An Efficient Technique for the Performance Evaluation of Antenna Arrays with Noisy Carrier Reference

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An efficient computational technique is developed to evaluate the performance of coherent receivers with noisy carrier reference and multiple antennas. The received signal is assumed to be uncoded residual carrier BPSK (binary phase shift keying), with a PLL used for extracting the carrier. Explicit relationships between the error probabilities and the various system parameters are given. Specific results are given for the performance gain of combined carrier referencing over baseband only combining when the channel alignment process is ideal. A simple asymptotic expression for the performance gain is determined when the number of antennas used is increased without bound. An example using a Block III DSN PLL illustrates the performance of each arraying structure. The technique used in this paper is applicable to the performance evaluation for other receivers having similar decision statistics.

I. Introduction

This paper presents a technique for computing the probability of an uncoded bit detection error for a coherent receiver with a noisy carrier reference. The received signal is assumed to be a BPSK (binary phase shift keying) waveform, with a residual carrier that is extracted by means of a phase locked loop (PLL). The generality of the technique allows the application to multiple antenna receivers. In this paper, we compare two basic designs for antenna arraying: combined carrier referencing and baseband only combining.

The structures of these two basic designs are illustrated in Figs. 1 and 2. Due to the different path lengths between the antennas and the transmitting source, the received RF signals will be delayed relative to each other. We assume that no attempt to compensate for these delays is made prior to the point of input illustrated in Figs. 1 and 2.

In baseband only combining (Fig. 1), each IF channel has its own PLL. Each PLL derives a reference which is used to demodulate that channel to baseband. The alignment of the baseband signals is then performed, after which they are coherently added together. Bit detection is then accomplished using the combined signal.

In combined carrier referencing (Fig. 2), the alignment is performed immediately at IF (intermediate frequency) thus allowing coherent summation and a resultant SNR gain at IF. A single PLL is now used to derive the carrier reference. The loop SNR is correspondingly higher than in the baseband case, hence the derived reference will have a smaller phase error. This reference is used to mix down to baseband, whereupon the detection is performed as in the baseband case.

This paper compares the performance of each of these structures under the assumption that the alignment processes are perfect. We will see that combined carrier referencing yields superior performance. The problem, of course, is that it is more difficult to align at IF than at baseband. Thus combined carrier referencing will be more costly and will depend more critically on alignment errors.

The development of the technique presented in this paper was motivated by the anticipated need for antenna arraying at Voyager Uranus encounter. For this mission $B_L T_b$ will be much smaller than unity, where B_L is the PLL loop bandwidth and T_b is the bit interval. The technique given provides an efficient means for evaluating the performance of a multiple antenna coherent receiver of uncoded BPSK data whenever $B_L T_B \ll 1$.

Lindsey (Ref. 1) showed that when $B_L T_b \ll 1$ the performance evaluation of a (single antenna) coherent receiver reduces to calculating an integral of the form

$$\int_0^{\pi} \frac{\exp[\rho \cos \phi]}{\pi I_0(\rho)} Q(\alpha \cos \phi)\, d\phi \tag{1}$$

where I_0 is the zeroth order modified Bessel function and

$$Q(x) = \int_x^{\infty} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt \tag{2}$$

The integral (1) may be computed by numerical integration. Alternatively, the integration may be carried out analytically, yielding an infinite series of Bessel functions. Neither of these methods are as efficient as the one described in this paper.

Layland (Ref. 2) essentially showed that for baseband only combining (1) becomes a volume integral of dimensionality equal to the number of antennas. Results are given in Ref. 2 for the two antenna baseband array.

In the case of $B_L T_b \gg 1$ the performance is easily computed; expressions for the multiple antenna case are given in this paper. Exact performance evaluation for intermediate values of $B_L T_b$ is difficult, and has been approached using a Gaussian approximation for the phase error as well as the approximation $\cos \phi \cong 1 - \phi^2/2$ for the detector characteristic (Refs. 3-5). Layland used these ideas for some of his two antenna baseband array results. We will not be concerned with these intermediate ranges in this paper. However, it should be noted that the performance for intermediate values of $B_L T_b$ can be bounded by the results obtained using the $B_L T_b \ll 1$ and $B_L T_b \gg 1$ techniques; this is verified using convexity arguments (Ref. 3).

Recent work on evaluating the performance of antenna arrays has been done by Divsalar, Hansen and Yuen (Ref. 12), and Deutsch, Miller and Butman (Ref. 13). Both of these papers are concerned with coded BPSK signals, and use numerical multidimensional integration to obtain specific results. Additionally, simulation results are reported in Ref. 13. Here we evaluate the performance for the uncoded BPSK signal. An upper bound on the bit error probability for the coded case can be obtained by applying the union bound to the uncoded results (Ref. 14).

This paper is organized into six sections. The mathematical model will be developed in section II. The derivation and general form for the basic computational technique is presented in section III. In section IV we determine the relative asymptotic performance between the two arraying structures as the number of antennas increases without bound. Section V gives numerical results using receiver specifications from the Deep Space Network (DSN). Finally, the last section presents a summary and describes possible extensions of the technique.

II. Mathematical Model

Suppose that the incoming IF signals illustrated in Figs. 1 and 2 are given by

$$y_m(t) = \sqrt{2P_m} \cos[\omega_I(t - t_m) + \theta_0 + \theta D(t - t_m)] + n_m(t), m = 1, \cdots, M \tag{3}$$

where

M = number of antennas

ω_I = radian IF

θ_0 = unknown constant phase

θ = modulation index

$D(t)$ = NRZ (nonreturn to zero) data stream, consisting of equiprobable + or −1 symbols during each interval $[(k-1)T_b, kT_b)$, k integer

T_b = bit interval

P_m = signal power in the m^{th} channel

t_m = delay in m^{th} channel due to path differences

$n_m(t)$ = white Gaussian thermal noise in m^{th} channel, assumed to be independent of $D(t)$ and of $n_l(t)$ for $l \neq m$

$N_m/2$ = two-sided power spectral density of $n_m(t)$

In our analysis we assume the alignment process is perfect in either case; thus we may take $t_m = 0$ for all m.

A. Baseband Only Combining

Each PLL forms a carrier reference

$$r_m(t) = \sqrt{2} \sin [\omega_I t + \theta_0 + \phi_m(t)] \quad m = 1, \cdots, M \tag{4}$$

where $\phi_m(t)$ is the phase reference error. From Ref. 6, the stochastic process $\phi_m(t)$ has a stationary probability density given by

$$p(\phi_m) = \lim_{t \to \infty} P(\phi_m(t)) = \frac{\exp [\rho_m \cos \phi_m]}{2\pi I_0 (\rho_m)}, \; -\pi \leqslant \phi_m < \pi \tag{5}$$

where I_0 is the zeroth order modified Bessel function, and where ρ_m is the loop equivalent SNR:

$$\rho_m = \frac{P_{cm}}{N_m B_m (P_{cm}/N_m)} \tag{6}$$

where

P_{cm} = m^{th} channel residual carrier power

$= P_m \cos^2 \theta$

and

$B_m(\cdot) = m^{\text{th}}$ PLL loop bandwidth, which is a function of P_{cm}/N_m

Mixing $r_m(t)$ and $y_m(t)$ yields the baseband signals

$$U_m(t) = \sqrt{P_m} \sin \theta \, D(t) \cos [\phi_m(t)] + q_m(t) \quad m = 1, \cdots, M \tag{7}$$

where $q_m(t)$ may be taken to still be white noise with density $N_m/2$. We approximate the situation by assuming $q_m(t)$ and $\phi_m(t)$ are independent processes. The bias term

$$\sqrt{P_m} \cos \theta \sin [\phi_m(t)]$$

has been subtracted out of (7). In practice, this term is spectrally isolated by multiplying $D(t)$ by a squarewave subcarrier.

At this point the signals are aligned. We then form the weighted sum

$$\sum_{m=1}^{M} W_m U_m(t)$$

(Note that the values of the optimal weights are not obvious due to the noisy phase term.)

The detector then finds (assuming perfect bit synchronization)

$$Y_B = \int_0^{T_b} \sum_{m=1}^{M} W_m \left\{ \sqrt{P_m} \sin \theta \, D(t) \cos [\phi_m(t)] + q_m(t) \right\} dt$$

$$= D(0) \sum_{m=1}^{M} W_m \sqrt{P_m} \sin \theta \int_0^{T_b} \cos [\phi_m(t)] \, dt + \nu \tag{8}$$

where ν is a zero mean Gaussian random variable with variance

$$\sigma_B^2 = \sum_{m=1}^{M} W_m^2 \frac{N_m T_b}{2} \tag{9}$$

The Bayes estimate of $D(0)$ is

$$\hat{D}(0) = \begin{cases} +1 & \text{if } Y_B > 0 \\ -1 & \text{if } Y_B < 0 \end{cases} \tag{10}$$

Assuming $D(0) = +1$ was sent, a bit detection error is made with probability

$$PE_B = Pr \{ Y_B < 0 \mid D(0) = +1 \} \tag{11}$$

B. Combined Carrier Referencing

In this case the signals are aligned immediately at IF. Then the weighted sum is formed

$$z(t) = \sum_{m=1}^{M} W'_m y_m(t) \tag{12}$$

The optimum weights are given by

$$W'_m = c\frac{\sqrt{P_m}}{N_m} \tag{13}$$

for any constant $c \neq 0$.

The total signal power is

$$\left(\sum_{m=1}^{M} W'_m \sqrt{P_m}\right)^2 \tag{14}$$

and the residual carrier power is

$$P_c = \left(\sum W'_m \sqrt{P_m} \cos\theta\right)^2 \tag{15}$$

Since the noise adds noncoherently, the resultant two-sided noise power spectral density is

$$\frac{N}{2} = \sum_{m=1}^{M} W'^2_m \frac{N_m}{2} \tag{16}$$

The case of combined carrier referencing is equivalent to a single antenna with the above power levels. The reference phase error ϕ has a density of the form given by (5) where the loop SNR is

$$\rho_c = \frac{P_c}{NB_L(P_c/N)} \tag{17}$$

where $B_L(\cdot)$ is the (single) PLL loop bandwidth. If optimal weighting is used then

$$\rho_c = \sum_{m=1}^{M} \frac{P_m \cos^2\theta}{N_m B_L(P_c/N)} \tag{18}$$

The detector output is then

$$Y_c = D(0)\int_0^{T_b} \cos[\phi(t)]\,dt \left(\sum_{m=1}^{M} W'_m \sqrt{P_m} \sin\theta\right) + \nu' \tag{19}$$

where ν' is a zero mean Gaussian random variable with variance

$$\sigma_c^2 = \sum_{m=1}^{M} W'^2_m \frac{N_m T_b}{2} \tag{20}$$

Using a decision rule similar to (10), we obtain a bit detection error with probability

$$PE_c = Pr\{Y_c < 0 \mid D(0) = +1\} \tag{21}$$

III. Performance Evaluation

The decision statistics given by (8) and (19) depend on the evaluation of the integral

$$\frac{1}{T_b}\int_0^{T_b} \cos[\phi(t)]\,dt \tag{22}$$

In general, we require the joint density function for every finite set of points to characterize the stochastic process $\phi(t)$. However, if we assume $B_L T_b \ll 1$, the integrand will essentially be constant over the entire bit duration. Since the decision process is independent from one interval to the other, we can write

$$\frac{1}{T_b}\int_0^{T_b} \cos[\phi(t)]\,dt \approx \cos\phi \tag{23}$$

where ϕ is a random variable with density function given by (5). Hence for baseband only combining, from (8)

$$Y_B = D(0)\sum_{m=1}^{M} \alpha_m \cos\phi_m + \nu \tag{24}$$

where ϕ_m is the phase reference error from the m^{th} PLL, and

$$\alpha_m = W_m T_b \sqrt{P_m} \sin\theta \quad, \quad m = 1, \ldots, M.$$

Also, for combined carrier referencing, from (19)

$$Y_c = D(0)\,\alpha' \cos\phi' + \nu' \tag{25}$$

where ϕ' is the phase reference error from the single PLL and

$$\alpha' = T_b \sum_{m=1}^{M} W'_m \sqrt{P_m} \sin\theta$$

As was pointed out in section II, determination of the performance of a carrier array is mathematically equivalent to that of a single antenna with the appropriate parameter settings. Therefore, we will concentrate on evaluating the baseband array performance, and identify the carrier array result as a special case.

For baseband only combining, if we condition on $D(0) = +1$, the decision statistic Y_B involves a random variable of the form

$$X = \sum_{m=1}^{M} \alpha_m \cos\phi_m$$

where the α_m's are constants and the ϕ_m's are independent random variables with density functions $p(\phi_m)$ given by (5).

We extend the density function $p(\phi_m)$ to $\hat{p}(\phi_m)$ such that

$$\hat{p}(\phi_m) = \sum_{n=-\infty}^{\infty} p(\phi_m + 2n\pi)$$

Hence

$$\hat{p}(\phi_m) = \hat{p}(\phi_m + 2\pi)$$

and

$$\hat{p}(\phi_m) = \sum_{n=-\infty}^{\infty} C_{n,m} e^{jn\phi_m}$$

where

$$C_{n,m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\phi_m) e^{-jn\phi_m} d\phi_m$$

From (5)

$$C_{n,m} = \frac{1}{2\pi} \frac{I_n(\rho_m)}{I_0(\rho_m)} \tag{26}$$

where $I_n(\rho)$ is the n^{th} order modified Bessel function.

The characteristic function of the random variable X can be found to be

$$\Phi_X(u) = E[e^{-juX}] = \prod_{m=1}^{M} \sum_{n=-\infty}^{\infty} 2\pi j^n C_{n,m} J_n(-u\alpha_m) \tag{27}$$

where $J_n(\cdot)$ is the n^{th} order Bessel function.

Substituting (26) into (27) and using the identities (Ref. 7, p. 30)

$$I_n(\rho) = j^n J_n(-j\rho)$$

and

$$J_0(x+y) = \sum_{n=-\infty}^{\infty} J_{-n}(x) J_n(y) \quad ,$$

we obtain

$$\Phi_X(u) = \prod_{m=1}^{M} \frac{I_0(\rho_m - ju\alpha_m)}{I_0(\rho_m)} \tag{28}$$

From (24) (or (25)), the decision statistic is of the form

$$Y = X + n \tag{29}$$

where X and n are independent and where n is a Gaussian random variable with zero mean and variance σ_n^2.

The characteristic function of the random variable Y will be given by

$$\Phi_Y(u) = \Phi_X(u) e^{-\frac{1}{2}\sigma_n^2 u^2}$$

$$= e^{-\frac{1}{2}\sigma_n^2 u^2} \prod_{m=1}^{M} \frac{I_0(\rho_m - ju\alpha_m)}{I_0(\rho_m)} \tag{30}$$

From (11) (or (21)), the probability of making an error based on the statistic Y in (29) can be written as

$$PE = \int_{-\infty}^{0} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_Y(u) e^{juy} du\right] dy \tag{31}$$

Assuming we can interchange the order of integration, and recognizing (Ref. 8, p. 42)

$$\int_{-\infty}^{0} e^{juy}\, dy = \pi\delta(u) + \frac{1}{ju}$$

we can write

$$PE = \frac{1}{2} + \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{1}{u} e^{-\frac{1}{2}\sigma_n^2 u^2} \Phi_X(u)\, du$$

From the Taylor Series Expansion of $\Phi_X(u)$, after some mathematical manipulation, we obtain (Ref. 9, p. 302)

$$PE = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k E[X^{2k-1}]}{(2k-1)!} \int_{0}^{\infty} u^{2(k-1)} e^{-\frac{1}{2}\sigma_n^2 u^2} du$$

$$= \frac{1}{2} - \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)} E\left[\left(\frac{X}{\sqrt{2}\sigma_n}\right)^{2n+1}\right] \quad (32)$$

The probability of error for baseband only combining is determined using (32) with the appropriate moments. From (24) and (9) these can be written as

$$E\left[\left(\frac{X}{\sigma_n}\right)^{2n+1}\right] = E\left[\left(\sum_{m=1}^{M} \beta_m \cos\phi_m\right)^{2n+1}\right] \quad (33)$$

where

$$\beta_m = W_m \sqrt{\frac{2E_m}{\sum_{k=1}^{M} W_k^2 N_k}} \sin\theta, \quad m = 1, \ldots, M \quad (34)$$

and

$$E_m = P_m T_b$$

is the energy per bit at the m^{th} antenna input. The moments (33) can be evaluated by the iterative procedure shown in the Appendix. The technique given there is similar to the one used by (Ref. 10, 11).

The probability of error for combined carrier referencing is also computed using (32). In this case (25) and (20) imply

$$E\left[\left(\frac{X}{\sigma_n}\right)^{2n+1}\right] = E\left[(\beta' \cos\phi')^{2n+1}\right] \quad (35)$$

where

$$\beta' = \frac{\sum_{k=1}^{M} W_k' \sqrt{2E_k}}{\sqrt{\sum_{l=1}^{M} W_l'^2 N_l}} \sin\theta \quad (36)$$

It is interesting to note that if X in (32) is almost surely constant then

$$E\left[\left(\frac{X}{\sqrt{2}\sigma_n}\right)^{2n+1}\right] = \left(\frac{X}{\sqrt{2}\sigma_n}\right)^{2n+1}$$

and

$$PE = \frac{1}{2} - \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)} \left(\frac{X}{\sqrt{2}\sigma_n}\right)^{2n+1} = Q\left(\frac{X}{\sigma_n}\right) \quad (37)$$

where $Q(x)$ is given by (2). Equation (37) is the well known result for the error probability of using BPSK across a white Gaussian noise channel with no phase reference error.

In the next section, we will show that for a large number of antennas, essentially the Strong Law of Large Numbers guarantees that the k^{th} moment converges to the first moment to the k^{th} power. This has been observed in computations using the algorithm of the appendix.

IV. Asymptotic Performance

In this section we determine the performance of baseband only combining as the number of antennas approaches infinity, and compare it to that of combined carrier referencing. The antennas are assumed to be identical and are weighted equally. For baseband only combining all the β_m's of (34) are equal and given by

$$\beta_m = \frac{1}{M}\sqrt{\frac{2ME_b}{N_0}} \sin\theta \quad (38)$$

where $E_b = E_m$ and $N_o = N_m$ for $m = 1, 2, \cdots, M$. Using (8) and (11), by first conditioning on the vector of phase errors, we obtain

$$PE_B = E[Q(B)] \quad (39)$$

where $Q(\cdot)$ is given by (2), and B is the random variable

$$B = \sqrt{\frac{2ME_b}{N_0}} \sin\theta \frac{1}{M} \sum_{m=1}^{M} \frac{1}{T_b} \int_0^{T_b} \cos\phi_m(t)\, dt \quad (40)$$

The variance of B is

$$\mathrm{Var}[B] = \frac{2E_b}{N_0} \sin^2\theta \ \mathrm{Var}\,[Z_1] \quad (41)$$

where

$$Z_1 = \frac{1}{T_b} \int_0^{T_b} \cos\phi_1(t)\, dt \quad (42)$$

Since $|Z_1| \leqslant 1$, Var $[Z_1] \leqslant 1$. If we fix the total system energy per bit

$$E_s = ME_b$$

then

$$\mathrm{Var}\,[B] \leqslant \frac{1}{M} \frac{2E_s}{N_0} \sin^2\theta$$

tends to zero as M increases, and we may approximate B by its mean. Taking the expectation inside the integral (42) and using the density function (5) we obtain

$$E[Z_1] = \frac{I_1(\rho_B)}{I_0(\rho_B)} \quad (43)$$

where ρ_B is the common loop SNR for each PLL. Thus for $E_b/N_0 << 1$ (typically true for $M >> 1$ and fixed E_s/N_0)

$$PE_B \approx Q\left[\sqrt{\frac{2ME_b}{N_0}} \sin\theta \frac{I_1(\rho_B)}{I_0(\rho_B)}\right], \frac{E_b}{N_0} << 1 \quad (44)$$

To compare this with the performance of combined carrier referencing, PE_c can be evaluated by (32) and (35), and is available in graphical form (Ref. 6). Alternatively, it may be possible to approximate PE_c as well. Assuming a similar PLL is used, $M >> 1$ implies $\rho_c >> \rho_B$ so that the phase error will be essentially zero. Then from (35) and (37)

$$PE_c \approx Q\left[\sqrt{\frac{2ME_b}{N_0}} \sin\theta\right], \rho_c >> 1 \quad (45)$$

From (44) and (45) we can define the asymptotic power loss L_∞ due to baseband only combining relative to combined carrier referencing as

$$L_\infty(\rho_B) = \left[\frac{I_1(\rho_B)}{I_0(\rho_B)}\right]^2, \frac{E_b}{N_0} << 1, \rho_c >> 1 \quad (46)$$

The conditions required by (46) will typically be satisfied when $M >> 1$.

The results given thus far in this section are independent of any assumption on $B_L T_B$. If $B_L T_B >> 1$ (slow rate model) then (42) can be treated as a time average, and (44) will be true for any M and E_b/N_0. PE_c may be obtained using (44) and replacing ρ_B by ρ_C. Thus we may define the relative power loss for this case as

$$L(\rho_B, \rho_C) = \frac{\left[\frac{I_1(\rho_B)}{I_0(\rho_B)}\right]^2}{\left[\frac{I_1(\rho_c)}{I_0(\rho_c)}\right]^2} \quad (47)$$

V. Numerical Example

In both (33) and (35), the density functions of ϕ' and ϕ_m, $m = 1, \ldots, M$, depend on the corresponding loop SNRs. The loop SNR depends on the loop bandwidth B_L, which in turn is a nonlinear function of the input signal and noise levels. The specific PLL design we use for our numerical example is based on Deep Space Network Block III receiver data. The nominal loop bandwidth is $2B_{LO} = 12$ Hz. Hence for any given T_b, θ, and set of E_m/N_m and W_m, $m = 1, 2, \ldots, M$, the corresponding loop SNR can be determined from either (6) or (17).

We consider an example where all antennas are identical and have unity weights (the theory does not require this assumption). In this case, all E_m/N_m, $m = 1, 2, \ldots, M$ become identical and are given by E_b/N_0, where E_b/N_0 is the energy per bit to noise ratio at each antenna input. Figure 3 shows the error probability based on the bit interval being 50 μs and the modulation index being 80°. Since $B_L T_b$ is much smaller than unity, (32) will yield exact performance results under the assumption the channel alignment process is ideal.

Figure 4 shows the comparative performance difference between the two arraying structures for the same example. Since combined carrier referencing always gives superior performance over baseband only combining, Fig. 4 shows the additional amount of signal energy required at each antenna input of the baseband array in order to obtain the same error probabilities as the carrier array. Note that when E_b/N_0 is small, there is not enough signal power for the PLL to extract the phase reference in the baseband array. A large number of antennas is then necessary to match the performance of combined carrier referencing. However, when E_b/N_0 increases, the performance of baseband only combining matches that of combined carrier referencing with a small number of antennas.

VI. Summary and Future Extension

In this paper, we have examined the performance of two antenna arraying techniques which may be used in the Deep Space Network. The computational technique developed can be applied to any combination of antennas. These antennas need not be identical. A simple expression is given to determine the asymptotic comparative performance between the two arraying structures.

Specifically, we have shown that with a Block III DSN PLL and three identical antennas, combined carrier referencing will provide a 0.2 to 0.3 dB improvement over baseband only combining for the range of interest. Explicit error probability versus E_b/N_o is given for either configuration. As shown in Fig. 4, the ultimate gain to be achieved by using combined carrier referencing over baseband only combining is quickly approached with a small number of antennas.

Throughout this paper it has been assumed that a BPSK signal is used. It would seem likely that the same technique can be used to determine the error probability for DPSK (differential phase shift keying). Also, in attempting to study the case of MPSK (M-ary phase shift keying) signals, one encounters the following generalization of (31):

$$PE(k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_Y(u, v) \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_k(x,y)\, e^{j(ux+vy)}\, dxdy\right] du\, dv$$

where $PE(k)$ is the error probability conditioned on the k^{th} message being sent, Φ_Y is the characteristic function for the two-dimensional statistic $\underline{Y}$, and I_k is the indicator function for the k^{th} decision region. Evaluation of the inner double integral via generalized functions may provide an easily evaluatable expression for $PE(k)$.

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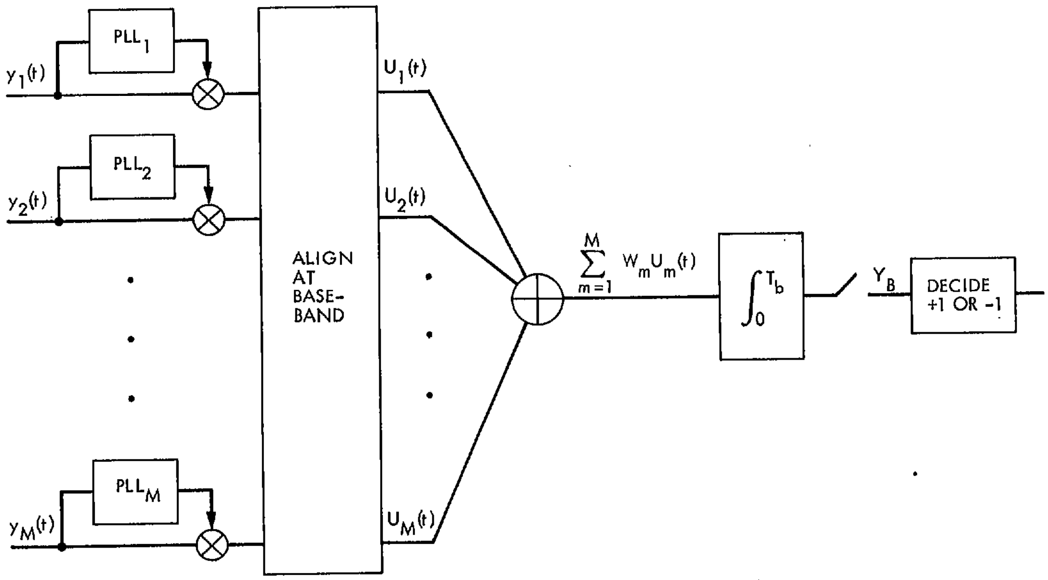

Fig. 1. Baseband only combining model

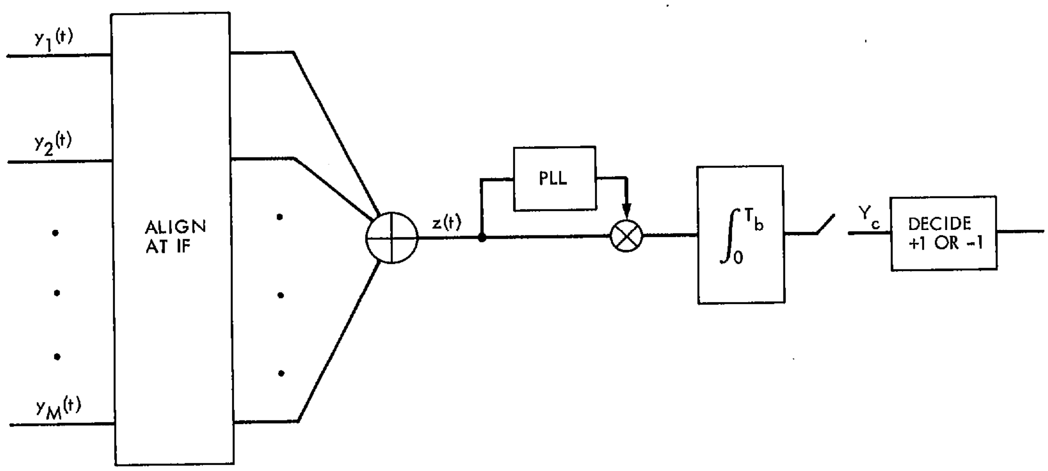

Fig. 2. Combined carrier referencing model

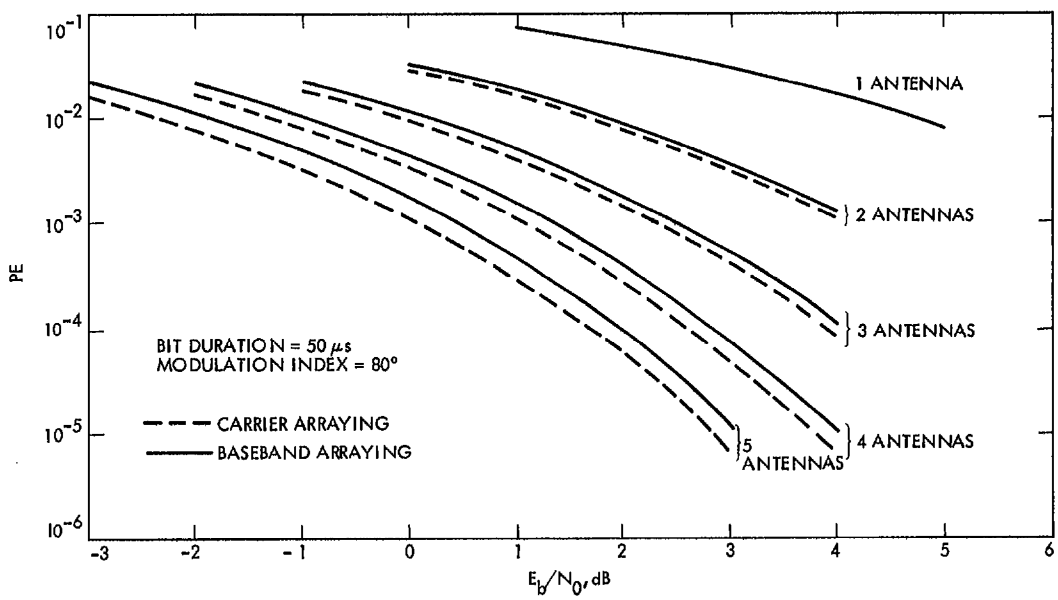

Fig. 3. Error probability versus E_b/N_0 for antenna arrays

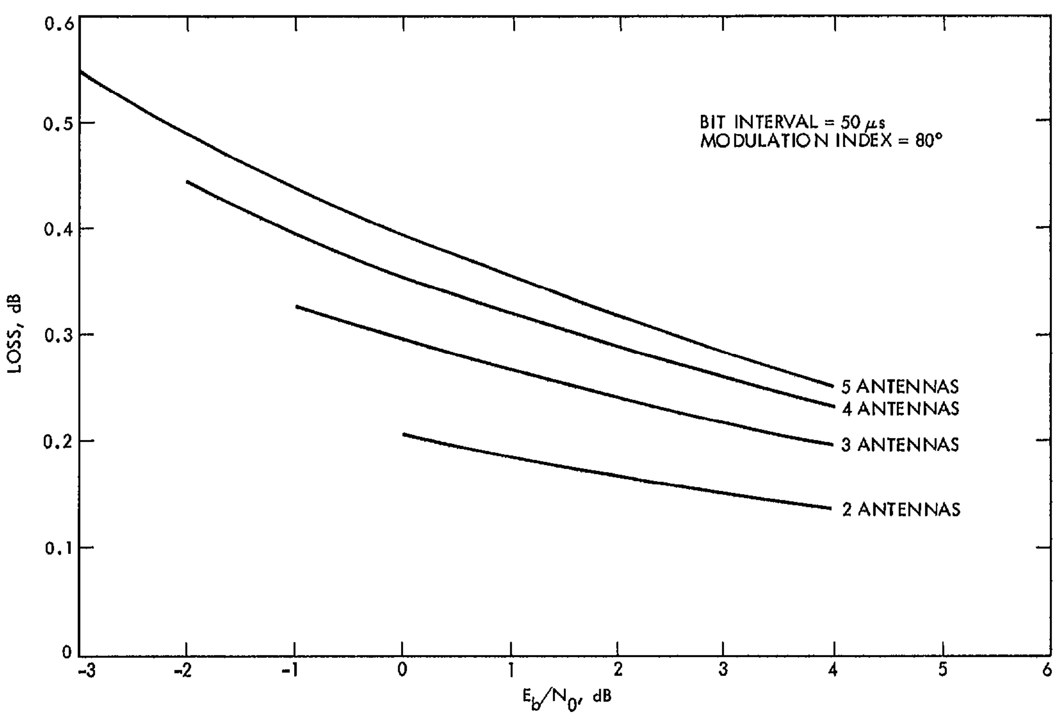

Fig. 4. Comparative performance loss between combined carrier referencing and baseband only combining

Appendix A
Evaluation of Moments

This appendix describes the computation of moments for the random variable.

$$X = \sum_{m=1}^{M} \beta_m \cos\phi_m$$

where β_m's are scalar constants and ϕ_m's are independent random variables with the Tikhonov density functions given by (5).

Define

$$Y_0 = 0$$

and

$$Y_n = Y_{n-1} + \beta_n \cos\phi_n \quad n = 1, \ldots, M \tag{A-1}$$

Then

$$X = Y_M$$

and

$$EX^k = EY_M^k$$

From (A-1)

$$EY_n^k = \sum_{j=0}^{k} \binom{k}{j} \beta_n^{k-j} E(Y_{n-1}^j) E \cos^{(k-j)}\phi_n \tag{A-2}$$

$$n = 1, \ldots, M \quad k = 0, 1, 2, \cdots$$

Hence all the moments of the random variable X can be evaluated recursively by (A-2) given $E(\cos^l \phi_n)$, $l = 1, \ldots, K$, $n = 1, \ldots, M$, where K is the largest moment required. From Ref. 9, p. 376,

$$\frac{d^k I_n(u)}{du^k} = \frac{1}{2^k}\left\{ I_{n-k}(u) + \binom{k}{1} I_{n-k+2}(u) + \binom{k}{2} I_{n-k+4}(u) + \cdots + I_{n+k}(u) \right\}$$

$$k = 0, 1, 2, \cdots \tag{A-3}$$

Then

$$E[\cos^l \phi_n] = \left(\frac{1}{2}\right)^l \sum_{m=0}^{l} \binom{l}{m} \frac{I_{2m-l}(\rho_n)}{I_0(\rho_n)} \tag{A-4}$$

for $l = 1, \cdots, K$ and $n = 1, \cdots, M$

Therefore

$$EY_n^k = \sum_{j=0}^{k} \binom{k}{j} \left(\frac{\beta_n}{2}\right)^{k-j} F_{k-j}(\rho_n)\,[E\,Y_{n-1}^j] \tag{A-5}$$

where

$$F_m(\rho) = \sum_{j=0}^{m} \binom{m}{j} \frac{I_{2j-k}(\rho)}{I_0(\rho)}$$

Define

$$E\overline{Y}_n = (EY_n^1, EY_n^2, EY_n^3, \cdots, EY_n^k)' \tag{A-6}$$

Then

$$E\overline{Y}_n = \underline{A}_n E\overline{Y}_{n-1} + \overline{B}_n \quad \text{for } n \geqslant 1 \tag{A-7}$$

and

$$E\overline{Y}_0 = \overline{0}$$

where

$$\underline{A}_n = \left\{ a_{i,j}^{(n)} \right\}, \overline{B}_n = \left\{ b_i^{(n)} \right\}$$

and

$$a_{i,j}^{(n)} = \begin{cases} 0 & \text{for } j > i \\ 1 & \text{for } j = i \\ \binom{i}{j}\left(\frac{\beta_n}{2}\right)^{(i-j)} F_{(i-j)}(\rho_n) & \text{for } i > j \end{cases}$$

$$b_i^{(n)} = \left(\frac{\beta_n}{2}\right)^i F_i(\rho_n)$$

Using (A-7)

$$E\overline{X} = E\overline{Y}_M$$

where

$$E\overline{X} = \left(EX, EX^2, EX^3, \cdots, EX^K\right)' .$$