \epsilon^4$$

$$\mu_6' = \bar{\mu}_6' - \frac{5}{4}\bar{\mu}_4' \epsilon^2 + \frac{7}{16}\bar{\mu}_2' \epsilon^4 - \frac{31\epsilon^6}{1344} \tag{8}$$

IV. Conclusion

This article has presented the corrections for quantization effects in the estimation of distribution moments. The limitations are that the quantization steps be equally spaced and that the continuous functions whose integrals are estimated must in some sense be smooth with respect to quantization grid (see Ref. 3). The extension of this approach to absolute moments (which are not "smooth" at the origin) and to unequal quantization spacing (companding converters) is currently under study. The effect of these corrections on specific SNR estimation algorithms depends upon the detailed representation of the estimator in terms of distribution moments. Results will be reported as they become available.

References

1. Howard, L. D., Prototype Real-Time Baseband Signal Combiner, in *The Telecommunications and Data Acquisition Progress Report 42-60*, pp. 145-151, Jet Propulsion Laboratory, Pasadena, Calif., Dec. 15, 1980.
2. Boole, George, *A Treatise on the Calculus of Finite Differences* (Second Ed., 1872), Dover Publications, Inc., New York, 1960.
3. Kendall, M. G., and Stuart, A., *The Advanced Theory of Statistics*, Vol. 1, Ch. 3, Hafner Publishing Co., New York, 1969.