# On Counters Used for Node Synchronization

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*A node synchronization algorithm for a quick-look convolutional decoder was given in a previous article, which left two assertions unproved. The present article proves these assertions and gives an estimate for the distribution of the time to false alarm.*

## I. Introduction

A suboptimal quick-look decoding algorithm for the Deep Space Network (DSN) (7, 1/2) convolutional code is discussed in Refs. 1 to 3. Figure 1 shows the encoding and decoding schemes (without error correction, which does not concern us here). To detect node synchronization, one can use an up-down counter driven by the syndrome bits $p_n$ as follows: If $p_n = 0$, then the counter is decremented by 1; if $p_n = 1$, then the counter is incremented by a fixed positive integer $k-1$. The counter is not allowed to become negative, however, and a false-sync condition is declared if the counter reaches a certain threshold $T$.

The probability of false alarm, $P_{FA}$, is the probability of reaching $T$ during the total time of use, given that sync is true. We want $P_{FA}$ to be small. References 2 and 3 give estimates for $E_{FA}$, the expected time to false alarm, and execute a counter design based in part on the requirement $E_{FA} >> n_b$, the total number of bits seen by the decoder (specifically, $E_{FA} > 100\ n_b$). This is dangerous because the ratio $E_{FA}/n_b$ by itself gives no information about $P_{FA}$.

We have three aims here. First, the behavior of the node sync counter, called Counter 1, is estimated in Ref. 2 by comparing it to a certain random walk with independent steps, called Counter 2. Reference 2 asserts that Counter 1 is never above Counter 2. At the time, we carelessly regarded this assertion as obvious; in fact, it requires a substantial proof, which we give below. Second, we prove that the first-passage times of Counter 2 have finite expectation; Ref. 2 gives estimates for these expectations without proving their existence. Third, we give a crude (but still useful) estimate for $P_{FA}$.

## II. Proof That Counter 1 $\leq$ Counter 2

First, we review the generation of the syndrome $(p_n)$. According to Fig. 1, the syndrome is obtained by combining the outputs of two shift registers fed by the corrupted channel symbols $s^*_{1n}$, $s^*_{2n}$. The shift register taps are given by the polynomials

$$C_1(x) = 1 + x^2 + x^3 + x^5 + x^6$$

$$C_2(x) = 1 + x + x^2 + x^3 + x^6$$

Let $e_{1n}$, $e_{2n}$ be the binary channel symbol *errors*, with associated formal power series

$$E_1(x) = \sum e_{1n}x^n, E_2(x) = \sum e_{2n}x^n.$$

Then, if

$$P(x) = \sum p_n x^n,$$

we have

$$P(x) = C_2(x)E_1(x) + C_1(x)E_2(x) \pmod 2 \qquad (1)$$

Next, we define the counters. Let $\epsilon_t$ ($t = 1/2, 1, 3/2, 2, \cdots$) be the multiplexed symbol error stream, that is, $\epsilon_{n-1/2} = e_{1n}$, $\epsilon_n = e_{2n}$. Counters 1 and 2 both start at zero. Let $K_1(n)$ = counter 1 state at bit time $n$, $K_2(t)$ = counter 2 state at time $t = 1/2, 1, 3/2, 2, \cdots$. For our purpose we can ignore the absorbing barrier at $T$. Let $k$ be a fixed integer $\geqslant 2$. By definition,

$$K_1(0) = K_2(0) = 0$$

$$\begin{aligned} K_1(n) - K_1(n-1) &= k-1 \quad &&\text{if } p_n = 1 \\ &= -1 &&\text{if } p_n = 0, K_1(n-1) > 0 \\ &= 0 &&\text{if } p_n = 0, K_I(n-1) = 0 \end{aligned}$$

$$\begin{aligned} K_2(t) - K_2\left(t-\frac{1}{2}\right) &= 5k - \frac{1}{2} \quad &&\text{if } \epsilon_t = 1 \\ &= -\frac{1}{2} &&\text{if } \epsilon_t = 0, K_2\left(t-\frac{1}{2}\right) > 0 \\ &= 0 &&\text{if } \epsilon_t = 0, K_2\left(t-\frac{1}{2}\right) = 0 \end{aligned}$$

**Theorem**: Assume that $e_{1n} = e_{2n} = 0$ for $n \leqslant 0$. For any symbol error sequence ($e_{1n}, e_{2n}: n \geqslant 1$), we have

$$K_1(n) \leqslant K_2(n) \quad (n = 1, 2, \cdots)$$

If there were no reflecting barrier, the theorem would be obvious, for let $K_i'$ be Counter $i$ without the barrier. For example, $K_1'(n) - K_1'(n-1) = kp_n - 1$ for all $n$. Then, as Ref. 2 points out,

$$K_1'(n) = k\sum_{j=1}^{n} p_j - n \leqslant 5k \sum_{t \leqslant n} \epsilon_t - n = K_2'(n)$$

since each $\epsilon_t = 1$ propagates a pattern of 5 parity errors into the future, and these patterns are added modulo 2.

To prove the theorem with the barrier, we introduce another sequence $q_1, q_2, \cdots$ and a third counter $K_3$. The formal power series

$$Q(x) = \sum q_n x^n$$

is defined by

$$Q(x) = C_2(x)E_1(x) + C_1(x)E_2(x)$$

which is just Eq. (1), except that now the arithmetic is *not* performed modulo 2. Thus, $p_n = q_n \bmod 2 \leqslant q_n$. The counter $K_3$ is driven from the $q_n$ just as $K_1$ is driven from the $p_n$. By definition, $K_3(0) = 0$ and

$$\begin{aligned} K_3(n) - K_3(n-1) &= kq_n - 1 \quad &&\text{if } q_n > 0 \\ &= -1 &&\text{if } q_n = 0, K_3(n-1) > 0 \\ &= 0 &&\text{if } q_n = 0, K_3(n-1) = 0 \end{aligned}$$

Our purpose is to prove that

$$K_1(n) \leqslant K_3(n), K_3(n) \leqslant K_2(n) \text{ for all } n \qquad (2)$$

Since $p_n \leqslant q_n$, we have $K_1(n) - K_1(n-1) \leqslant K_3(n) - K_3(n-1)$. This proves the first half of Eq. (2).

To prove the second half, we introduce the notion of *burst event*. We shall say that a burst event starts at the integer $m$ if $e_{1m} = 1$ or $e_{2m} = 1$, and the previous 6 bit times are free of symbol errors. It ends (at integer time $r > m$) as soon as 6 consecutive error-free bit times have occurred (at times $r - 5, \cdots, r$). (The event goes on forever if a run of 6 good bit times never occurs after $m$.)

Let a burst event start at $m$. Let $K_3'$ be $K_3$ without the reflecting barrier. We shall prove that

$$K_3(n) - K_3(m-1) = K_3'(n) - K_3'(m-1) \qquad (3)$$

for all $n$ in the burst event. This means that the barrier does not influence the motion of $K_3$ during the burst event. If Eq. (3) holds for $k = 2$, then it holds for all $k > 2$ because the counter increments are greater. So assume $k = 2$.

The proof goes by induction on $n$. Equation (3) holds for $n = m - 1$. Let $n$ be in the burst event and assume that Eq. (3)

holds through time $n - 1$. There is an integer $i$ between $n - 6$ and $n$ such that $e_{1i} = 1$ or $e_{2i} = 1$. By assumption,

$$K_3(i-1) - K_3(m-1) = K_3'(i-1) - K_3'(m-1) \qquad (4)$$

If $e_{1i} = 1$, then $C_2 = 1111001$ propagates into the $q_n$ stream. If there are no other symbol errors from time $i$ onward, then $K_3(j) - K_3(i-1)$ takes values 1, 2, 3, 4, 3, 2, 3 for $j = i, \cdots, i+6$. Similarly, $e_{2i} = 1$ by itself propagates $C_1 = 1011011$ and causes $K_3(j) - K_3(i-1)$ to take values 1, 0, 1, 2, 1, 2, 3. Although the counter dips to zero in this case (if $K_3(i-1) = 0$), the next increment, being positive, moves the counter away from the barrier. Since any other symbol errors between $i$ and $i+6$ cause the counter to take values above those just displayed, we have shown that

$$K_3(j) - K_3(i-1) = K_3'(j) - K_3'(i-1)$$

for $i \leqslant j \leqslant i+6$, in particular, for $j = n$. With Eq. (4), this completes the induction and proves Eq. (3) over the whole burst event.

Consider now the behavior of $K_2$ during a burst event starting at $m$. Each symbol error (at time $n$ or $n - 1/2$) contributes $5k$ to $K_2'$ immediately (combined with a constant drift of $-1$ per bit), whereas the $5k$-contribution to $K_3'$ is spread over the times $n, n+1, \cdots, n+6$. Therefore

$$K_3'(n) - K_3'(m-1) \leqslant K_2'(n) - K_2'(m-1)$$

(In fact, the two sides are equal at the end of the burst event.) In view of Eq. (3) and the relation

$$K_2'(t) - K_2'(s) \leqslant K_2(t) - K_2(s)$$

valid whenever $s \leqslant t$, we have

$$K_3(n) - K_3(m-1) \leqslant K_2(n) - K_2(m-1) \qquad (5)$$

for all $n$ in the burst event.

We are almost done. Before the first burst event (if it exists), $K_3(n) = K_2(n) = 0$. During the first event, $K_3(n) \leqslant K_2(n)$ by Eq. (5). If the first event ends, then $K_3$ and $K_2$ both start to decrease at the same rate until they hit zero or the second burst event starts (if it exists). Just before the start of the second event, $K_3 \leqslant K_2$. By Eq. (5), $K_3 \leqslant K_2$ during the second event, and so on. This proves the second half of Eq. (2), and completes the proof of the theorem.

## III. Proof That the Mean Absorption Times of Counter 2 Are Finite

Since Counter 2 takes half-integral values with time steps of length 1/2, a simple change of variables (as in Ref. 2) brings the notation into line with the discussions of integer-valued random walks in Feller (Ref. 4). When we do this, we have a random walk, with independent steps, starting at height 1. Each step is equal to $d = 10\,k - 1$ with probability $p$, and $-1$ with probability $q = 1 - p$. The walk is not allowed to go below 1 (reflecting barrier at 0) and stops if it reaches or exceeds an absorbing barrier at $a = 2T + 1$.

Reference 2 uses the difference-equation method of Ref. 4 to get bounds on the expected absorption time (without first proving that the expectation exists). Here, we use the same method to estimate the generating function of the absorption-time distribution. For $1 \leqslant j \leqslant a - 1$ and $n \geqslant 1$, let $u_{j,n}$ be the probability that the walk is absorbed at time $n$, given that it starts at height $j$. The first step is to $j + d$ or $j - 1$ and so

$$u_{j,n+1} = pu_{j+d,n} + qu_{j-1,n} \qquad (6)$$

for $2 \leqslant j \leqslant a - d - 1$, $n \geqslant 1$. If we account for the absorbing and reflecting barriers by imposing the boundary conditions

$$\begin{aligned} u_{0,n} &= u_{1,n}, u_{j,n} = 0 \quad (a \leqslant j \leqslant a+d-1, n \geqslant 1) \\ u_{j,0} &= 0 \qquad\qquad (0 \leqslant j \leqslant a-1) \qquad (7) \\ u_{j,0} &= 1 \qquad\qquad (a \leqslant j \leqslant a+d-1) \end{aligned}$$

then Eq. (6) holds for $1 \leqslant j \leqslant a - 1$, $n \geqslant 0$. Introduce the generating functions

$$U_j(s) = \sum_{n=0}^{\infty} u_{j,n} s^n \quad (0 \leqslant j \leqslant a+d-1)$$

which converge at least for $|s| \leqslant 1$. Equations (6) and (7) are equivalent to the equations

$$\begin{aligned} U_j(s) &= psU_{j+d}(s) + qsU_{j-1}(s) \quad (1 \leqslant j \leqslant a-1) \qquad (8) \\ U_0(s) &= U_1(s) \\ U_j(s) &= 1 \qquad\qquad (a \leqslant j \leqslant a+d-1) \end{aligned}$$

Fix an $s$, $0 < s < 1$. The characteristic equation of Eq. (8),

$$pz^d + qz^{-1} = \frac{1}{s} \tag{9}$$

has exactly two real, positive roots, $\lambda_1(s)$, $\lambda_2(s)$, which satisfy $0 < \lambda_1(s) < 1 < \lambda_2(s)$. The sequence

$$E_j(s) = \frac{(\lambda_2 - 1)\lambda_1{}^j + (1 - \lambda_1)\lambda_2{}^j}{\lambda_1{}^a(\lambda_2 - 1) + \lambda_2{}^a(1 - \lambda_1)}$$

satisfies an equation analogous to Eq. (8), plus the boundary conditions

$$E_0(s) = E_1(s), E_a(s) = 1$$

Because $E_j(s)$ is also convex in $j$, we have

$$E_j(s) \geqslant 1 \quad (a \leqslant j \leqslant a + d - 1)$$

Let $\Delta_j(s) = E_j(s) - U_j(s)$ for $0 \leqslant j \leqslant a + d - 1$. Then

$$p\Delta_{j+d}(s) + q\Delta_{j-1}(s) = \frac{1}{s}\Delta_j(s) \quad (1 \leqslant j \leqslant a - 1) \tag{10}$$

$$\Delta_0(s) = \Delta_1(s) \tag{11}$$

$$\Delta_j(s) \geqslant 0 \qquad (a \leqslant j \leqslant a + d - 1) \tag{12}$$

We assert that $\Delta_j(s) \geqslant 0$ for $0 \leqslant j \leqslant a + d - 1$. To prove this let

$$m = \Delta_r(s) = \min\{\Delta_j(s)\colon\ 0 \leqslant j \leqslant a + d - 1\}$$

We want to show $m \geqslant 0$. If $a \leqslant r \leqslant a + d - 1$, we are done, by Eq. (12). Otherwise, we can assume $r \geqslant 1$ because of Eq. (11), and we have, from Eq. (10),

$$p\left(\Delta_{r+d} - \frac{m}{s}\right) + q\left(\Delta_{r-1} - \frac{m}{s}\right) = 0$$

Since $\Delta_{r+d} \geqslant m$, $\Delta_{r-1} \geqslant m$, we have

$$pm\left(1 - \frac{1}{s}\right) + qm\left(1 - \frac{1}{s}\right) \geqslant 0$$

and so $m \geqslant 0$.

We have thus derived the bound

$$U_1(s) \leqslant E_0(s) \tag{13}$$

By a similar argument,

$$U_1(s) \geqslant F_0(s) \tag{14}$$

where $F_0(s)$ is like $E_0(s)$ except that $a$ is replaced by $a + d - 1$.

From now on, assume that $q > pd$. An inspection of Eq. (9) shows that

$$\frac{1 - \lambda_1(s)}{1 - s} \to \frac{1}{q - pd}, \lambda_2(s) \to \lambda > 1$$

as $s \to 1-$. From this we see that $(1 - E_0(s))/(1 - s)$ and $(1 - F_0(s))/(1 - s)$ both tend to finite limits as $s \to 1-$. Hence, $(1 - U_1(s))/(1 - s)$ tends to a finite limit $D_1$. This shows, first, that the absorption time is finite with probability 1, and second, that its expectation is $D_1$. In fact, the above limits give the same upper and lower bounds on $D_1$ as Ref. 2 gives, namely

$$\frac{1}{q - pd}\left(\frac{\lambda^a - 1}{\lambda - 1} - a\right) \leqslant D_1 \leqslant \frac{1}{q - pd}\left(\frac{\lambda^b - 1}{\lambda - 1} - b\right) \tag{15}$$

where $b = a + d - 1$, and $\lambda$ is the unique real number satisfying $\lambda > 1, p\lambda^d + q\lambda^{-1} = 1$. Therefore, as in Ref. 2, we have

$$E_{FA} \geqslant \frac{1}{2(q - pd)}\left(\frac{\lambda^a - 1}{\lambda - 1} - a\right)$$

because Counter 2 operates twice each bit time.

## IV. A Tail Estimate for the Absorption Time

Let $\tau$ be the absorption time for the random walk discussed in the last section, where the walk starts at height 1. Equation (15) gives bounds for $E(\tau) = D_1$, and we now desire a bound for the left-hand tail probabilities $P\{\tau < n\}$.

We say that our random walk $X_n$ is reflected at time $n \geqslant 1$ if $X_{n-1} = 1$, $X_n = 1$. In other words, the walk returns to 1 and then tries to get to 0. There is a certain probability $\alpha$ that the random walk is absorbed at $a$ without ever undergoing a reflection. If, however, the walk is reflected, it "starts from scratch;" again it has probability $\alpha$ of being absorbed before reflection. Thus, if $N$ is the number of reflections before final absorption, we have

$$P\{N = 0\} = \alpha, P\{N = 1\} = (1 - \alpha)\alpha, \cdots$$

$$P\{N = n\} = (1 - \alpha)^n\alpha, \cdots$$

We invoke the absurdly simple inequality

$$\tau \geqslant N$$

and its consequence

$$P\{\tau < n\} \leqslant P\{N < n\} = 1 - (1 - \alpha)^n \tag{16}$$

For our situation this estimate is not bad; because $p \ll 1$ and the average drift rate $pd - q$ is negative, most of the intervals between reflections have length 1. To use Eq. (16) we need to compute $\alpha$. This is the familiar gambler's ruin problem with barriers at 0 and $a$. Again using the difference equation technique, Ref. 4, Chap. XIV, Eq. (8.12) gives

$$\frac{\lambda - 1}{\lambda^{a+d-1} - 1} \leqslant \alpha \leqslant \frac{\lambda - 1}{\lambda^a - 1} \tag{17}$$

Letting $\alpha^* = (\lambda - 1)/(\lambda^a - 1)$, we have

$$P\{\tau < n\} \leqslant 1 - (1 - \alpha^*)^n$$

$$\approx 1 - e^{-n\alpha^*}$$

for $n\alpha^{*2} \ll 1$. Since Counter 2 operates twice each bit time, the false-alarm probability $P_{FA}$ for Counter 1 satisfies

$$\begin{aligned} P_{FA} &\lessapprox 1 - \exp(-2n_b \alpha^*) \\ &\approx 2n_b \alpha^* \text{ for } 2n_b\alpha^* \ll 1 \end{aligned} \tag{18}$$

Finally, observe that

$$E(\tau) \geqslant E(N) = \frac{1}{\alpha} - 1 \geqslant \frac{\lambda^a - \lambda}{\lambda - 1} \tag{19}$$

The quality of Eq. (16) can be judged by comparing Eq. (19) with Eq. (15). Essentially, we are giving up a factor $q - pd$ in the mean.

## V. Numerical Example

Let us substitute numbers from the design given in Ref. 2. The parameters are $p = 6.13 \times 10^{-3}$, $k = 8$, $T = 511$. Then we have $d = 79$, $a = 1023$, $q - pd = 0.5096$, $\lambda = 1.016408599$, $\alpha^* = (\lambda - 1)/(\lambda^a - 1) = 1/(1.037 \times 10^9)$.

For the false-alarm probability during $n_b$ bits, and the expected false-alarm time, we have

$$P_{FA} \lessapprox \frac{2n_b}{10^9} \text{ for } 2n_b \ll 10^9 \tag{20}$$

$$E_{FA} \gtrapprox \frac{\alpha^*}{2(0.5096)} \approx 10^9 \text{ bits} \tag{21}$$

In particular, if $n_b = 10^9/100 = 10^7$ bits, then $P_{FA} \lessapprox 0.02$.

## VI. Conclusions

We have seen that it is not difficult to get practical estimates for the behavior of Counter 2, a random walk with independent steps. It appears that the false-alarm time for Counter 2 is approximately exponentially distributed; estimates for the distribution and its mean have been given. Although these estimates could be refined, we think that the real loss comes from the estimate "Counter 1 $\leqslant$ Counter 2;" a brief simulation showed that the excursions of Counter 2 were much greater than those of Counter 1. The real $P_{FA}$ of Counter 1 is probably much less than the 0.02 upper bound based on Counter 2 theory.

## Acknowledgment

I thank Gary Lorden for helping me with the estimation of absorption time probabilities.

## References

1. Greenhall, C. A., and Miller, R. L., "Quick-Look Decoding Schemes for DSN convolutional Codes," *DSN Progress Report 42-51*, 162-166, Jet Propulsion Laboratory, Pasadena, Calif., 1979.
2. Greenhall, C. A., and Miller, R. L., "Design of a Quick-Look Decoder for the DSN (7, 1/2) Convolutional Code," *DSN Progress Report 42-53*, 93-101, Jet Propulsion Laboratory, Pasadena, Calif., 1979.
3. Greenhall, C. A., Miller, R. L., and Butman, S. A., "A Quick-Look Decoder With Isolated Error Correction and Node Synchronization," *IEEE Transactions on Communications*, July 1981 (to appear).
4. Feller, W., *An Introduction to Probability Theory and Its Applications*, Vol. 1, 3rd Edition, Wiley and Sons, New York, N. Y., 1968.

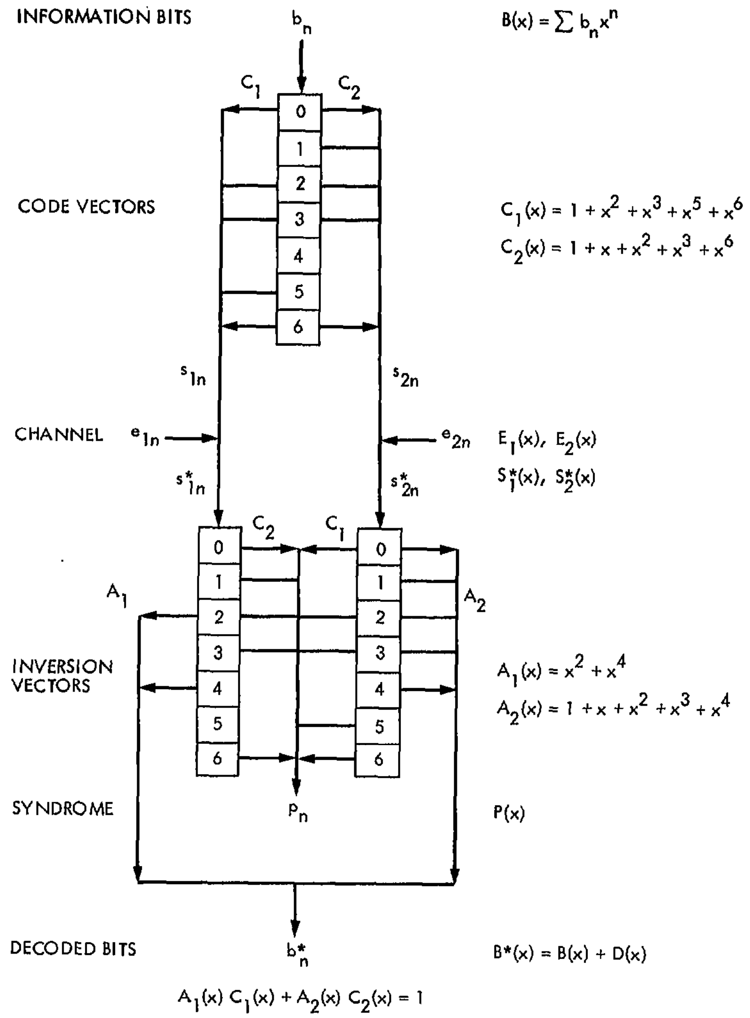


**Fig. 1. Quick-look decoder for the DSN (7, 1/2) code**